Geometric Quantum Field Theories

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Based on: K Finn, S Karamitsos, AP, PRD102 (2020) 045014 [arXiv:1910.06661]
EPJC81 (2021) 572 [arXiv:2006.05831]

$\Longrightarrow$ Viola Gattus, AP, arXiv:2307.01126

Related works: S Karamitsos, AP, NPB907 (2016) 785; NPB927 (2018) 219

Outline:

- Brief History on Covariant or Geometric Methods in QFT
- Grand Covariance in Quantum Gravity
- The Fermion problem and Supermanifolds
- Supergeometric Effective Actions
- Conclusions & Future Directions

• Brief History on Covariant Methods in QFT

- B DeWitt '67: . . . *The Manifestly Covariant Quantum Field Theory*, introducing the well-known *Background Field Method*.

- S Weinberg '68, . . . J Honerkamp '72 . . . : *Geometry of Field Space and Amplitudes in Non-linear σ-Models*

  $\Rightarrow$ Little Higgs Models, SMEFT $\subset$ HEFT, Multifield Inflation . . .

- L Alvarez-Gaumé, DZ Friedman, S Mukhi '81: *Geometry and UV Finiteness of SUSY Non-linear σ-Models*

- G Vilkovisky '84, B DeWitt '85, *Gauge Independence of Covariant VDW Effective Action*

- MK Gaillard '86: *Gauge-Covariant Derivative Approach to EFT*

  $\Rightarrow$ Universal effective action, threshold effects in SMEFT . . .

- JM Cornwall '82, JM Cornwall, J Papavassiliou '89, AP '97:
  Pinch Technique: a Gauge-Covariant Diagrammatic Approach to Off-Shell Amplitudes

  $\Rightarrow$ Effective gluon mass, Non-Abelian effective charges,
  Unstable particle dynamics and High-energy unitarity of Amplitudes

- G Sigl, G Raffelt '93, PSB Dev, P Millington, AP, D Teresi '14:
  Flavour Covariant Transport Equations

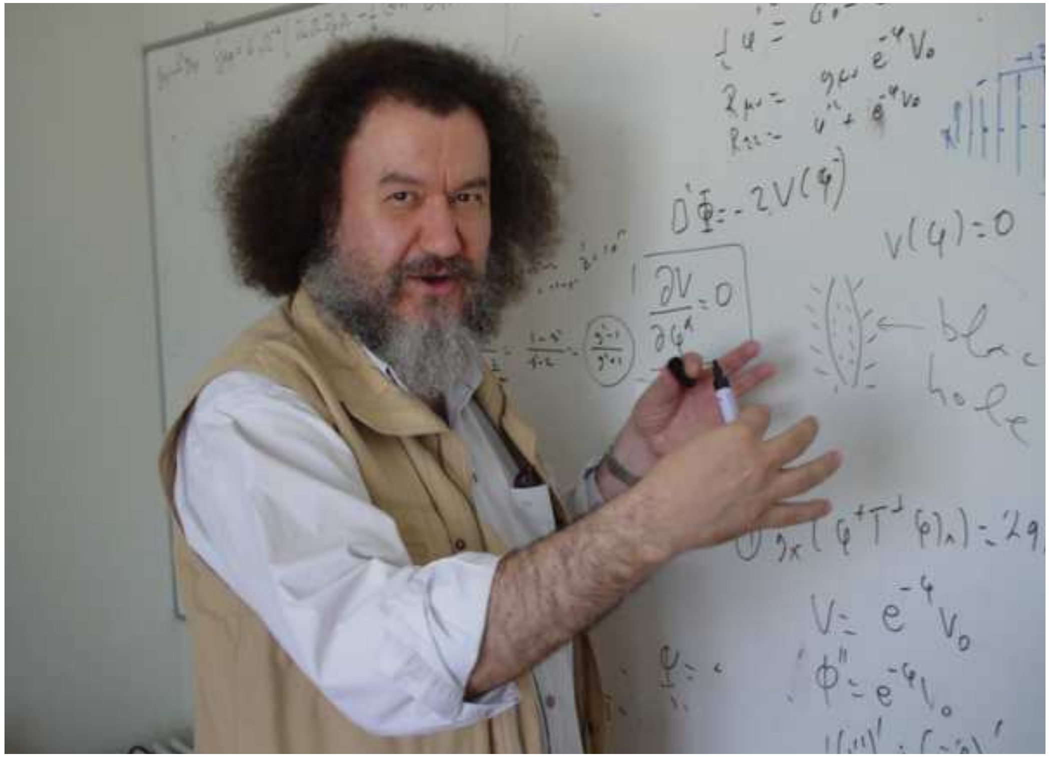
  $\Rightarrow$ Neutrino flavour dynamics in astrophysics and cosmology,
  Leptogenesis . . .

- D Teresi, AP '13:
  Symmetry Improved Cornwall-Jackiw-Tomboulis Effective Action

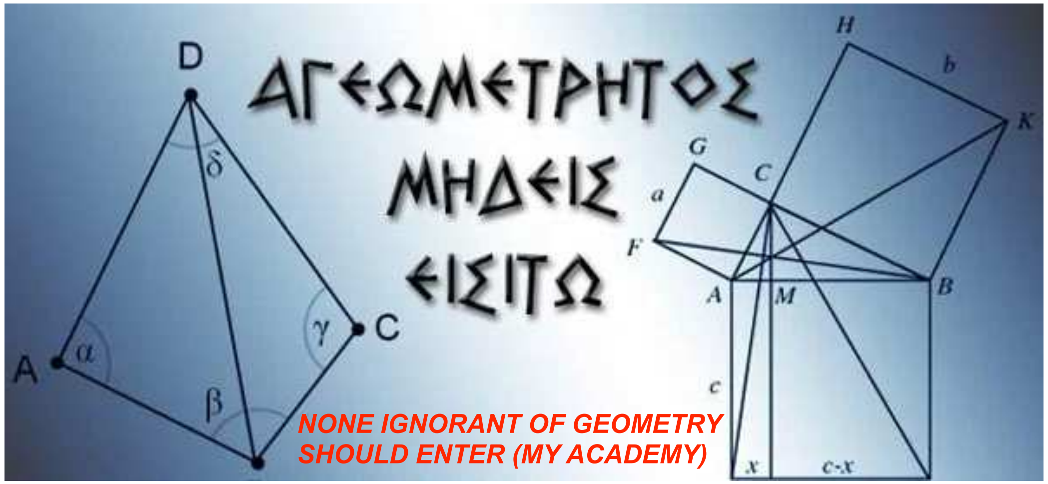
  $\Rightarrow$ Massless Goldstones, 2nd order phase transitions for $\mathbb{O}(N)$ theories,
  RG exactness and IR safe effective potentials . . .

$\vdots$

Goerge's Legacy of Corfu Summer Institutes



from Plato



• Grand Covariance in Quantum Gravity

[G Vilkovisky '84, B DeWitt, '85;
K Finn, S Karamitsos, AP, '19]

$$S \;=\; \int d^Dx\,\sqrt{-g}\,\left[-\,\frac{f(\boldsymbol{\varphi})}{2}R \;+\; \frac{1}{2}k_{AB}(\boldsymbol{\varphi})\,g^{\mu\nu}(\nabla_\mu\varphi^A)(\nabla_\nu\varphi^B) \;-\; V(\boldsymbol{\varphi})\right],$$

$S = S\big[g_{\mu\nu}, \boldsymbol{\varphi}; f(\boldsymbol{\varphi}), k(\boldsymbol{\varphi}), V(\boldsymbol{\varphi})\big]$: *classical action*

$f(\boldsymbol{\varphi}),\; k_{AB}(\boldsymbol{\varphi}),\, V(\boldsymbol{\varphi})$: *model functions*

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Grand *or* Frame Covariance:

(i) *Spacetime diffeomorphisms*

$$x^\mu \;\to\; \tilde{x}^\mu = \tilde{x}^\mu(x^\nu), \quad \text{with} \;\; ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \widetilde{g}_{\mu\nu}\, d\tilde{x}^\mu d\tilde{x}^\nu$$

(ii) *Field reparametrizations*

$$\begin{aligned} g_{\mu\nu} \;\; &\to \;\; \widetilde{g}_{\mu\nu} = \widetilde{g}_{\mu\nu}(g_{\kappa\lambda}, \boldsymbol{\varphi}) = \Omega^2(\boldsymbol{\varphi})\, g_{\mu\nu} \\ \varphi^A \;\; &\to \;\; \widetilde{\varphi}^A = \widetilde{\varphi}^A(g_{\mu\nu}, \boldsymbol{\varphi}) = \widetilde{\varphi}^A(\boldsymbol{\varphi}) \end{aligned}$$

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$$S = \int d^D x \sqrt{-g}\left[-\frac{f(\varphi)}{2}R + \frac{1}{2}k_{AB}(\varphi)\, g^{\mu\nu}(\nabla_\mu\varphi^A)(\nabla_\nu\varphi^B) - V(\varphi)\right],$$

$S = S\big[g_{\mu\nu}, \varphi; f(\varphi), k(\varphi), V(\varphi)\big]$: *classical action*

$f(\varphi),\ k_{AB}(\varphi),\ V(\varphi)$: *model functions*

Grand *or* Frame Covariance:

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(ii) *Field reparametrizations*

$$g_{\mu\nu} \to \widetilde{g}_{\mu\nu} = \widetilde{g}_{\mu\nu}(g_{\kappa\lambda}, \varphi) = \underline{\Omega^2(\varphi)\, g_{\mu\nu}}$$

$$\varphi^A \to \widetilde{\varphi}^A = \widetilde{\varphi}^A(g_{\mu\nu}, \varphi) = \widetilde{\varphi}^A(\varphi)$$

– Introduce new model function $\ell(\varphi)$ to restore (i)

$$ds^2 = \bar{g}_{\mu\nu}\, dx^\mu dx^\nu\,, \quad \text{with} \;\; \bar{g}_{\mu\nu} \;\equiv\; \frac{g_{\mu\nu}}{\ell^2(\varphi)}$$

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– **Transformation of model functions**

$$\begin{aligned}
\tilde{\ell}(\varphi) &= \Omega\, \ell(\varphi)\,, \\
\tilde{f}(\varphi) &= \Omega^{-2}\, f(\varphi)\,, \\
\tilde{k}_{\widetilde{A}\widetilde{B}}(\varphi) &= \Big[k_{AB} - 6f(\ln\Omega)_{,A}(\ln\Omega)_{,B} + 3f_{,A}(\ln\Omega)_{,B} + 3(\ln\Omega)_{,A}f_{,B}\Big]\partial^A\varphi_{\widetilde{A}}\,\partial^B\varphi_{\widetilde{B}}\,, \\
\widetilde{V}(\varphi) &= \Omega^{-4}\, V(\varphi)\,.
\end{aligned}$$

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\end{aligned}$$

– Frame invariance of the classical action S:

$$\boxed{S[\tilde{g}_{\mu\nu}, \widetilde{\varphi}; \tilde{\ell}(\varphi), \tilde{f}(\varphi), \tilde{k}(\varphi), \widetilde{V}(\varphi)] \;=\; S[g_{\mu\nu}, \varphi; \ell(\varphi), f(\varphi), k(\varphi), V(\varphi)] \;\; *}$$

Models related by a frame transformation define an *equivalence class*

[K Falls, M Herrero-Valea '19]

---

* $S = \int d^Dx\, \sqrt{-g}\mathcal{L} = \int d^Dx \sqrt{-\bar{g}}\, \overline{\mathcal{L}}$ is independent of $\ell(\varphi)$ (only at tree level).

– Coordinates of the Grand Configuration Space

$$\Phi^i \equiv \Phi^I(x_I) = \begin{pmatrix} g^{\mu\nu}(x) \\ \phi^A(x_A) \end{pmatrix}, \text{ with } i = \{I\,,\, x_I\}\,,\; I = \{\mu\nu\,,\, A\}\,,\; x_I = \{x\,,\, x_A\}\,.$$

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– The Grand Configuration Space Metric

$$\mathcal{G}_{ij} \equiv \frac{\bar{g}_{\mu\nu}}{D}\frac{\bar{\delta}^2 S}{\bar{\delta}(\partial_\mu\Phi^i)\bar{\delta}(\partial_\nu\Phi^j)} = \ell^2 \begin{pmatrix} f P_{\mu\nu\rho\sigma} & -\frac{3}{4}f_{,B}\, g_{\mu\nu} \\ -\frac{3}{4}f_{,A}\, g_{\rho\sigma} & k_{AB} \end{pmatrix} \bar{\delta}^{(D)}(x_I - x_J),$$

where $\bar{\delta}^{(D)}(x_I - x_J) \equiv \delta^{(D)}(x_I - x_J)/\sqrt{-g}$ is *frame invariant*, and

$$P_{\mu\nu\rho\sigma} \equiv G_{(\mu\nu)(\rho\sigma)} = \frac{1}{2}\Big(g_{\mu\rho}\, g_{\sigma\nu} + g_{\mu\sigma}\, g_{\rho\nu} - \alpha\, g_{\mu\nu}\, g_{\rho\sigma}\Big)$$

is the *gravitational field-space metric*.

Condition on the inverse metric $G^{(\mu\nu)(\rho\sigma)}$:

$$G^{(\mu\nu)(\rho\sigma)} = g^{\alpha\mu}g^{\beta\nu}g^{\kappa\rho}g^{\lambda\sigma}\, G_{(\alpha\beta)\,(\kappa\lambda)} \implies \alpha = 0 \text{ or } 1.$$

– Quantum Effective Action

$$\exp\left(\frac{i}{\hbar}\Gamma[\boldsymbol{\varphi}]\right) = \int [\mathcal{D}\boldsymbol{\phi}] \exp\left\{\frac{i}{\hbar}\Big[S[\boldsymbol{\phi}] + \frac{\delta\Gamma[\boldsymbol{\varphi}]}{\delta\varphi^a}(\varphi^a - \phi^a)\Big]\right\}.$$

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Not invariant under frame transformations

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– VDW Quantum Effective Action

$$\exp\left(\frac{i}{\hbar}\Gamma[\boldsymbol{\varphi}]\right) = \int [\overline{\mathcal{D}}\boldsymbol{\Phi}]\, \mathcal{M}[\boldsymbol{\Phi}]\, \exp\left\{\frac{i}{\hbar}\Big[S[\boldsymbol{\Phi}] + \frac{\bar{\delta}\Gamma[\boldsymbol{\varphi}]}{\bar{\delta}\varphi^i}\,\Sigma^i[\boldsymbol{\varphi},\boldsymbol{\Phi}]\Big]\right\},$$

with $\boldsymbol{\varphi} = (g^{\mu\nu},\ \phi)$,

$$[\overline{\mathcal{D}}\boldsymbol{\Phi}] = \exp\Big[\sum_I \int d^D x\, \sqrt{-\bar{g}(x)}\, \ln \mathcal{D}\Phi^I(x)\Big], \quad \mathcal{M}[\boldsymbol{\Phi}] = V_{\rm FP}\sqrt{\overline{\det}\,(\mathcal{G}_{ij})}\,,$$

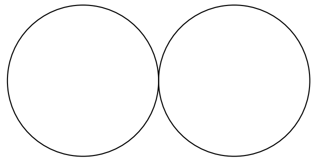
and $V_{\rm FP}$ is the *Faddeev–Popov determinant* [for SU(N), see Rebhan '87].

– One- and Two-Loop VDW Effective Actions

[consistent with P Ellicott, T Toms '89]

$$\Gamma^{(1)}[\boldsymbol{\varphi}] \;=\; \frac{i}{2}\ln\overline{\det}\big(\nabla^a\nabla_b S\big)\,,$$

$$\Gamma^{(2)}[\boldsymbol{\varphi}] \;=$$



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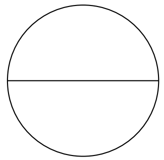

$$= -\frac{1}{8}\Delta^{ab}\Delta^{cd}\,\nabla_{(a}\nabla_b\nabla_c\nabla_{d)}S + \frac{1}{12}\Delta^{ab}\Delta^{cd}\Delta^{ef}\big(\nabla_{(a}\nabla_c\nabla_{e)}S\big)\big(\nabla_{(b}\nabla_d\nabla_{f)}S\big)\,,$$

with $\Delta^{ab} \equiv \big(\nabla_a\nabla_b S\big)^{-1} = \Delta^{ba}$.

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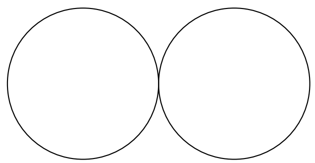
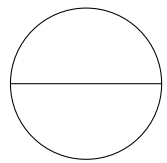
 $+$ 

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– Grand Covariance of the VDW Effective Action

[K Finn, S Karamitsos, AP, '19]

$$\boxed{\Gamma[\boldsymbol{\varphi};\ell(\phi),f(\phi),k_{AB}(\phi),V(\phi)\,] \;=\; \Gamma[\widetilde{\boldsymbol{\varphi}}(\boldsymbol{\varphi});\tilde{\ell}(\phi),\tilde{f}(\phi),\tilde{k}_{AB}(\phi),\widetilde{V}(\phi)\,]}$$

with $\boldsymbol{\varphi} = (g^{\mu\nu},\ \phi)$.

– Recent works

- J. Honerkamp, Nucl. Phys. B36 (1972) 130; G. Ecker, J. Honerkamp, Phys. Lett. B42 (1972) 253.
- $\vdots$
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- S. Karamitsos, AP, *Frame Covariant Nonminimal Multifield Inflation*, Nucl. Phys. B927 (2018) 219.
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- R. Nagai, M. Tanabashi, K. Tsumura, Y. Uchida, Phys. Rev. D100 (2019) 075020
- A. Helset, A. Martin, M. Trott, JHEP03 (2020) 163
- K. Finn, S. Karamitsos, AP, *Frame covariant formalism for fermionic theories*, Eur. Phys. J. C81 (2021) 572
- T. Cohen, N. Craig, X. Lu, D. Sutherland, JHEP12 (2021) 003
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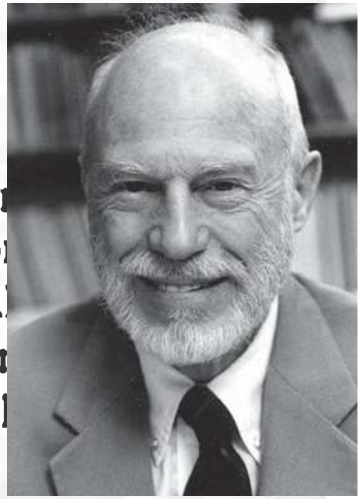
- A. Helset, E.E. Jenkins, A.V. Manohar, *Geometry in scattering amplitudes*, Phys. Rev. D106 (2022) 116018
- V. Gattus and AP, *Minimal Supergeometric Quantum Field Theories*, arXiv:2307.01126.

• The Fermion problem and Supermanifolds

The Effective Action

Bryce De Witt '85



14 DISCUSSION

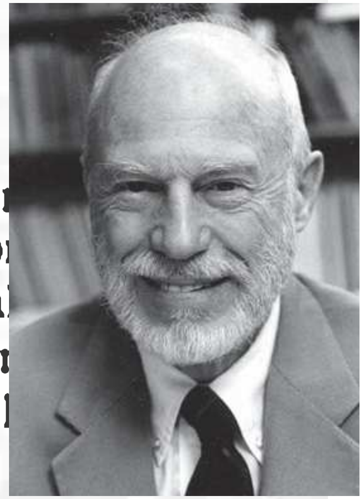
This completes the basic outline of how Vilkovisky's idea fo invariant scalar effective action works, or can be made to wo discussing the advantages of such an effective action, I shall ta some of its possible defects. First of all, how unique is it? I do it is *in principle* as unique as Vilkovisky has claimed. Choices made for three quantities: the starting metric γ_{IJ}, the functional measure $\mu_K[I,K]$; all else follows from these. The last two have no effect on the final form of Γ. The measure μ_I, and hence μ, is determined by unitarity requirements. Expression (13.11) for μ appears to depend on a fourth arbitrary quantity, g^+, but in fact is independent of g^+. To show this just vary ${}_\alpha f_\beta$ and use (13.15).

That leaves γ_{IJ}. Vilkovisky (1984) has suggested that γ_{IJ} should be determined by the coefficient of the highest derivatives in the superclassial field equations. This cannot be correct in the fermion sector of supergravity theory, where the highest-order derivative is first order, because the coefficient of a first-order derivative cannot yield a tensor of even rank having

• The Fermion problem and Supermanifolds

The Effective Action

Bryce De Witt '85



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$\Longrightarrow$ **No metric for <u>non-SUSY theories</u> with fermions:** $g_{XY}(\phi)\,\overline{\psi}^X i\gamma^\mu\partial_\mu\psi^Y$

- **Supergeometry on a Supermanifold** [K Finn, S Karamitsos, AP, EPJC81 (2021) 572; V Gattus, AP, arXiv:2307.01126]

– **Fermions as Coordinates in the Field-Space Supermanifold**

$$\mathbf{\Phi} \equiv \{\Phi^\alpha\} = \big(\phi^A, \quad \psi_a^1, \quad \overline{\psi}_{\dot{a}}^1, \quad \psi_a^2, \quad \overline{\psi}_{\dot{a}}^2, \quad \ldots\big)^{\mathsf{T}},$$

where ϕ^A are scalars and ψ_a^X are Dirac (or Weyl) fermions.

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where ϕ^A are scalars and ψ^X_a are Dirac (or Weyl) fermions.

– Frame-covariant Lagrangian of a scalar theory with fermions

$$\mathcal{L} = \underbrace{\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi^\alpha \; {}_\alpha k_\beta(\mathbf{\Phi}) \; \partial_\nu \Phi^\beta}_{:\ \text{scalars}} + \underbrace{\frac{i}{2} \zeta^\mu_\alpha(\mathbf{\Phi}) \; \partial_\mu \Phi^\alpha}_{:\ \text{fermions}} - U(\mathbf{\Phi}).$$

Model functions:

${}_\alpha k_\beta(\mathbf{\Phi})$: *rank-2 tensor* of the would-be bosonic metric (with ${}_\alpha k_X = 0$)
$\zeta^\mu_\alpha(\mathbf{\Phi})$: mixed spacetime and field-space *vector*
$U(\mathbf{\Phi})$: a *scalar* describing the potential and Yukawa sector

– Extracting the model functions ${}_{\alpha}k_{\beta}$ and ζ^{μ}_{α}

$$
{}_{\alpha}k_{\beta} \;=\; \frac{g_{\mu\nu}}{D}\frac{\overrightarrow{\partial}}{\partial(\partial_{\mu}\Phi^{\alpha})}\,\mathcal{L}\,\frac{\overleftarrow{\partial}}{\partial(\partial_{\nu}\Phi^{\beta})}\,, \qquad \zeta^{\mu}_{\alpha} \;=\; \frac{2}{i}\left(\mathcal{L}-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\Phi^{\alpha}\;{}_{\alpha}k_{\beta}\;\partial_{\nu}\Phi^{\beta}\right)\frac{\overleftarrow{\partial}}{\partial(\partial_{\mu}\Phi^{\alpha})}
$$

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$$
{}_\alpha k_\beta \;=\; \frac{g_{\mu\nu}}{D}\frac{\overrightarrow{\partial}}{\partial(\partial_\mu \Phi^\alpha)}\,\mathcal{L}\,\frac{\overleftarrow{\partial}}{\partial(\partial_\nu \Phi^\beta)}\,, \qquad \zeta^\mu_\alpha \;=\; \frac{2}{i}\left(\mathcal{L} - \frac{1}{2}g^{\mu\nu}\partial_\mu \Phi^\alpha\;{}_\alpha k_\beta\;\partial_\nu \Phi^\beta\right)\frac{\overleftarrow{\partial}}{\partial(\partial_\mu \Phi^\alpha)}
$$

– Free theory as an example

$$
\begin{aligned}
\mathcal{L} \;&=\; \sum_{A\in N\text{scalars}} \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^A - \frac{1}{2}m_A^2(\phi^A)^2 \\
&\quad + \sum_{X\in M\text{fermions}} \frac{i}{2}\Big(\overline{\psi}^X\gamma^\mu\partial_\mu\psi^X - \partial_\mu\overline{\psi}^X\gamma^\mu\psi^X\Big) - m_X\overline{\psi}^X\psi^X\,.
\end{aligned}
$$

Model functions:

$$
\begin{aligned}
{}_\alpha k_\beta \;&=\; \begin{pmatrix} \delta_{AB} & \mathbf{0}_{N\times 8M} \\ \mathbf{0}_{8M\times N} & \mathbf{0}_{8M\times 8M} \end{pmatrix}, \\
\zeta^\mu_\alpha \;&=\; \Big(\mathbf{0}_N,\; \overline{\psi}^1_{\dot a}\gamma^\mu_{\dot a a},\; \gamma^\mu_{\dot a a}\psi^1_a,\; \overline{\psi}^2_{\dot b}\gamma^\mu_{\dot b b},\; \gamma^\mu_{\dot b b}\psi^2_b,\; \dots\Big)
\end{aligned}
$$

– Deriving the Field-Space Metric

Define the rank-1 field-superspace tensor,

[improved by V Gattus, AP, arXiv:2307.01126]

$$\zeta_\alpha(\mathbf{\Phi}) \;=\; \frac{1}{4}\,\frac{\delta\zeta^\mu_\alpha(\mathbf{\Phi})}{\delta\gamma^\mu} \quad \xrightarrow{\text{improved}} \quad \zeta_\alpha(\mathbf{\Phi}) \;=\; \zeta^\mu_\beta(\mathbf{\Phi})\,\overleftarrow{\Sigma}^{\beta}_{\mu,\alpha}\,,$$

to derive the rank-2 anti-supersymmetric tensor (in analogy to $F_{\mu\nu}$ in QED)

$${}_\alpha\lambda_\beta(\mathbf{\Phi}) = \frac{1}{2}\left(\frac{\overrightarrow{\partial}}{\partial\Phi^\alpha}\zeta_\beta(\mathbf{\Phi}) \;-\; (-1)^{\alpha+\beta+\alpha\beta}\frac{\overrightarrow{\partial}}{\partial\Phi^\beta}\zeta_\alpha(\mathbf{\Phi})\right)\,, \quad \text{with } \lambda^{\mathsf{sT}} = -\lambda.$$

Introduce the *non*-singular rank-2 tensor:

$${}_\alpha\Lambda_\beta \;\equiv\; {}_\alpha k_\beta \;+\; {}_\alpha\lambda_\beta \;\xrightarrow{\text{free theory}}\; {}_\alpha N_\beta \equiv \begin{pmatrix} 1_N & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1_4 & 0 & 0 & \cdots \\ 0 & 1_4 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1_4 & \cdots \\ 0 & 0 & 0 & 1_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

– Properties of the Field-Space Super-Metric ${}_\alpha G_\beta(\mathbf{\Phi})$

The Super-Metric ${}_\alpha G_\beta(\mathbf{\Phi})$ should:

1. Be *uniquely* determined from the *action*.

2. Transform as a proper *rank-2 field-space tensor*.

3. Be *supersymmetric* and *non-singular* to produce a non-zero line element.

4. Be *ultralocal*, i.e. it should not depend on $\partial_\mu \mathbf{\Phi}$.

5. Have the *local form* on each point of the field-space Supermanifold

$$
{}_aH_b \equiv \begin{pmatrix}
1_N & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1_4 & 0 & 0 & \cdots \\
0 & -1_4 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1_4 & \cdots \\
0 & 0 & 0 & -1_4 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} .
$$

– The Field-Space Super-Metric

Determine first the *field-space vielbeins* ${}_\alpha e^a(\mathbf{\Phi})$ from

$$ {}_\alpha\Lambda_\beta(\mathbf{\Phi}) \;=\; {}_\alpha e^a(\mathbf{\Phi})\; {}_a N_b\; {}^b e^{\mathsf{sT}}_\beta(\mathbf{\Phi})\,, $$

and use these to obtain the Field-Space Super-Metric:

$$ \boxed{\; {}_\alpha G_\beta(\mathbf{\Phi}) \;=\; {}_\alpha e^a(\mathbf{\Phi})\; {}_a H_b\; {}^b e^{\mathsf{sT}}_\beta(\mathbf{\Phi}) \;} $$

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– The Christoffell Symbols

$$ {}^\alpha\Gamma_{\beta\gamma} \;=\; \frac{1}{2}\, G^{\alpha\delta}\Big[{}_\delta G_\beta\, \overleftarrow{\partial}_\gamma + (-1)^{\beta\gamma}\, {}_\delta G_\gamma\, \overleftarrow{\partial}_\beta - (-1)^\beta\, \overrightarrow{\partial}_\delta\, {}_\beta G_\gamma \Big] $$

– The Riemann Tensor

$$ \begin{aligned} R^\alpha{}_{\beta\gamma\delta} \;=\; & -\,{}^\alpha\Gamma_{\beta\gamma}\overleftarrow{\partial}_\delta \;+\; (-1)^{\gamma\delta}\,{}^\alpha\Gamma_{\beta\delta}\overleftarrow{\partial}_\gamma \;+\; (-1)^{\gamma(\beta+\epsilon)}\,{}^\alpha\Gamma_{\epsilon\gamma}\,{}^\epsilon\Gamma_{\beta\delta} \\ & -\,(-1)^{\delta(\epsilon+\beta+\gamma)}\,{}^\alpha\Gamma_{\epsilon\delta}\,{}^\epsilon\Gamma_{\beta\gamma} \end{aligned} $$

• Supergeometric Effective Action with Fermions

[K Finn, S Karamitsos, AP, EPJC81 (2021) 572]

$$\exp(i\Gamma[\mathbf{\Phi}]) = \int [D\mathbf{\Phi}_q]\,\sqrt{|\mathsf{sdet}G|}\,\exp\left(iS[\mathbf{\Phi}_q] + i\int d^4x\sqrt{-g}\,\Gamma[\mathbf{\Phi}]\frac{\overleftarrow{\partial}}{\partial\Phi^\alpha}\Sigma^\alpha[\mathbf{\Phi},\mathbf{\Phi}_q]\right)$$

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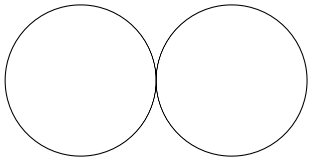
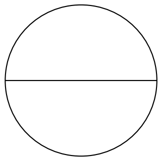
$$\exp(i\Gamma[\mathbf{\Phi}]) = \int [D\mathbf{\Phi}_q]\,\sqrt{|\mathrm{sdet}G|}\,\exp\left(iS[\mathbf{\Phi}_q] + i\int d^4x\sqrt{-g}\,\Gamma[\mathbf{\Phi}]\frac{\overleftarrow{\partial}}{\partial\Phi^\alpha}\Sigma^\alpha[\mathbf{\Phi},\mathbf{\Phi}_q]\right)$$

– One- and Two-Loop Grand Covariant Effective Actions

$$\Gamma^{(1)}[\mathbf{\Phi}] \;=\; \tfrac{i}{2}\ln\mathrm{sdet}\left(\overrightarrow{\nabla}^{\widehat{\alpha}}\,S\,\overleftarrow{\nabla}_{\widehat{\beta}}\right) \;=\; \tfrac{i}{2}\mathrm{str}\ln\left(\overrightarrow{\nabla}^{\widehat{\alpha}}\,S\,\overleftarrow{\nabla}_{\widehat{\beta}}\right)$$

$\Gamma^{(2)}[\mathbf{\Phi}] \;=$  $+$ 

$$= -\tfrac{1}{8}S\overleftarrow{\nabla}_{\{\widehat{\alpha}}\overleftarrow{\nabla}_{\widehat{\beta}}\overleftarrow{\nabla}_{\widehat{\gamma}}\overleftarrow{\nabla}_{\widehat{\delta}\}}\,\Delta^{\widehat{\delta}\widehat{\gamma}}\,\Delta^{\widehat{\beta}\widehat{\alpha}}$$

$$+(-1)^{\widehat{\gamma}\widehat{\beta}+\widehat{\epsilon}(\widehat{\delta}+\widehat{\beta})}\tfrac{1}{12}\left(S\overleftarrow{\nabla}_{\{\widehat{\epsilon}}\overleftarrow{\nabla}_{\widehat{\gamma}}\overleftarrow{\nabla}_{\widehat{\alpha}\}}\right)\Delta^{\widehat{\alpha}\widehat{\beta}}\,\Delta^{\widehat{\gamma}\widehat{\delta}}\,\Delta^{\widehat{\epsilon}\widehat{\zeta}}\left(\overrightarrow{\nabla}_{\{\widehat{\zeta}}\overrightarrow{\nabla}_{\widehat{\delta}}\overrightarrow{\nabla}_{\widehat{\beta}\}}S\right)$$

${}_{\widehat{\alpha}}\Delta^{-1}_{\widehat{\beta}} \equiv \overrightarrow{\nabla}_{\widehat{\alpha}}\,S\overleftarrow{\nabla}_{\widehat{\beta}}$, with ${}^{\widehat{\alpha}}\Delta^{\widehat{\beta}} = \Delta^{\widehat{\alpha}\widehat{\beta}}$ defined through: ${}^{\widehat{\alpha}}\Delta^{\widehat{\gamma}}{}_{\widehat{\gamma}}\Delta^{-1}_{\widehat{\beta}} = {}^{\widehat{\alpha}}\delta_{\widehat{\beta}}$.

– Single Fermion Model

$$\mathcal{L} \;=\; \frac{1}{2}k(\phi)\,\partial_\mu\phi\,\partial^\mu\phi \;+\; \frac{i}{2}g(\phi)\left(\overline{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\overline{\psi}\,\gamma^\mu\psi\right) \\ -\,\frac{1}{2}h(\phi)\,\overline{\psi}\gamma^\mu\psi\,\partial_\mu\phi \;-\; Y(\phi)\,\overline{\psi}\psi \;-\; V(\phi)$$

Supermanifold: $\Phi^\alpha = \left(\phi, \psi, \bar{\psi}\right)$, with grand field-space metric

$${}_\alpha G_\beta = \begin{pmatrix} k - \dfrac{g'^2 + h^2}{2g}\,\overline{\psi}\psi & -\frac{1}{2}(g' - ih)\,\overline{\psi} & \frac{1}{2}(g' + ih)\,\psi \\ \frac{1}{2}(g' - ih)\,\overline{\psi} & 0_4 & g\mathbb{1}_4 \\ -\frac{1}{2}(g' + ih)\,\psi & -g\mathbb{1}_4 & 0_4 \end{pmatrix}$$

But, $\quad R^\alpha{}_{\beta\gamma\delta} \,=\, 0 \quad \Longrightarrow \quad$ field-space is flat

– **Frame-reparametrization to a Cartesian Frame**, $\widetilde{\Phi}^{\alpha} = \big(\widetilde{\phi},\ \widetilde{\psi},\ \widetilde{\overline{\psi}}\big)^{\mathsf{T}}$:

$$\phi \;\to\; \widetilde{\phi} = \int_0^{\phi} \sqrt{k(\phi)}\, d\phi\,, \quad \psi \;\to\; \widetilde{\psi} = \sqrt{g(\phi)}\, \exp\Big(\frac{i}{2}\int_0^{\phi} \frac{h(\phi)}{g(\phi)} d\phi\Big)\, \psi$$

Lagrangian in the Cartesian Frame:

$$\mathcal{L} \,=\, \frac{1}{2}\partial_\mu \widetilde{\phi}\, \partial^\mu \widetilde{\phi} \,+\, \frac{i}{2}\Big(\widetilde{\overline{\psi}}\gamma^\mu \partial_\mu \widetilde{\psi} - \partial_\mu \widetilde{\overline{\psi}}\, \gamma^\mu \widetilde{\psi}\Big) \,-\, \widetilde{Y}(\widetilde{\phi})\, \widetilde{\overline{\psi}}\widetilde{\psi} \,-\, \widetilde{V}(\widetilde{\phi})\,,$$

with $\widetilde{Y}(\widetilde{\phi}) = Y(\phi)/g(\phi)$ and $\widetilde{V}(\widetilde{\phi}) = V(\phi)$.

– Frame-reparametrization to a Cartesian Frame, $\widetilde{\Phi}^\alpha = (\widetilde{\phi},\ \widetilde{\psi},\ \widetilde{\overline{\psi}})^{\mathsf{T}}$:

$$\phi \ \to\ \widetilde{\phi} = \int_0^{\phi} \sqrt{k(\phi)}\, d\phi\,, \quad \psi \ \to\ \widetilde{\psi} = \sqrt{g(\phi)}\ \exp\left(\frac{i}{2}\int_0^{\phi}\frac{h(\phi)}{g(\phi)}d\phi\right)\,\psi$$

Lagrangian in the Cartesian Frame:

$$\mathcal{L} \,=\, \frac{1}{2}\partial_\mu\widetilde{\phi}\,\partial^\mu\widetilde{\phi} \,+\, \frac{i}{2}\left(\widetilde{\overline{\psi}}\gamma^\mu\partial_\mu\widetilde{\psi} - \partial_\mu\widetilde{\overline{\psi}}\,\gamma^\mu\widetilde{\psi}\right) \,-\, \widetilde{Y}(\widetilde{\phi})\,\widetilde{\overline{\psi}}\widetilde{\psi} \,-\, \widetilde{V}(\widetilde{\phi})\,,$$

with $\widetilde{Y}(\widetilde{\phi}) = Y(\phi)/g(\phi)$ and $\widetilde{V}(\widetilde{\phi}) = V(\phi)$.

– One-Loop Effective Action

$$\Gamma[\Phi] = S[\Phi] \,+\, \frac{i}{2}\mathrm{Tr}\ln\left\{\Box + \widetilde{V}'' - \widetilde{\overline{\psi}}\left[2\,\widetilde{Y}'^{\,2}\left(-\,i\partial\!\!\!/ + \widetilde{Y}\right)^{-1} - \widetilde{Y}''\right]\widetilde{\psi}\right\}$$
$$-\, i\mathrm{Tr}\ln\left(-\,i\partial\!\!\!/ + \widetilde{Y}\right)$$

– Model with Multiple Fermions (up to quadratic kinetic terms)

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} k_{AB}(\mathbf{\Phi})\, \partial_\mu \phi^A\, \partial_\nu \phi^B \;+\; \frac{i}{2}\, g_{XY}(\mathbf{\Phi}) \left(\overline{\psi}^X \gamma^\mu \partial_\mu \psi^Y - \partial_\mu \overline{\psi}^X \gamma^\mu \psi^Y \right)$$
$$-\; \frac{1}{2}\, h_{AXY}(\mathbf{\Phi})\, \overline{\psi}^X \gamma^\mu \psi^Y \partial_\mu \phi^A \;+\; \cdots \;-\; U(\mathbf{\Phi})$$

– No-Go Theorem for Fermionic Curvature

[V Gattus, AP, arXiv:2307.01126]

Field-space super-metric for single scalar ϕ:

$$_\alpha G_\beta = \begin{pmatrix} k - \frac{1}{2}\overline{\boldsymbol{\psi}}(\boldsymbol{g}' - i\boldsymbol{h})\, \boldsymbol{g}^{-1}\, (\boldsymbol{g}' + i\boldsymbol{h})\boldsymbol{\psi} & -\frac{1}{2}\overline{\boldsymbol{\psi}}\,(\boldsymbol{g}' - i\boldsymbol{h}) & \frac{1}{2}\boldsymbol{\psi}^{\mathsf{T}}(\boldsymbol{g}'^{\mathsf{T}} + i\boldsymbol{h}^{\mathsf{T}}) \\ \frac{1}{2}\,(\boldsymbol{g}'^{\mathsf{T}} - i\boldsymbol{h}^{\mathsf{T}})\,\overline{\boldsymbol{\psi}}^{\mathsf{T}} & \mathbf{0} & \boldsymbol{g}^{\mathsf{T}}\,\mathbf{1}_4 \\ -\frac{1}{2}\,(\boldsymbol{g}' + i\boldsymbol{h})\,\boldsymbol{\psi} & -\boldsymbol{g}\,\mathbf{1}_4 & \mathbf{0} \end{pmatrix},$$

with $\boldsymbol{\psi} = \{\psi^X\}$, $\boldsymbol{g}(\phi) = \{g_{XY}\} = \boldsymbol{g}^\dagger(\phi)$ and $\boldsymbol{h}(\phi) = \{h_{XY}\} = \boldsymbol{h}^\dagger(\phi)$.

But, $R^\alpha{}_{\beta\gamma\delta} = 0 \quad \Longrightarrow \quad$ field super-space has zero curvature.

– Minimal 2D Model with Fermionic Curvature

[V Gattus, AP, arXiv:2307.01126]

$$\mathcal{L} \;=\; \tfrac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi \;+\; \tfrac{i}{2}\Big(g_0(\phi) + \underline{\boldsymbol{g}_1(\phi)\,\overline{\boldsymbol{\psi}}\boldsymbol{\psi}}\Big)\Big(\overline{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\overline{\psi}\,\gamma^\mu\psi\Big)$$
$$-\,Y(\phi)\,\overline{\psi}\psi \;-\; V(\phi)\,, \qquad \text{with } \gamma^\mu = \Big(\sigma^1\,, -i\sigma^2\Big)\,.$$

Chart: $\mathbf{\Phi} = \big(\phi\,,\psi^{\mathsf{T}}\,,\overline{\psi}\big)$, $\psi^{\mathsf{T}} = \big(\psi_1\,,\psi_2\big)$ and $\overline{\psi} \equiv \psi^\dagger\gamma^0$.

Field-space super-metric:

$$_\alpha G_\beta = \begin{pmatrix} 1 + \boldsymbol{b}^{\mathsf{T}}\,(\boldsymbol{d}^{-1})^{\mathsf{T}}\,\boldsymbol{a}^{\mathsf{T}} - \boldsymbol{a}\,\boldsymbol{d}^{-1}\,\boldsymbol{b} & -\boldsymbol{a} & \boldsymbol{b}^{\mathsf{T}} \\ \boldsymbol{a}^{\mathsf{T}} & \mathbf{0} & \boldsymbol{d}^{\mathsf{T}} \\ -\boldsymbol{b} & -\boldsymbol{d} & \mathbf{0} \end{pmatrix}\,,$$

where $\boldsymbol{d} = \big(g_0 + g_1\,\overline{\psi}\psi\big)\,\mathbf{1}_2 + g_1\,\psi\overline{\psi}$,

$\boldsymbol{a} = \overline{\psi}\big(g_0' + g_1'\,\overline{\psi}\psi\big)$, $\quad \boldsymbol{b} = \big(g_0' + g_1'\,\overline{\psi}\psi\big)\,\psi \quad$ (with $g_{0,1}' \equiv \partial g_{0,1}/\partial\phi$),

Set $g_0(\phi) = 1$ for simplicity:

Super-Ricci scalar: $\mathfrak{R} \;=\; 4g_1 \;-\; \tfrac{1}{2}\,g_1'^2\,\big(\overline{\psi}\psi\big)^2 \;\neq\; 0 \quad\Longrightarrow\quad$ field super-space is curved.

• Conclusions

- Complete formulation of Geometric Effective Actions for Scalar–Tensor Theories, with a complete set of model functions

- New model function $\ell = \ell(\Phi)$ ensures the uniqueness of the VDW path-integral measure, with $ds^2 = g_{\mu\nu}/\ell^2\, dx^\mu dx^\nu$

- Extension of the VDW formalism to Supermanifolds to describe realistic theories that include fermions, such as the SM

- Minimal Supergeometric QFT Models with non-zero fermionic curvature [See talk by V Gattus]

- Analytic expressions of One- and Two-Loop Supergeometric Effective Actions for Theories with Fermions

• Future Research Directions

- UV completions of Minimal Supergeometric QFTs (Plan A)
  (avoiding the Haag-Lopuszanski-Sohnius theorem)

- Application of Supergeometric QFTs to SMEFT and HEFT (Plan B)

- Include gauge and other symmetries $\Longleftrightarrow$ Isometries on Supermanifold

- Phenomenological/cosmological applications of Supergeometric QFTs

$$\vdots$$

Supergeometry on a Supermanifold

$$\boxed{D\Sigma^2 \;=\; d\Phi^{\widehat{\alpha}}\ {}_{\widehat{\alpha}}G_{\widehat{\beta}}(\mathbf{\Phi})\ d\Phi^{\widehat{\beta}}}$$

[K Finn, S Karamitsos, AP, EPJC81 (2021) 572;
Viola Gattus, AP, arXiv:2307.01126]

Supergeometry on a Supermanifold

$$\boxed{D\Sigma^2 \;=\; d\Phi^{\widehat{\alpha}}\ {}_{\widehat{\alpha}}G_{\widehat{\beta}}(\mathbf{\Phi})\ d\Phi^{\widehat{\beta}}}$$

[K Finn, S Karamitsos, AP, EPJC81 (2021) 572;
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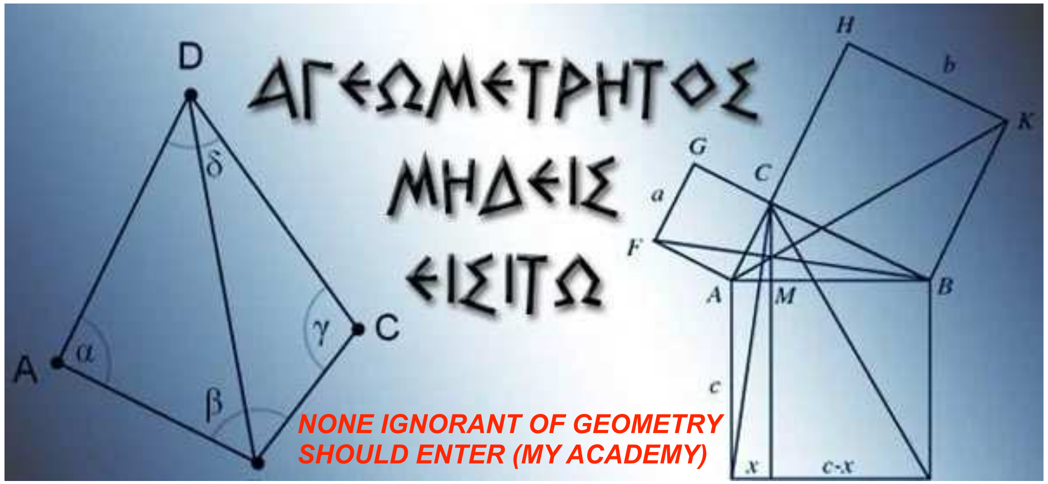
from Plato



Back-Up Slides

• From Geometrizing the Cosmos to Micro-Cosmos

- **Geometrizing the Cosmos**: . . . , Pythagoras (5c BC)

- **Geocentric versus Heliocentric System** [e.g., Van der Waerden '87]

  Geocentric: Anaximander (6c BC), . . . , Plato (4c BC),
  Aristotle (3c BC), Ptolemy (2c AD), . . . Tycho (16c AD)

  Heliocentric: Aristarchus (3c BC), Seleucus (2c BC),
  Copernicus (15c AD), Kepler (16c AD), Galileo (16c AD), . . .

- **Absolute versus Relative/Local Inertial Frame in Gravitation**

  Absolute: Newton (17c AD), . . .

  Relative: Einstein (20c AD), . . .

- **Geometrizing the Micro-Cosmos** as a solution to frame problems in Quantum Field Theory and Quantum Gravity

– Einstein versus Jordan Frame

Action in Einstein Frame: $S^{\mathrm{EF}}[g_{\mu\nu},\varphi] = \int_x \left[-\frac{1}{2}M_P^2 R + \frac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi)\right]$

Action in Jordan Frame: $S^{\mathrm{JF}}[\tilde{g}_{\mu\nu},\widetilde{\varphi}] = \int_x \left[-\frac{1}{2}f(\widetilde{\varphi})\widetilde{R} + \frac{1}{2}(\partial_\mu\widetilde{\varphi})^2 - \widetilde{V}(\widetilde{\varphi})\right]$

Frame equivalence $\Longrightarrow$ $S^{\mathrm{JF}}[\tilde{g}_{\mu\nu},\widetilde{\varphi}] = S^{\mathrm{EF}}[g_{\mu\nu},\varphi]$ [R. H. Dicke '62]

$$\begin{tikzcd}
S^{\mathrm{JF}} \arrow[rr, "\text{Field reparametrization}"] \arrow[d] & & S^{\mathrm{EF}} \arrow[d] \\
& \text{Quantization} & \\
\Gamma^{\mathrm{JF}}_{1-\mathrm{loop}} \arrow[rr, "\text{Field reparametrization}"'] & & \Gamma^{\mathrm{EF}}_{1-\mathrm{loop}}
\end{tikzcd}$$

$\boldsymbol{\Gamma}^{\mathrm{JF}}_{1-\mathrm{loop}}[\tilde{\boldsymbol{g}}_{\mu\nu},\widetilde{\varphi}] \neq \boldsymbol{\Gamma}^{\mathrm{EF}}_{1-\mathrm{loop}}[\boldsymbol{g}_{\mu\nu},\varphi]$: **Effective action is frame dependent, except at extrema of the action.**

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- S. Karamitsos and A. Pilaftsis, *Frame Covariant Nonminimal Multifield Inflation*, Nucl. Phys. B927 (2018) 219.

- **Effects of $\ell(\varphi)$ (from $\bar{g}_{\mu\nu} \equiv g_{\mu\nu}/\ell^2(\varphi)$)**

– **Frame-invariant Dirac-delta function**

$$\int d^D x\, \sqrt{-\bar{g}}\ \bar{\delta}^{(D)}(x) \;=\; 1\,, \qquad \text{with} \quad \bar{\delta}^{(D)}(x_I - x_J) \;\equiv\; \frac{\delta^{(D)}(x_I - x_J)}{\sqrt{-\bar{g}}}$$

– **Functional derivative**

$$\frac{\bar{\delta} F[\Phi(x)]}{\bar{\delta}\Phi(y)} \;\equiv\; \lim_{\epsilon\to 0} \frac{F[\Phi(x) + \epsilon\bar{\delta}^{(D)}(x-y)] - F[\Phi(x)]}{\epsilon}$$

– **Functional determinant**

$$\overline{\det}(M_{xy}) \;\equiv\; \exp\left[i\int d^D x \sqrt{-\bar{g}}\ \ln(M)_{xx}\right]$$

• Field-Space Riemann Tensor $\mathfrak{R}^{(\mu\nu)}{}_{(\alpha\beta)(\rho\sigma)(\gamma\delta)}$ for General Relativity

[K Finn, S Karamitsos, AP, '19]

$$
\begin{aligned}
\mathfrak{R}^{(\mu\nu)}{}_{(\alpha\beta)(\rho\sigma)(\gamma\delta)} = & -\frac{1}{32}\delta^\mu_\rho\delta^\nu_\beta g_{\sigma\gamma}g_{\alpha\delta} - \frac{1}{32}\delta^\mu_\sigma\delta^\nu_\beta g_{\rho\gamma}g_{\alpha\delta} - \frac{1}{32}\delta^\mu_\beta\delta^\nu_\sigma g_{\rho\gamma}g_{\alpha\delta} - \frac{1}{32}\delta^\mu_\beta\delta^\nu_\rho g_{\sigma\gamma}g_{\alpha\delta} \\
& -\frac{1}{32}\delta^\mu_\rho\delta^\nu_\beta g_{\sigma\delta}g_{\alpha\gamma} - \frac{1}{32}\delta^\mu_\sigma\delta^\nu_\beta g_{\rho\delta}g_{\alpha\gamma} - \frac{1}{32}\delta^\mu_\beta\delta^\nu_\sigma g_{\rho\delta}g_{\alpha\gamma} - \frac{1}{32}\delta^\mu_\beta\delta^\nu_\rho g_{\sigma\delta}g_{\alpha\gamma} \\
& -\frac{1}{32}\delta^\mu_\alpha\delta^\nu_\rho g_{\sigma\gamma}g_{\beta\delta} - \frac{1}{32}\delta^\mu_\alpha\delta^\nu_\sigma g_{\rho\gamma}g_{\beta\delta} - \frac{1}{32}\delta^\mu_\rho\delta^\nu_\alpha g_{\sigma\gamma}g_{\beta\delta} - \frac{1}{32}\delta^\mu_\sigma\delta^\nu_\alpha g_{\rho\gamma}g_{\beta\delta} \\
& -\frac{1}{32}\delta^\mu_\alpha\delta^\nu_\rho g_{\sigma\delta}g_{\beta\gamma} - \frac{1}{32}\delta^\mu_\alpha\delta^\nu_\sigma g_{\rho\delta}g_{\beta\gamma} - \frac{1}{32}\delta^\mu_\rho\delta^\nu_\alpha g_{\sigma\delta}g_{\beta\gamma} - \frac{1}{32}\delta^\mu_\sigma\delta^\nu_\alpha g_{\rho\delta}g_{\beta\gamma} \\
& +\frac{1}{32}\delta^\mu_\gamma\delta^\nu_\beta g_{\rho\delta}g_{\sigma\alpha} + \frac{1}{32}\delta^\mu_\delta\delta^\nu_\beta g_{\rho\gamma}g_{\sigma\alpha} + \frac{1}{32}\delta^\mu_\beta\delta^\nu_\delta g_{\rho\gamma}g_{\sigma\alpha} + \frac{1}{32}\delta^\mu_\beta\delta^\nu_\gamma g_{\rho\delta}g_{\sigma\alpha} \\
& +\frac{1}{32}\delta^\mu_\gamma\delta^\nu_\beta g_{\rho\alpha}g_{\sigma\delta} + \frac{1}{32}\delta^\mu_\delta\delta^\nu_\beta g_{\rho\alpha}g_{\sigma\gamma} + \frac{1}{32}\delta^\mu_\beta\delta^\nu_\delta g_{\rho\alpha}g_{\sigma\gamma} + \frac{1}{32}\delta^\mu_\beta\delta^\nu_\gamma g_{\rho\alpha}g_{\sigma\delta} \\
& +\frac{1}{32}\delta^\mu_\alpha\delta^\nu_\gamma g_{\rho\delta}g_{\sigma\beta} + \frac{1}{32}\delta^\mu_\alpha\delta^\nu_\delta g_{\rho\gamma}g_{\sigma\beta} + \frac{1}{32}\delta^\mu_\gamma\delta^\nu_\alpha g_{\rho\delta}g_{\sigma\beta} + \frac{1}{32}\delta^\mu_\delta\delta^\nu_\alpha g_{\rho\gamma}g_{\sigma\beta} \\
& +\frac{1}{32}\delta^\mu_\alpha\delta^\nu_\gamma g_{\rho\beta}g_{\sigma\delta} + \frac{1}{32}\delta^\mu_\alpha\delta^\nu_\delta g_{\rho\beta}g_{\sigma\gamma} + \frac{1}{32}\delta^\mu_\gamma\delta^\nu_\alpha g_{\rho\beta}g_{\sigma\delta} + \frac{1}{32}\delta^\mu_\delta\delta^\nu_\alpha g_{\rho\beta}g_{\sigma\gamma} \\
& +\frac{1}{4D}g_{\rho\gamma}g^{\mu\nu}g_{\sigma\beta}g_{\alpha\delta} + \frac{1}{4D}g_{\rho\delta}g^{\mu\nu}g_{\sigma\beta}g_{\alpha\gamma} + \frac{1}{4D}g_{\rho\alpha}g^{\mu\nu}g_{\sigma\gamma}g_{\beta\delta} + \frac{1}{4D}g_{\rho\alpha}g^{\mu\nu}g_{\sigma\delta}g_{\beta\gamma} \\
& +\frac{1}{4D}g_{\rho\gamma}g^{\mu\nu}g_{\sigma\alpha}g_{\beta\delta} + \frac{1}{4D}g_{\rho\delta}g^{\mu\nu}g_{\sigma\alpha}g_{\beta\gamma} + \frac{1}{4D}g_{\rho\beta}g^{\mu\nu}g_{\sigma\delta}g_{\alpha\gamma} + \frac{1}{4D}g_{\rho\beta}g^{\mu\nu}g_{\sigma\gamma}g_{\alpha\delta} \\
& -\frac{1}{4D}g^{\mu\nu}g_{\rho\beta}g_{\sigma\gamma}g_{\alpha\delta} - \frac{1}{4D}g^{\mu\nu}g_{\rho\beta}g_{\sigma\delta}g_{\alpha\gamma} - \frac{1}{4D}g^{\mu\nu}g_{\rho\gamma}g_{\sigma\alpha}g_{\beta\delta} - \frac{1}{4D}g^{\mu\nu}g_{\rho\delta}g_{\sigma\alpha}g_{\beta\gamma} \\
& -\frac{1}{4D}g^{\mu\nu}g_{\rho\alpha}g_{\sigma\gamma}g_{\beta\delta} - \frac{1}{4D}g^{\mu\nu}g_{\rho\alpha}g_{\sigma\delta}g_{\beta\gamma} - \frac{1}{4D}g^{\mu\nu}g_{\rho\delta}g_{\sigma\beta}g_{\alpha\gamma} - \frac{1}{4D}g^{\mu\nu}g_{\rho\gamma}g_{\sigma\beta}g_{\alpha\delta}
\end{aligned}
$$

– Field-Space Ricci Tensor $\mathfrak{R}_{(\mu\nu)(\rho\sigma)}$

$$\mathfrak{R}_{(\mu\nu)(\rho\sigma)} \;=\; \frac{1}{4} g_{\mu\nu} g_{\rho\sigma} - \frac{D}{8} g_{\mu\rho} g_{\nu\sigma} - \frac{D}{8} g_{\mu\sigma} g_{\nu\rho}$$

– Field-Space Ricci Scalar $\mathfrak{R}$

$$\mathfrak{R} \;=\; \frac{D}{4} - \frac{D^2}{8} - \frac{D^3}{8} \;<\; 0\,,$$

for spacetime dimensions $D > 1$.

$\Longrightarrow$ Gravity has a curved field space, with *negative* scalar curvature.

• Partial differentation with respect to γ^μ

Any matrix $M_{a\dot{a}}$ in spinor space can be uniquely expressed as

$$M_{a\dot{a}} \;=\; \sum_{i=S,P,V,A,T} a_{(i)}\, \Gamma^{(i)}_{a\dot{a}}, \qquad \text{with} \quad \Gamma^{(i)} \in \; \{I_4, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}\,,$$

where $\Gamma^{(i)}$ are the 16 orthogonal Lorentz-covariant bilinears, $a_{(i)}$ is a coefficient, and $\sigma^{\mu\nu} \equiv i/2\,[\gamma^\mu, \gamma^\nu]$.

– Γ-matrix partial differentiation:

$$\frac{\delta M}{\delta \Gamma^{(i)}} \;\equiv\; \lim_{\epsilon^{(i)}\to 0} \frac{M[\Gamma^{(i)} \to \Gamma^{(i)} + \epsilon^{(i)} I_4] \;-\; M[\Gamma^{(i)}]}{\epsilon^{(i)}}\;.$$

For $\Gamma^{(i)} = \Gamma^{(V)} = \gamma^\mu$,

$$\frac{\delta M_{a\dot{a}}}{\delta \gamma^\mu} = a_\mu\, \delta_{a\dot{a}}\;.$$