Bulk Locality on the Celestial Sphere

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2106.11948

Motivation :

Great progress in the S-matrix bootstrap

Unitarity + Causality + Crossing

→ carving out the space of consistent gravity amp

(See Simon's Talk)

But not enough !

Maximal SUSY 10D $M(s,t) = R^4 \left(\frac{1}{stu} + g_0 + \cdots \right)$

$0 \lesssim g_0 \lesssim 3 \frac{8\pi G}{M^6}$ (2102.08951)

$2\zeta(3) \sim 2.4$

We're missing control over the H.E. behavior.

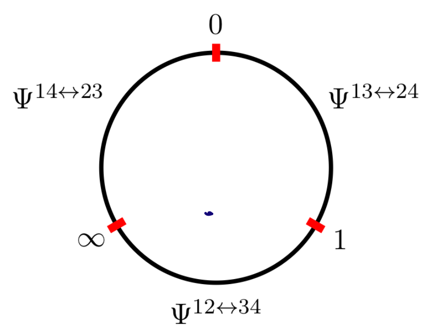
CA has the appealing feature that H.E. fixed angle (t/s) soft-ness is built in.

→ can we build a bootstrap program

We will focus on the 4-pt amp.

$$\Psi(\beta, \ell_i, z) \propto \int_0^\infty d\omega\, \omega^{\beta-1} T_{\ell_i}(s,t)$$

Importantly, there are 3 distinct functions on the equator.



kinematics	$12 \leftrightarrow 34$	$13 \leftrightarrow 24$	$14 \leftrightarrow 23$
physical region	$z \geq 1$ $s \geq 0 \geq u, t$	$1 \geq z \geq 0$ $u \geq 0 \geq s, t$	$0 \geq z$ $t \geq 0 \geq s, u$
ω	$\frac{2\tilde{\omega}_4}{\sqrt{z-1}}$	$\frac{2\tilde{\omega}_4}{\sqrt{z(1-z)}}$	$\frac{2\tilde{\omega}_4}{\sqrt{-z}}$
(s,u,t)	$(\omega^2, -\frac{1}{z}\omega^2, -\frac{(z-1)}{z}\omega^2)$	$(-z\omega^2, \omega^2, -(1-z)\omega^2)$	$(-\frac{(-z)}{1-z}\omega^2, -\frac{1}{1-z}\omega^2, \omega^2)$

$$\Psi^{12\leftrightarrow 34}(\boldsymbol{\Delta}, \ell_i, z) = \frac{1}{2^{\boldsymbol{\Delta}-7}} z^2 \int_0^\infty d\omega\, \omega^{\boldsymbol{\Delta}-5} T_{\ell_i}\left(\omega^2, -\frac{(z-1)}{z}\omega^2\right)$$

$$\Psi^{13\leftrightarrow 24}(\boldsymbol{\Delta}, \ell_i, z) = \frac{1}{2^{\boldsymbol{\Delta}-7}} z^{\frac{\boldsymbol{\Delta}}{2}} \int_0^\infty d\omega\, \omega^{\boldsymbol{\Delta}-5} T_{\ell_i}\left(-z\omega^2, (z-1)\omega^2\right)$$

$$\Psi^{14\leftrightarrow 23}(\boldsymbol{\Delta}, \ell_i, z) = \frac{1}{2^{\boldsymbol{\Delta}-7}} (-z)^{\frac{\boldsymbol{\Delta}}{2}} (1-z)^{2-\frac{\boldsymbol{\Delta}}{2}} \int_0^\infty d\omega\, \omega^{\boldsymbol{\Delta}-5} T_{\ell_i}\left(\frac{z}{1-z}\omega^2, \omega^2\right)$$

How do we see $T|_{s\to m^2} \sim \frac{Res}{s-M^2}$ in $\Psi(\beta, \ell_i, z)$??

Consider massless scalar. (3D. Lam Shao 1711.06138)

$$T\left(s,t\right) = -g^2\left(\frac{1}{s-m^2+i\varepsilon}+\frac{1}{u-m^2+i\varepsilon}+\frac{1}{t-m^2+i\varepsilon}\right)$$

Take s-channel kinematics

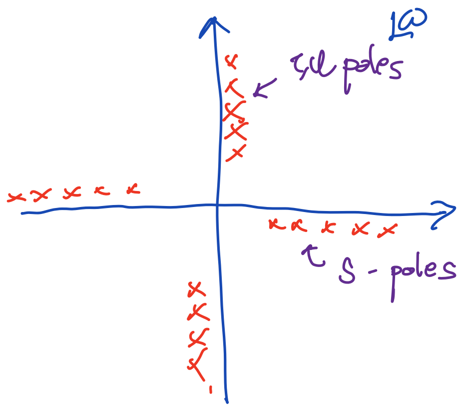
$$\Psi^{12\leftrightarrow 34}(\beta,z) = \frac{2^{3-\beta}z^2}{(1-e^{i\pi\beta})}\int_{-\infty}^{\infty} d\omega\,\omega^{\beta-1}T\left(\omega^2,-\frac{(z-1)}{z}\omega^2\right)$$

$$\Downarrow$$

unphysical threshol produces real

$$\Psi^{12\leftrightarrow 34}_{\text{scalar}}(\beta,z) = \frac{\pi g^2}{\sin\frac{\pi\beta}{2}}\left(\frac{m}{2}\right)^{\beta-2}z^2\left[e^{\frac{1}{2}\pi i\beta}+z^{\frac{\beta}{2}}+\left(\frac{z}{z-1}\right)^{\frac{\beta}{2}}\right] \quad (z\geq 1)$$

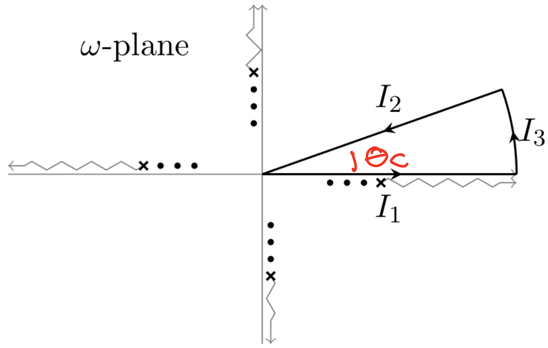
physical threshold produces imaginary part



$$\textbf{Im}\,\Psi^{12\leftrightarrow 34}_{\text{scalar}}(\beta,z) = \pi g^2\left(\frac{m}{2}\right)^{\beta-2}z^2 = 2^{3-\beta}\pi z^2 g^2\,\textbf{Res}_{\omega=m}\left[\omega^{\beta-1}T(\omega,z)\right]$$

Claim: this is universal

First step: define Im Ψ via countour integral

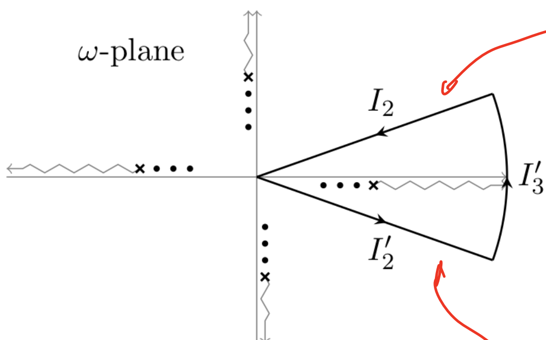


← assuming I_3 vanishes for $\theta < \theta_c$.

Then

$$\Psi(\beta, z) = B(z) \int_{I_1} \mathrm{d}\omega\, \omega^{\beta-1} T(\omega, z) = -B(z) \int_{I_2} \mathrm{d}\omega\, \omega^{\beta-1} T(\omega, z)$$

Now instead consider



$$\int_{I_2'} \mathrm{d}\omega\, \omega^{\beta-1} T(\omega, z) = -\Big[\int_{I_2} \mathrm{d}\omega\, \omega^{\beta-1} T(\omega, z) \Big]^*$$

We arrive at

$$\begin{aligned} \mathbf{Im}\, \Psi(\beta, z) &= -\frac{1}{2i} B(z) \int_{I_2 + I_2'} \mathrm{d}\omega\, \omega^{\beta-1} T(\omega, z) \\ &= -B(z) \left\{ \pi \sum_i \underset{\omega \to m_i}{\mathbf{Res}} \left[\omega^{\beta-1} T(\omega, z) \right] + \int_M^\infty \mathrm{d}\omega\, \omega^{\beta-1}\, \mathbf{Disc}\, [T(\omega, z)] \right\} \end{aligned}$$

Verification :

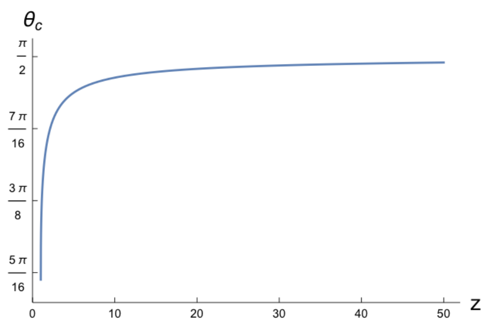
String theory type-I : $\frac{\Gamma[-s]\Gamma[-t]}{\Gamma[1+u]}$, type-II : $\frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]}{\Gamma[1+s]\Gamma[1+u]\Gamma[1+t]}$,

with $S = \omega^2 = (re^{i\theta})^2$ (Stieberger Taylor 1806.05688)

$$T(r^2 e^{2i\theta}, z) \sim \exp\left[g(\theta, z) r^2 + \mathcal{O}(\log r)\right]$$

$g(\theta, z) < 0$ for $0 < \theta < \theta_c$

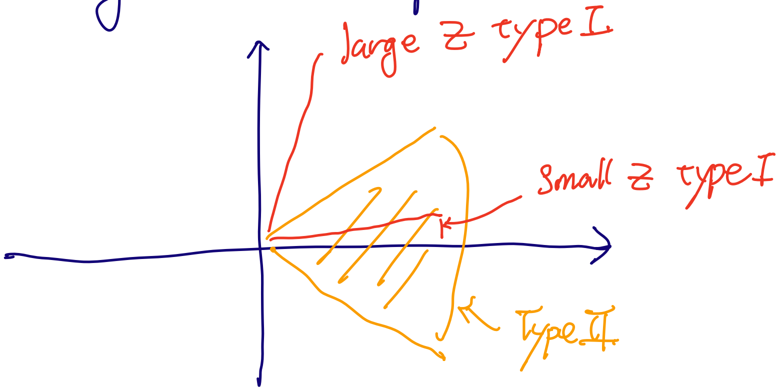
type I



type II

z- independent

Convergence on ω-plane.

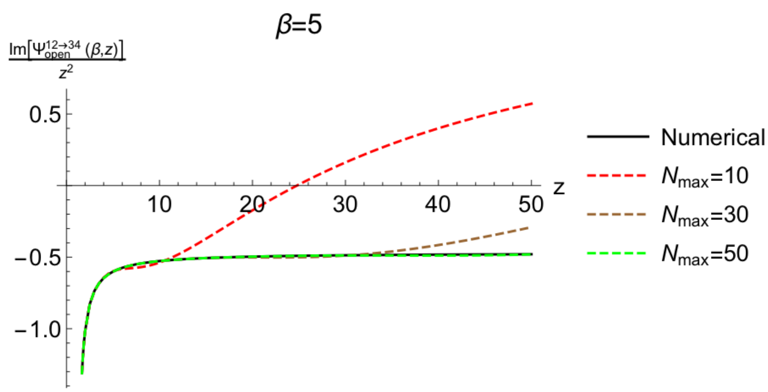


$$\mathbf{Res}_{s=n}[T_{\text{open}}(s,t)] = \frac{\prod_{k=1}^{n-1}(t+k)}{n!} \; = \frac{\left[n\left(\frac{1}{z}-1\right)+1\right]_{n-1}}{2\sqrt{n}n!}$$

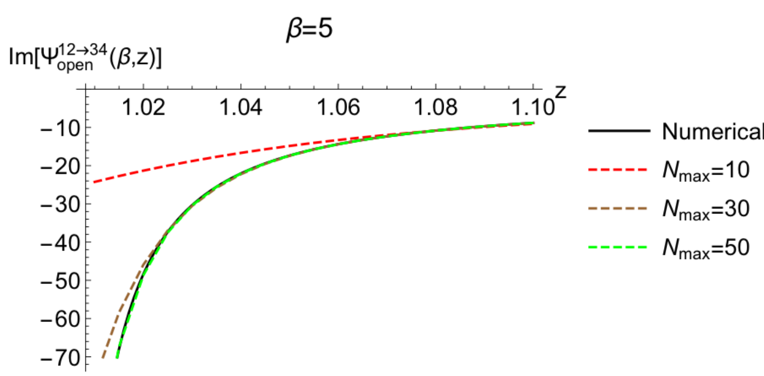
We expect.

$$\mathbf{Im}\,\Psi^{12\leftrightarrow 34}_{\text{open}}(\beta,z) \; = -\pi 2^{3-\beta} z^2 \sum_{n=1}^{\infty} (\sqrt{n})^{\beta-2} \frac{\left[n\left(\frac{1}{z}-1\right)+1\right]_{n-1}}{2n!} \quad (z>1)$$

$z >> 1$



$z \to 1$

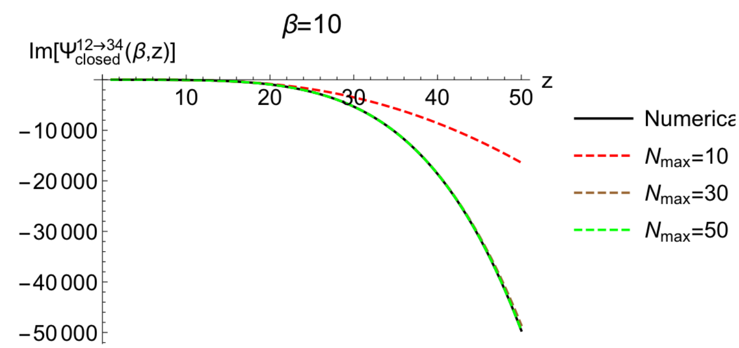


$$\mathbf{Res}_{s=n}[T_{\text{closed}}(s,t)] = \frac{\prod_{k=1}^{n-1}(t+k)^2}{(n!)^2} = \frac{\left[\left(1+\frac{1-z}{z}n\right)_{n-1}\right]^2}{2\sqrt{n}n(n!)^2}$$

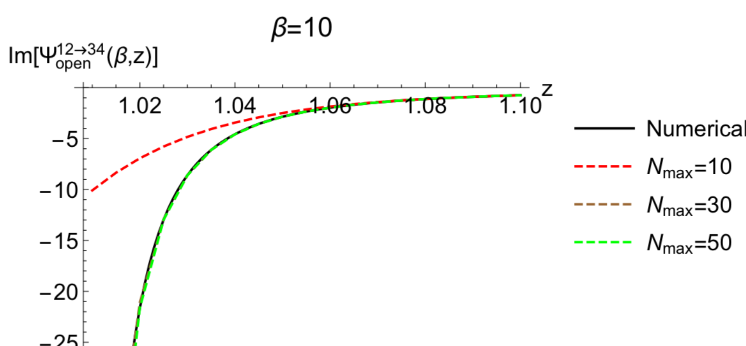
We expect

$$\mathbf{Im}\,\Psi^{12\leftrightarrow 34}_{\text{closed}}(\beta,z) = -\pi 2^{3-\beta} z^2 \sum_{n=1}^{\infty} \frac{(\sqrt{n})^{\beta-2}}{2} \left(\frac{\left[n\left(\frac{1}{z}-1\right)+1\right]_{n-1}}{n!}\right)^2$$

$z >> 1$



$z \to 1$



We have

$$\mathbf{Im}\,\Psi(\beta,z) = -B(z)\left\{\pi\sum_i \underset{\omega\to m_i}{\mathbf{Res}}\left[\omega^{\beta-1}T(\omega,z)\right] + \int_M^\infty \mathrm{d}\omega\,\omega^{\beta-1}\,\mathbf{Disc}\,[T(\omega,z)]\right\}$$

There is a natural POSITIVE expansion basis !! $P_\ell(\cos\theta)$

Thus we have positivity in the imaginary part of $\Psi(\beta,z)$

$p_i > 0$

$$\mathbf{Im}\,\Psi^{12\leftrightarrow34}(\beta,z) = \pi z^2 \sum_{i\in\mathcal{I}} p_i P_{J_i}\left(\frac{z-2}{z}\right) \qquad (z\geq 1),$$

$$\mathbf{Im}\,\Psi^{13\leftrightarrow24}(\beta,z) = \pi z^{\frac{\beta}{2}+2} \sum_{i\in\mathcal{I}} p_i P_{J_i}(1-2z) \qquad (1\geq z\geq 0),$$

$$\mathbf{Im}\,\Psi^{14\leftrightarrow23}(\beta,z) = \pi(-z)^{\frac{\beta}{2}+2}(1-z)^{-\frac{\beta}{2}} \sum_{i\in\mathcal{I}} p_i P_{J_i}\left(\frac{z+1}{z-1}\right) \qquad (0\geq z),$$

$$p_i = g_i^2\left(\tfrac{m_i}{2}\right)^{\beta-3} m_i^{2J_i+1}$$

It is straight ~~forward~~ forward to generalize to helicity states

$$T_{\ell_i}(s,t)|_{s\to m^2} = -\frac{m^{2J} d^J_{\ell_{34},\ell_{12}}\left(\arccos(\frac{u-t}{m^2})\right)}{s-m^2+i\varepsilon}.$$

$$\mathbf{Im}\,\Psi^{12\leftrightarrow34}(\beta,z) = \sum_i p_{i12;34}\ z^{2-\ell_{12}}(z-1)^{\frac{\ell_{12-34}}{2}} \mathcal{J}^{\ell_{12+34},\ell_{12-34}}_{J_i-\ell_{12}}\left(\frac{z-2}{z}\right) \qquad (z\geq 1),$$

$$\mathbf{Im}\,\Psi^{13\leftrightarrow24}(\beta,z) = \sum_i p_{i13;24}\ z^{\frac{\beta+\ell_{13+24}}{2}+2}(1-z)^{\frac{\ell_{13-24}}{2}} \mathcal{J}^{\ell_{13+24},\ell_{13-24}}_{J_i-\ell_{13}}(1-2z) \qquad (1\geq z\geq 0),$$

$$\mathbf{Im}\,\Psi^{14\leftrightarrow23}(\beta,z) = \sum_i p_{i14;23}\ \frac{(-z)^{\frac{\beta+\ell_{14+23}}{2}+2}}{(1-z)^{\frac{\beta}{2}+\ell_{23}}} \mathcal{J}^{\ell_{14+23},\ell_{14-23}}_{J_i-\ell_{23}}\left(\frac{z+1}{z-1}\right) \qquad (0\geq z),$$

$\mathcal{J}_\ell^{\alpha,\beta}$ are Jacoby polynomials $p_{i12;34} = \pi g_i\left(\frac{m}{2}\right)^{\beta-3} m_i^{2J_i+1} B^{J_i}_{12;34} > 0$

Note that these basis polynomials are nothing but the Poincaré Partial Waves (Law, Zlotnikov 2008.02331)

$$\tilde{\mathcal{A}}^{12\leftrightarrow 34}_{\Delta_i,\ell_i;m,J}(z_i,\bar{z}_i) = \langle \Delta_3, z_3, \bar{z}_3, \ell_3, \Delta_4, z_4, \bar{z}_4, \ell_4 | \mathbb{P}_{m,J} | \Delta_1, z_1, \bar{z}_1, \ell_1, \Delta_2, z_2, \bar{z}_1, \ell_2 \rangle$$

$$(\mathcal{P}_1^\mu + \mathcal{P}_2^\mu - \mathcal{P}_3^\mu - \mathcal{P}_4^\mu)\tilde{\mathcal{A}}^{12\leftrightarrow 34}_{\Delta_i,\ell_i;m,J}(z_i,\bar{z}_i) = 0\,,$$
$$(\mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu} + \mathcal{M}_3^{\mu\nu} + \mathcal{M}_4^{\mu\nu})\tilde{\mathcal{A}}^{12\leftrightarrow 34}_{\Delta_i,\ell_i;m,J}(z_i,\bar{z}_i) = 0\,,$$

$$\begin{aligned}
\mathcal{M}^{01} &= \frac{i}{2}\left[(\bar{w}^2-1)\bar{\partial} + (w^2-1)\partial + 2(\bar{h}\bar{w} + hw)\right]\,,\\
\mathcal{M}^{02} &= -\frac{1}{2}\left[(\bar{w}^2+1)\bar{\partial} - (w^2+1)\partial + 2(\bar{h}\bar{w} - hw)\right]\,,\\
\mathcal{M}^{03} &= i(\bar{w}\bar{\partial} + w\partial + \bar{h} + h)\,,\\
\mathcal{M}^{12} &= -\bar{w}\bar{\partial} + w\partial - \bar{h} + h\,,\\
\mathcal{M}^{13} &= \frac{i}{2}\left[(\bar{w}^2+1)\bar{\partial} + (w^2+1)\partial + 2(\bar{h}\bar{w} + hw)\right]\,,\\
\mathcal{M}^{23} &= -\frac{1}{2}\left[(\bar{w}^2-1)\bar{\partial} - (w^2-1)\partial + 2(\bar{h}\bar{w} - hw)\right]\,,
\end{aligned}$$

$$\mathcal{P}^\mu = 2q^\mu e^{\partial_\Delta}\,.$$

We have 2 Casmirs

$$(\mathcal{P}_1+\mathcal{P}_2)^\mu(\mathcal{P}_1+\mathcal{P}_2)_\mu \tilde{\mathcal{A}}^{12\leftrightarrow 34}_{\Delta_i,\ell_i;m,J}(z_i,\bar{z}_i) \;= -m^2\tilde{\mathcal{A}}^{12\leftrightarrow 34}_{\Delta_i,\ell_i;m,J}(z_i,\bar{z}_i)$$

$$-\,4e^{2\partial_{\boldsymbol{\Delta}}}\Phi^{12\leftrightarrow 34}_{m,J}(\boldsymbol{\Delta},\ell_i,z) = -m^2\Phi^{12\leftrightarrow 34}_{m,J}(\boldsymbol{\Delta},\ell_i,z)$$

$$(\mathcal{W}_1+\mathcal{W}_2)^\mu(\mathcal{W}_1+\mathcal{W}_2)_\mu \tilde{\mathcal{A}}^{12\leftrightarrow 34}_{\Delta_i,\ell_i;m,J}(z_i,\bar{z}_i) = m^2 J(J+1)\tilde{\mathcal{A}}^{12\leftrightarrow 34}_{\Delta_i,\ell_i;m,J}(z_i,\bar{z}_i)$$

$$\begin{aligned}
&\left[\frac{(\frac{1}{4}\ell^2_{12+34}-4)z^2 + (10-\ell_{12}\ell_{34})z - 6}{z-1} + (3z-4)z\partial - (z-1)z^2\partial^2\right]\Phi^{12\leftrightarrow 34}_{m,J}(\boldsymbol{\Delta},\ell_i,z)\\
&= J(J+1)\Phi^{12\leftrightarrow 34}_{m,J}(\boldsymbol{\Delta},\ell_i,z)\,.
\end{aligned}$$

So we find

$$\Phi^{\ell_i}_{m,J}(\beta,z)=\left(\frac{m}{2}\right)^{\beta}\sqrt{\frac{(2J+1)}{m}}\begin{cases} B^J_{12,34} z^{2-\ell_{12}}(z-1)^{\frac{\ell_{12-34}}{2}}\mathcal{J}^{\ell_{12+34},\ell_{12-34}}_{J-\ell_{12}}\left(\frac{z-2}{z}\right) & (z\geq 1)\,,\\ B^J_{13,24} z^{\frac{\beta+\ell_{13+24}}{2}+2}(1-z)^{\frac{\ell_{13-24}}{2}}\mathcal{J}^{\ell_{13+24},\ell_{13-24}}_{s-\ell_{13}}(1-2z) & (1\geq z\geq 0)\,,\\ B^J_{14,23}\frac{(-z)^{\frac{\beta+\ell_{14+23}}{2}+2}}{(1-z)^{\frac{\beta}{2}+\ell_{23}}}\mathcal{J}^{\ell_{14+23},\ell_{14-23}}_{J-\ell_{23}}\left(\frac{z+1}{z-1}\right) & (0\geq z)\,. \end{cases}$$

$\updownarrow$ The same basis

$$\mathbf{Im}\,\Psi^{12\leftrightarrow 34}(\beta,z)=\sum_i p_{i12;34}\; z^{2-\ell_{12}}(z-1)^{\frac{\ell_{12-34}}{2}}\mathcal{J}^{\ell_{12+34},\ell_{12-34}}_{J_i-\ell_{12}}\left(\frac{z-2}{z}\right)\qquad (z\geq 1)\,,$$

$$\mathbf{Im}\,\Psi^{13\leftrightarrow 24}(\beta,z)=\sum_i p_{i13;24}\; z^{\frac{\beta+\ell_{13+24}}{2}+2}(1-z)^{\frac{\ell_{13-24}}{2}}\mathcal{J}^{\ell_{13+24},\ell_{13-24}}_{J_i-\ell_{13}}(1-2z)\qquad (1\geq z\geq 0)\,,$$

$$\mathbf{Im}\,\Psi^{14\leftrightarrow 23}(\beta,z)=\sum_i p_{i14;23}\;\frac{(-z)^{\frac{\beta+\ell_{14+23}}{2}+2}}{(1-z)^{\frac{\beta}{2}+\ell_{23}}}\mathcal{J}^{\ell_{14+23},\ell_{14-23}}_{J_i-\ell_{23}}\left(\frac{z+1}{z-1}\right)\qquad (0\geq z)\,,$$

The imaginary part of the celestial sphere amplitude has a positive expansion in terms of Poincaré partial waves

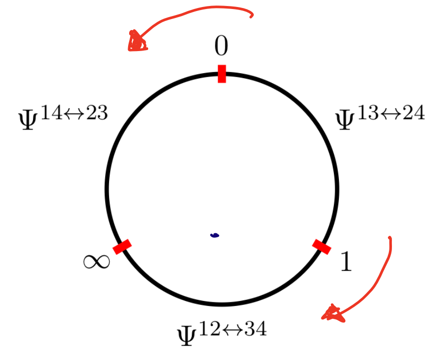
$$\mathbf{Im}\,\Psi_{>\Lambda}(\beta,\ell_i,z)=\sum_a p_a\Phi_{m_a,J_a}(\beta,\ell_i,z),\quad p_a>0$$

1. What is the image of dispersion relations for $T(s,t)$

2. Since poles at $\beta=-2N$ are EFT coefficients what is the image of EFThedron?

3. Do we have a bootstrap (Arkani-Hamed, Pate, Raclariu, Strominger 2102.04208)

Problem: we have 3 distinct functions!



We need to analytic continue to unphysical regions

Back to scalar

$$\Psi^{12\leftrightarrow 34}_{\text{scalar}}(\beta,z)=\frac{\pi g^2}{\sin\frac{\pi\beta}{2}}\left(\frac{m}{2}\right)^{\beta-2}z^2\left[e^{\frac{1}{2}\pi i\beta}+z^{\frac{\beta}{2}}+\left(\frac{z}{z-1}\right)^{\frac{\beta}{2}}\right]\qquad(z\geq 1)$$

$$\Psi^{13\leftrightarrow 24}(\beta,z)=\frac{\pi g^2}{\sin\frac{\pi\beta}{2}}\left(\frac{m}{2}\right)^{\beta-2}z^2\left[1+e^{\frac{1}{2}\pi i\beta}z^{\frac{\beta}{2}}+(z(1-z))^{\frac{\beta}{2}}\right]\qquad(1\geq z\geq 0)\,,$$

$$\Psi^{14\leftrightarrow 23}(\beta,z)=\frac{\pi g^2}{\sin\frac{\pi\beta}{2}}\left(\frac{m}{2}\right)^{\beta-2}z^2\left[1+(-z)^{\frac{\beta}{2}}+e^{\frac{1}{2}\pi i\beta}\left(\frac{-z}{1-z}\right)^{\frac{\beta}{2}}\right]\qquad(0\geq z)\,.$$

Consider analytic continuing $\Psi^{12\leftrightarrow 34}$

$z-1\to e^{-\pi i}(1-z)$ $\quad z\to 1$

$z\to e^{\pi i}(-z)$ $\quad z\to\infty$

$$\Psi^{12\leftrightarrow 34}_{\text{scalar}}(\beta,z)=\frac{\pi g^2}{\sin\frac{\pi\beta}{2}}\left(\frac{m}{2}\right)^{\beta-2}z^2\begin{cases}e^{\frac{1}{2}\pi i\beta}+z^{\frac{\beta}{2}}+e^{\frac{1}{2}\pi i\beta}\left(\frac{z}{1-z}\right)^{\frac{\beta}{2}} & (1>z>0)\,,\\ e^{\frac{1}{2}\pi i\beta}+e^{\frac{1}{2}\pi i\beta}(-z)^{\frac{\beta}{2}}+\left(\frac{z}{z-1}\right)^{\frac{\beta}{2}} & (0>z)\,.\end{cases}$$

The prescription is such that the continued form matches irrespective of it's origin.

$$\begin{aligned}\mathbf{Im}\,\Psi^{12\leftrightarrow 34}_{\text{scalar}}(\beta,z)&=\pi g^2\left(\frac{m}{2}\right)^{\beta-2}z^2\begin{cases}1+\left(\frac{z}{1-z}\right)^{\frac{\beta}{2}} \quad (s,t) & (1>z>0)\,,\\ 1+(-z)^{\frac{\beta}{2}} \quad (s,u) & (0>z)\,,\end{cases}\\ &=2^{3-\beta}\pi z^2g^2\begin{cases}\underset{\omega\to m,\sqrt{\frac{z}{1-z}}m}{\mathbf{Res}}\left[\omega^{\beta-1}T^{12\leftrightarrow 34}_{\text{scalar}}(\omega,z)\right] & (1>z>0)\,,\\ \underset{\omega\to m,\sqrt{-z}m}{\mathbf{Res}}\left[\omega^{\beta-1}T^{12\leftrightarrow 34}_{\text{scalar}}(\omega,z)\right] & (0>z)\,.\end{cases}\end{aligned}$$

The Imaginary part of the continued CA reflects the unphysical poles of the

' regicu!

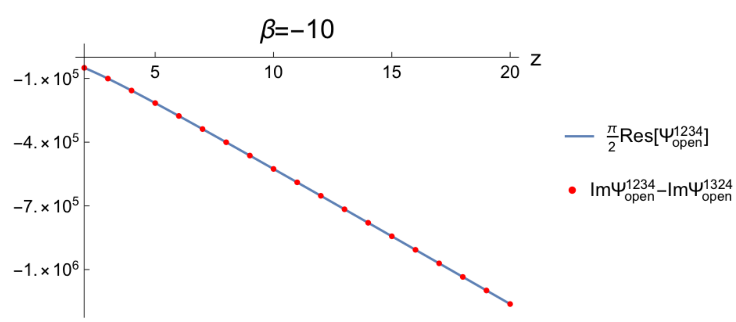
The dispersion relation for the flat space amplitude

$$\sum_{k,q} g_{k,q}\, s^{k-q} t^{q} = \sum_{n} \int \frac{ds}{s^{n+1}} \, \mathrm{Im}\left[M(s,t) \right]$$

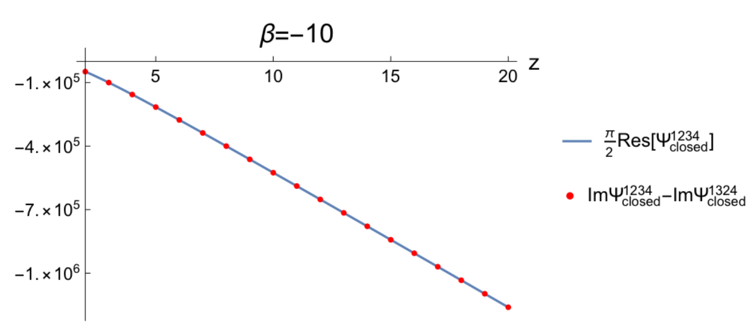
Now the image of this dispersion relation

$$\frac{\pi}{2}\underset{\beta\to-2n}{\mathbf{Res}}\left[\Psi^{12\leftrightarrow34}(\beta,z)\right]=\mathbf{Im}\left[\Psi^{12\leftrightarrow34}(\beta,z)+(-1)^{n}\Psi_{r}^{13\leftrightarrow24}(\beta,z)\right]\Big|_{\beta\to-2n}\quad(z\geq1)\,.$$

Test: (open string)

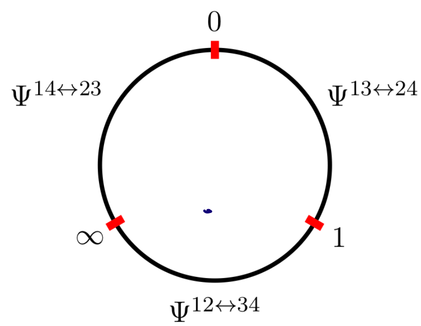


(Closed string)



Summary

1. The 4-pt CA is given by 3 seperate functions



The imaginary part has a postive exp.

$$
\begin{aligned}
\mathbf{Im}\,\Psi^{12\leftrightarrow 34}(\beta,z) &= \sum_i p_{i12;34}\; z^{2-\ell_{12}}(z-1)^{\frac{\ell_{12-34}}{2}}\mathcal{J}^{\ell_{12+34},\ell_{12-34}}_{J_i-\ell_{12}}\left(\frac{z-2}{z}\right) && (z\geq 1)\,,\\
\mathbf{Im}\,\Psi^{13\leftrightarrow 24}(\beta,z) &= \sum_i p_{i13;24}\; z^{\frac{\beta+\ell_{13+24}}{2}+2}(1-z)^{\frac{\ell_{13-24}}{2}}\mathcal{J}^{\ell_{13+24},\ell_{13-24}}_{J_i-\ell_{13}}(1-2z) && (1\geq z\geq 0)\,,\\
\mathbf{Im}\,\Psi^{14\leftrightarrow 23}(\beta,z) &= \sum_i p_{i14;23}\; \frac{(-z)^{\frac{\beta+\ell_{14+23}}{2}+2}}{(1-z)^{\frac{\beta}{2}+\ell_{23}}}\mathcal{J}^{\ell_{14+23},\ell_{14-23}}_{J_i-\ell_{23}}\left(\frac{z+1}{z-1}\right) && (0\geq z)\,,
\end{aligned}
$$

2. Analytic Continuate to unphysical region leads to a celestial sphere dispersion relation

$$
\frac{\pi}{2}\underset{\beta\to -2n}{\mathbf{Res}}\left[\Psi^{12\leftrightarrow 34}(\beta,z)\right] = \mathbf{Im}\left[\Psi^{12\leftrightarrow 34}(\beta,z)+(-1)^n\Psi^{13\leftrightarrow 24}(\beta,z)\right]\Big|_{\beta\to -2n}\quad (z\geq 1)\,.
$$