# Neutrino mixing, entanglement
# and the gauge paradigm in quantum field theory *.

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*L.Smaldone, G.Vitiello, in preparation;
M.Blasone, M.V.Gargiulo, G.Vitiello, J.Phys.Conf.Ser.(2017); Phys Lett B (2016)
M.Blasone, M.Di Mauro, G.Vitiello, Phys. Lett. B (2011);
M.Blasone, P.Jizba, G.Vitiello, Quantum field theory and its macroscopic manifestations, Imperial College Press 2011

**Some aspects of flavor mixing allow to regard flavor oscillations as a sort of dissipative process.**

**The QFT of flavor states leads to a vacuum flavor state for the mixed fields, which is orthogonal to the vacuum state for the fields with definite masses.**

**Its use allows to define correctly flavor states as eigenstates of the flavor charges.**

**It is a generalized $SU(2)$ coherent state. Moreover it is an entangled state of massive neutrinos and anti-neutrinos.**

**However, Lorentz invariance is broken, since the flavor vacuum is explicitly time-dependent.**

**As a consequence, flavor states cannot be interpreted in terms of irreducible representations of the Poincaré group. A possible way to**

**recover Lorentz invariance for mixed fields has been explored with relation to nonlinear realizations of the Poincaré group[*].**

**The relation of neutrino masses and mixing with the violation of the Lorentz and $CPT$ symmetries has been the subject of many efforts[†].**

**A related research line concerns the use of neutrino mixing and oscillations as a probe for quantum gravity effects, as quantum gravity induced decoherence is expected to affect neutrino oscillations[‡].**

**Such effects have also been connected to the non trivial structure of the flavor vacuum[§].**

[*]M. Blasone, J. Magueijo and P. Pires-Pacheco, Europhys. Lett. (2005)

[†]V.A. Kostelecky and M. Mewes, Phys. Rev. (2004); Phys. Rev. (2004)

[‡]G. Amelino-Camelia, Nature Phys. (2007); J. Alfaro, H. A. Morales-Tecotl and L. F. Urrutia, Phys. Rev. Lett. (2000); G. Lambiase, Mod. Phys. Lett. (2003)

[§]N.E. Mavromatos and S. Sarkar, New J. Phys. (2008)

**A non-Abelian gauge structure appears naturally in connection with flavor mixing: a possible account for the violation of Lorentz invariance.**

**Consider the Lagrangian in the flavor basis**

$$\mathcal{L}(x) = \bar{\nu}_f(x)\,(i\,\partial\!\!\!/ - M)\,\nu_f(x),$$

$\nu_f = (\nu_e, \nu_\mu)^T$ **and** $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$, **with** $m_e = m_1 \cos^2\theta + m_2 \sin^2\theta$ **,** $m_\mu = m_1 \sin^2\theta + m_2 \cos^2\theta$ **,** $m_{e\mu} = (m_2 - m_1)\sin\theta\cos\theta$ **.**

**The field equations are**

$$\begin{aligned} i\partial_0 \nu_e &= (-i\alpha\cdot\nabla + \beta\mathbf{m_e})\nu_\mathbf{e} + \beta\mathbf{m_{e\mu}}\nu_\mu \\ i\partial_0 \nu_\mu &= (-i\alpha\cdot\nabla + \beta\mathbf{m_\mu})\nu_\mu + \beta\mathbf{m_{e\mu}}\nu_\mathbf{e}, \end{aligned}$$

$\alpha_i$**,** $i = 1, 2, 3$ **and** $\beta$ **are the Dirac matrices. We choose the representation**

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1\!\!\mathrm{I} & 0 \\ 0 & -1\!\!\mathrm{I} \end{pmatrix},$$

$\sigma_i$ **are the Pauli matrices and** $1\!\!\mathrm{I}$ **the** $2\times 2$ **identity matrix. In a compact form:**

$$iD_0\nu_f = (-i\alpha\cdot\nabla + \beta\mathbf{M_d})\nu_\mathbf{f},$$

$\nu_f = (\nu_e, \nu_\mu)^T$ **is the flavor doublet and** $M_d = \mathrm{diag}(m_e, m_\mu)$ **is a diagonal mass matrix.**

**Note that the mixing term, proportional to** $m_{e\mu}$**, is taken into account by the (non-Abelian) covariant derivative:**

$$D_0 \equiv \partial_0 + i\, m_{e\mu}\, \beta\, \sigma_1,$$

**where** $m_{e\mu} = \frac{1}{2}\tan 2\theta\, \delta m$**, and** $\delta m \equiv m_\mu - m_e$**.**

**We thus see that flavor mixing can be seen as an interaction of the flavor fields with an** $SU(2)$ **constant gauge field:**

$$A_\mu \;\equiv\; \frac{1}{2} A_\mu^a \sigma_a = n_\mu \delta m \,\frac{\sigma_1}{2} \in su(2), \qquad\qquad n^\mu \equiv (1,0,0,0)^T,$$

**i.e., having only the temporal component in spacetime and only the first component in** $su(2)$ **space. The covariant derivative can be written in the form:**

$$D_\mu = \partial_\mu + i\, g\, \beta\, A_\mu,$$

where we have defined $g \equiv \tan 2\theta$ as the coupling constant for the mixing interaction. The Lagrangian is thus written as

$$\mathcal{L} = \bar{\nu}_f(i\gamma^\mu D_\mu - M_d)\nu_f.$$

In the case of maximal mixing ($\theta = \pi/4$), the coupling constant grows to infinity while $\delta m$ goes to zero. Since the gauge connection is a constant, with just one non-zero component in group space, its field strength vanishes identically:

$$F^a_{\mu\nu} = \epsilon^{abc} A^b_\mu A^c_\nu = 0, \qquad a, b, c = 1, 2, 3.$$

Despite $F_{\mu\nu}$ vanishes identically, the gauge field has physical effects, this leads to an analogy with the Aharonov–Bohm effect.

The energy momentum tensor associated with the flavor neutrino fields in interaction with the external gauge field is:

$$\widetilde{T}_{\rho\sigma} = \bar{\nu}_f i\gamma_\rho D_\sigma \nu_f - \eta_{\rho\sigma}\bar{\nu}_f(i\gamma^\lambda D_\lambda - M_d)\nu_f.$$

$\eta_{\rho\sigma} =$ **diag**$(+1,-1,-1,-1)$ **is the Minkowskian metric tensor.** $\widetilde{T}_{\rho\sigma}$ **is to be compared with the "canonical" energy momentum tensor:**

$$T_{\rho\sigma} \;=\; \bar{\nu}_f i\gamma_\rho D_\sigma \nu_f - \eta_{\rho\sigma}\bar{\nu}_f(i\gamma^\lambda D_\lambda - M_d)\nu_f + \eta_{\rho\sigma} m_{e\mu}\bar{\nu}_f\sigma_1\nu_f.$$

**The difference between the two is just the presence of the interaction terms in the** $00$ **component, i.e.** $T_{00} - \widetilde{T}_{00} = m_{e\mu}(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e)$**, while we have** $T_{0i} = \widetilde{T}_{0i}$**,** $T_{ij} = \widetilde{T}_{ij}$**.**

**The tensor** $\widetilde{T}_{\mu\nu}$ **is not conserved on-shell since** $[\gamma_\mu, D_0] \neq 0$**:**

$$\partial^\rho \widetilde{T}_{\rho i} \;=\; 0; \qquad\qquad \partial^\rho \widetilde{T}_{\rho 0} \neq 0.$$

**Without the mixing term in the covariant derivative it would have been conserved**

$$\partial^\mu \widetilde{T}_{\mu\nu} = g F_{\mu\nu a} j^\mu_a \;=\; 0,$$

**The 4-momentum $\widetilde{P}^{\mu} \equiv \int d^3\mathbf{x}\, \widetilde{T}^{0\mu}$ gives a conserved 3−momentum**

$$
\begin{aligned}
\widetilde{P}^{i} &= i\int d^3\mathbf{x}\, \nu_f^{\dagger}\partial^{i}\nu_f \\
&= i\int d^3\mathbf{x}\, \nu_e^{\dagger}\partial^{i}\nu_e + i\int d^3\mathbf{x}\, \nu_\mu^{\dagger}\partial^{i}\nu_\mu \\
&\equiv \widetilde{P}_e^{i}(x_0) + \widetilde{P}_\mu^{i}(x_0), \qquad\qquad i = 1,2,3 \qquad (1)
\end{aligned}
$$

**and a non conserved Hamiltonian operator:**

$$
\begin{aligned}
\widetilde{P}^{0}(x_0) &\equiv \widetilde{H}(x_0) = \int d^3\mathbf{x}\, \bar{\nu}_f\, (i\gamma_0 D_0 - i\gamma^{\mu}D_\mu + M_d)\, \nu_f \\
&= \int d^3\mathbf{x}\, \nu_e^{\dagger}\, (-i\alpha\cdot\nabla + \beta\mathbf{m_e})\, \nu_e + \int d^3\mathbf{x}\, \nu_\mu^{\dagger}\, (-i\alpha\cdot\nabla + \beta\mathbf{m}_\mu)\, \nu_\mu \\
&\equiv \widetilde{H}_e(x_0) + \widetilde{H}_\mu(x_0). \qquad (2)
\end{aligned}
$$

**The Hamiltonian and the momentum operators split in a contribution involving only the electron neutrino field and in another with only the muon neutrino field.**

**Note that the Lorentz generators $\widetilde{M}^{\lambda\rho}(x_0)$ can also be defined.**

**We remark that the tilde Hamiltonian is *not* the generator of time translations. This role competes to the complete Hamiltonian $H = \int d^3\mathbf{x}\, T^{00}$.**

**We now show that it is possible to define flavor neutrino states which are simultaneous eigenstates of the 4$-$momentum operators above constructed and of the flavor charges. Such a nontrivial request requires a redefinition of the flavor vacuum. To this end, we expand the flavor neutrino field operators in a different mass basis:**

$$\nu_\sigma(x) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_r \left[ u^r_{\mathbf{k},\sigma}(x_0)\,\widetilde{\alpha}^r_{\mathbf{k},\sigma}(x_0) + v^r_{-\mathbf{k},\sigma}(x_0)\,\widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(x_0)\right] e^{i\mathbf{k}\cdot\mathbf{x}},$$

**with $u^r_{\mathbf{k},\sigma}(x_0) = u^r_{\mathbf{k},\sigma} e^{-i\omega_{\mathbf{k},\sigma} x_0}$, $v^r_{-\mathbf{k},\sigma}(x_0) = v^r_{-\mathbf{k},\sigma} e^{i\omega_{\mathbf{k},\sigma} x_0}$. The new spinors are defined as the solutions of the equations:**

$$\begin{aligned}
(-\alpha\cdot\mathbf{k} + m_\sigma \beta)\, u^r_{\mathbf{k},\sigma} \;&=\; \omega_{\mathbf{k},\sigma} u^r_{\mathbf{k},\sigma} \\
(-\alpha\cdot\mathbf{k} + m_\sigma \beta)\, v^r_{-\mathbf{k},\sigma} \;&=\; -\omega_{\mathbf{k},\sigma} v^r_{-\mathbf{k},\sigma},
\end{aligned}$$

**where $\omega_{\mathbf{k},\sigma} = \sqrt{\mathbf{k}^2 + m_\sigma^2}$ and $\sigma = e, \mu$.**

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**The tilde flavor operators are connected to the previous ones by a Bogoliubov transformation:**

$$\left( \begin{array}{c} \widetilde{\alpha}^r_{\mathbf{k},\sigma}(x_0) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(x_0) \end{array} \right) = J^{-1}(x_0) \left( \begin{array}{c} \alpha^r_{\mathbf{k},\sigma}(x_0) \\ \beta^{r\dagger}_{-\mathbf{k},\sigma}(x_0) \end{array} \right) J(x_0),$$

**with generator:**

$$J(x_0) = \prod_{\mathbf{k},r} \exp\left\{ i \sum_{(\sigma,j)} \xi^{\mathbf{k}}_{\sigma,j} \left[ \alpha^{r\dagger}_{\mathbf{k},\sigma}(x_0) \beta^{r\dagger}_{-\mathbf{k},\sigma}(x_0) + \beta^{r}_{-\mathbf{k},\sigma}(x_0) \alpha^{r}_{\mathbf{k},\sigma}(x_0) \right] \right\},$$

**with** $(\sigma, j) = (e, 1), (\mu, 2)$, **and** $\xi^{\mathbf{k}}_{\sigma,j} = (\chi_\sigma - \chi_j)/2$ **and** $\chi_\sigma = \arctan(m_\sigma/|\mathbf{k}|)$**,** $\chi_j = \arctan(m_j/|\mathbf{k}|)$. **The new flavor vacuum is given by**

$$|\widetilde{0}(x_0)\rangle_{e\mu} = J^{-1}(x_0)|0(x_0)\rangle_{e\mu}.$$

**The (non-conserved) flavor charges (associated to the flavor La-**

**The tilde flavor operators are connected to the previous ones by a Bogoliubov transformation:**

$$\begin{pmatrix} \widetilde{\alpha}^{r}_{\mathbf{k},\sigma}(x_0) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(x_0) \end{pmatrix} = J^{-1}(x_0) \begin{pmatrix} \alpha^{r}_{\mathbf{k},\sigma}(x_0) \\ \beta^{r\dagger}_{-\mathbf{k},\sigma}(x_0) \end{pmatrix} J(x_0),$$

**with generator:**

$$J(x_0) = \prod_{\mathbf{k},r} \exp\left\{ i \sum_{(\sigma,j)} \xi^{\mathbf{k}}_{\sigma,j} \left[ \alpha^{r\dagger}_{\mathbf{k},\sigma}(x_0)\beta^{r\dagger}_{-\mathbf{k},\sigma}(x_0) + \beta^{r}_{-\mathbf{k},\sigma}(x_0)\alpha^{r}_{\mathbf{k},\sigma}(x_0) \right] \right\},$$

**with** $(\sigma, j) = (e, 1), (\mu, 2)$, **and** $\xi^{\mathbf{k}}_{\sigma,j} = (\chi_\sigma - \chi_j)/2$ **and** $\chi_\sigma = \arctan(m_\sigma/|\mathbf{k}|)$, $\chi_j = \arctan(m_j/|\mathbf{k}|)$. **The new flavor vacuum is given by**

$$|\widetilde{0}(x_0)\rangle_{e\mu} = J^{-1}(x_0)|0(x_0)\rangle_{e\mu}.$$

**Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:**

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

**Mixing transformation = Rotation** $(\cos\theta\,,\,\sin\theta)$ **+ Bogoliubov transformation** $(U_{\mathbf{k}}\,,\,V_{\mathbf{k}})$**.**

$$\alpha_e(t)|0(t)\rangle_{e,\mu} = G^{-1}_\theta(t)\alpha_1\mathbf{G}_\theta(\mathbf{t})\;\mathbf{G}^{-1}_\theta(\mathbf{t})|0\rangle_{1,2} = 0.$$

**Recall that**

$$G(t;\theta,m_1,m_2) = B^{-1}(t;m_1,m_2)\,R(t;\theta)\,B(t;m_1,m_2)\,,$$

**and the expansion of $G(\theta)$ at t=0**

$$G(\theta) = \exp\Bigg\{\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\Big[U_\mathbf{k}\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2}+\beta^{r}_{-\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}-\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1}-\beta^{r}_{-\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right)$$
$$+\epsilon^r V_\mathbf{k}\left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}-\beta^{r}_{-\mathbf{k},1}\alpha^{r}_{\mathbf{k},2}+\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}-\beta^{r}_{-\mathbf{k},2}\alpha^{r}_{\mathbf{k},1}\right)\Big]\Bigg\}\,,$$

**where**

$$U_\mathbf{k} = \cos(\Theta_{\mathbf{k},2}-\Theta_{\mathbf{k},1})\,,\qquad V_\mathbf{k} = \sin(\Theta_{\mathbf{k},2}-\Theta_{\mathbf{k},1})\,.$$

**The (non-conserved) flavor charges (associated to the flavor Lagrangian) describing the phenomenon of neutrino oscillations are**

$$Q_\sigma(x_0) \;=\; \int d^3\mathbf{x}\, \nu_\sigma^\dagger(x)\; \nu_\sigma(x) \;\;, \qquad \sigma = e, \mu,$$

**with** $Q_e(x_0) + Q_\mu(x_0) = Q$**. Explicitly:**

$$\begin{aligned} Q_e(x_0) \;&=\; \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger(x)\nu_2(x) + \nu_2^\dagger(x)\nu_1(x)\right] \;, \\ Q_\mu(x_0) \;&=\; \sin^2\theta\, Q_1 + \cos^2\theta\, Q_2 - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger(x)\nu_2(x) + \nu_2^\dagger(x)\nu_1(x)\right] \;. \end{aligned}$$

**with** $Q_j$**,** $j = 1, 2$**, two conserved (Noether) charges :**

$$Q_j \;=\; \int d^3\mathbf{x}\, \nu_j^\dagger(x)\; \nu_j(x)\,, \qquad\qquad j = 1, 2,$$

**and** $Q = Q_1 + Q_2$**. Contributions in** $Q_\sigma(x_0)$**,** $\sigma = e, \mu$**, that cannot be written in terms of** $Q_j$ **are related to the non-trivial structure of the flavor Hilbert space. The above Bogoliubov transformations leave invariant the** $Q_\sigma(x_0)$**:** $\widetilde{Q}_\sigma = Q_\sigma$**, with:**

$$\widetilde{Q}_\sigma(x_0) = \sum_r \int d^3\mathbf{k} \left( \widetilde{\alpha}^{r\dagger}_{\mathbf{k}\sigma}(x_0)\widetilde{\alpha}^{r}_{\mathbf{k}\sigma}(x_0) - \widetilde{\beta}^{r\dagger}_{-\mathbf{k}\sigma}(x_0)\widetilde{\beta}^{r}_{-\mathbf{k}\sigma}(x_0)\right).$$

**In terms of the tilde flavor ladder operators, the (tilde-) Hamiltonian and momentum operators read:**

$$\widetilde{\mathbf{P}}_\sigma(x_0) \;=\; \sum_r \int d^3\mathbf{k}\; \mathbf{k}\left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(x_0)\widetilde{\alpha}^{r}_{\mathbf{k},\sigma}(x_0) + \widetilde{\beta}^{r\dagger}_{\mathbf{k},\sigma}(x_0)\widetilde{\beta}^{r}_{\mathbf{k},\sigma}(x_0)\right),$$

$$\widetilde{H}_\sigma(x_0) \;=\; \sum_r \int d^3\mathbf{k}\, \omega_{\mathbf{k},\sigma}\left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(x_0)\,\widetilde{\alpha}^{r}_{\mathbf{k},\sigma}(x_0) - \widetilde{\beta}^{r}_{\mathbf{k},\sigma}(x_0)\,\widetilde{\beta}^{r\dagger}_{\mathbf{k},\sigma}(x_0)\right). \qquad (5)$$

**Since these operators are diagonal, we can define common eigen-states:**

$$|\widetilde{\nu}^{r}_{\mathbf{k},\sigma}(x_0)\rangle = \widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(x_0)|\widetilde{0}(x_0)\rangle_{e\mu}.$$

**and similar ones for the antiparticles, and**

$$\begin{pmatrix} \widetilde{H}_\sigma(x_0) \\ \widetilde{\mathbf{P}}_\sigma(x_0) \end{pmatrix} |\widetilde{\nu}^{r}_{\mathbf{k},\sigma}(x_0)\rangle = \begin{pmatrix} \omega_{\mathbf{k},\sigma} \\ \mathbf{k} \end{pmatrix} |\widetilde{\nu}^{r}_{\mathbf{k},\sigma}(x_0)\rangle,$$

**making explicit the $4-$vector structure.**

**The flavor charges commute with the tilde Hamiltonian operator: $[\widetilde{Q}_\sigma(x_0), \widetilde{H}(x_0)] = 0$, since**

$$[\widetilde{Q}_\sigma(x_0), \widetilde{H}_{\sigma'}(x_0)] = 0, \qquad \sigma, \sigma' = e, \mu.$$

**This is of course a consequence of the fact that the flavor non-conservation is entirely due to the interaction term, which is absent in $\widetilde{H}$.**

**This fact ensures the existence of a common set of eigenstates of these operators. Indeed the flavor states (6) are seen to be also eigenstates of the flavor charges:**

$$\widetilde{Q}_\sigma(x_0)|\widetilde{\nu}^r_{\mathbf{k},\sigma}(x_0)\rangle = |\widetilde{\nu}^r_{\mathbf{k},\sigma}(x_0)\rangle,$$

**which confirming that these are precisely the states we were looking for.**

**Note that the above construction and the consequent Poincaré invariance, holds at a given time $x_0$. Thus, for each different time, we**

**have a different Poincaré structure. This reminds of the quantization in the curved space-time and of quantum dissipation.**

**Flavor neutrino fields behave (locally in time) as ordinary on-shell fields with definite masses $m_e$ and $m_\mu$, rather than those of the mass eigenstates of the standard approach, $m_1$ and $m_2$.**

**The Hamiltonian operator $\widetilde{H}$ does not take into account the interaction energy, i.e. the energy associated with mixing.**

**$\widetilde{H}$ appears as the sum of the kinetic energies of the flavor neutrinos, or equivalently as the energy which can be extracted from flavor neutrinos by scattering processes, the mixing energy being "frozen" (there's no way to turn off the mixing!).**

**This suggests the interpretation of such a quantity as a "free" energy $F \equiv \widetilde{H}$, so that we can write:**

$$H - F = TS,$$

**which defines an entropy associated with flavor mixing. It is natural to identify the "temperature" $T$ with the coupling constant $g = \tan 2\theta$, thus leading to:**

$$S \;=\; \int d^3\mathbf{x}\,\bar{\nu}_f A_0 \nu_f = \frac{1}{2}\,\delta m \int d^3\mathbf{x}\,(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e).$$

**The appearance of an entropy confirms that each of the two flavor neutrinos can be considered as an "open system" which presents some kind of (cyclic) dissipation.**

**These thermodynamical considerations well fit with the interpretation of the gauge field as a reservoir.**

**The gauge field structure in the neutrino evolution (one neutrino evolution intrinsically dependent on the other neutrino evolution) signals that an entanglement is present in the QFT neutrino mixing vacuum.**

**I will shortly summarize the QFT vacuum structure for mixed neutrinos. The case of two neutrinos is considered, but extension to three neutrinos has been worked out (extension also Majorana neutrino, boson mixing, neutral mixing particle is also reported in the literature).**

**Condensate structure of $|0\rangle_{e,\mu}$ (use $\epsilon^r = (-1)^r$ )**

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1-\sin^2\theta\,|V_\mathbf{k}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_\mathbf{k}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_\mathbf{k}||U_\mathbf{k}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_\mathbf{k}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

**- 4 kinds of particle-antiparticle pairs with zero momentum and spin.**

**Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:**

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

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**Mixing transformation = Rotation** $(\cos\theta\,,\,\sin\theta)$ **+ Bogoliubov transformation** $(U_{\mathbf{k}}\,,\,V_{\mathbf{k}})$**.**

$$\alpha_e(t)|0(t)\rangle_{e,\mu} = G^{-1}_\theta(t)\alpha_1\mathbf{G}_\theta(\mathbf{t})\;\mathbf{G}^{-1}_\theta(\mathbf{t})|0\rangle_{1,2} = 0.$$

The linear correlation coefficient $J(\widehat{n}_a, \widehat{n}_b)$ provides a measure of the particle entanglement in $|0\rangle_{e,\mu}$

$$J(\widehat{n}_a, \widehat{n}_b) = \frac{cov(\widehat{n}_a, \widehat{n}_b)}{(\langle(\Delta\widehat{n}_a)^2\rangle)^{1/2}(\langle(\Delta\widehat{n}_b)^2\rangle)^{1/2}},$$

$\langle * \rangle$ denotes expectation value in $|0\rangle_{e,\mu}$, $\widehat{n}_a$, $\widehat{n}_b$ number operators, the variance $\langle(\Delta\widehat{n})^2\rangle \equiv \langle(\widehat{n} - \langle\widehat{n}\rangle)^2\rangle = \langle\widehat{n}^2\rangle - \langle\widehat{n}\rangle^2$, the covariance $cov(\widehat{n}_a, \widehat{n}_b) \equiv \langle\widehat{n}_a\widehat{n}_b\rangle - \langle\widehat{n}_a\rangle\langle\widehat{n}_b\rangle$. For non-correlated modes $\langle\widehat{n}_a\widehat{n}_b\rangle = \langle\widehat{n}_a\rangle\langle\widehat{n}_b\rangle$, and $cov(\widehat{n}_a, \widehat{n}_b)$ is zero. On the contrary, the $(a, b)$-pair correlation is due to the coherent condensate structure of the vacuum $|0\rangle_{e,\mu}$ generated by the Bogoliubov transformations.

$J(\widehat{n}_{\alpha^r_{\mathbf{k},i}}, \widehat{n}_{\alpha^r_{\mathbf{k},j}}) = 0 = J(\widehat{n}_{\beta^r_{-\mathbf{k},i}}, \widehat{n}_{\beta^r_{-\mathbf{k},j}})$, for any $\mathbf{k}$, $i \neq j$ and $i, j = 1, 2$

as expected from inspection of the vacuum structure and of the operator transformations, these particles are involved together only by the rotation part of the transformation.

**Instead,**

$$J(\widehat{n}_{\alpha^r_{\mathbf{k},i}}, \widehat{n}_{\beta^r_{-\mathbf{k},j}}) = \frac{1}{1+\tan^2\theta|U|^2} \approx 1 - \tan^2\theta|U|^2, \; i \neq j.$$

**Considering that** $|U|^2 < 1$**, and using** $\tan^2\theta = 0.44$ **(**$\sin^2 2\theta = 0.846 \pm 0.021$**) we have a sensible entanglement value (**$> 0.5$**) for** $\alpha^r_{\mathbf{k},i} - \beta^r_{-\mathbf{k},j}$**,** $i \neq j$**, particles. Finally, a much lower value is obtained for** $\alpha^r_{\mathbf{k},i} - \beta^r_{-\mathbf{k},i}$**,** $i = 1, 2$**,**

$$J(\widehat{n}_{\alpha^r_{\mathbf{k},i}}, \widehat{n}_{\beta^r_{-\mathbf{k},i}}) = \tan^2\theta|U|^2(1 - \tan^2\theta|U|^2) \ll J(\widehat{n}_{\alpha^r_{\mathbf{k},i}}, \widehat{n}_{\beta^r_{-\mathbf{k},j}})|_{i\neq j},$$

## Hilbert space for mixed neutrinos

**Mixing relations can be written as***

$$\nu_e^\alpha(x) \;=\; G_\theta^{-1}(t)\,\nu_1^\alpha(x)\,G_\theta(t)$$

$$\nu_\mu^\alpha(x) \;=\; G_\theta^{-1}(t)\,\nu_2^\alpha(x)\,G_\theta(t)$$

**with generator given by:**

$$G_\theta(t) = \exp[\theta\left(S_+(t) - S_-(t)\right)]$$

$$S_+(t) \equiv \int d^3\mathbf{x}\,\nu_1^\dagger(x)\nu_2(x) \quad , \quad S_-(t) \equiv \int d^3\mathbf{x}\,\nu_2^\dagger(x)\nu_1(x)$$

***M.Blasone and G.Vitiello, Annals Phys. (1995)**

**Introducing:**

$$S_3 \equiv \frac{1}{2}\int d^3\mathbf{x}\,\Big(\nu_1^\dagger(x)\nu_1(x) - \nu_2^\dagger(x)\nu_2(x)\Big)$$

$$S_0 \equiv \frac{1}{2}\int d^3\mathbf{x}\,\Big(\nu_1^\dagger(x)\nu_1(x) + \nu_2^\dagger(x)\nu_2(x)\Big)$$

**the $su(2)$ algebra is closed:**

$$[S_+(t)\,,\,S_-(t)] = 2S_3 \quad , \quad [S_3\,,\,S_\pm(t)] = \pm S_\pm(t)$$

**Verify above eqs. For $\nu_e$ we get**

$$\frac{d^2}{d\theta^2}\,\nu_e^\alpha = -\nu_e^\alpha$$

**with the initial conditions**

$$\nu_e^\alpha|_{\theta=0} = \nu_{_1}^\alpha \quad , \quad \frac{d}{d\theta}\nu_e^\alpha\Big|_{\theta=0} = \nu_{_2}^\alpha$$

$\nu_i$ **($i = 1, 2$) are free Dirac field operators:**

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} [\, u^r_{\mathbf{k},i}(t)\, \alpha^r_{\mathbf{k},i} \, + \, v^r_{-\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i} \,]$$

**with** $u^r_{\mathbf{k},i}(t) \, = \, e^{-i\omega_{k,i}t}\, u^r_{\mathbf{k},i}$ **,** $v^r_{\mathbf{k},i}(t) \, = \, e^{i\omega_{k,i}t}\, v^r_{\mathbf{k},i}$ **and** $\omega_{k,i} = \sqrt{k^2 + m_i^2}$**.**

**Anticommutation relations:**

$$\{\nu^\alpha_i(x), \nu^{\beta\dagger}_j(y)\}_{t=t'} = \delta^3(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta}\delta_{ij}$$

$$\{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij} \quad ; \quad \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}$$

**Orthonormality and completeness relations:**

$$u^{r\dagger}_{\mathbf{k},i} u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i} v^s_{\mathbf{k},i} = \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} = 0 \quad , \quad \sum_r (u^{r\alpha *}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha *}_{-\mathbf{k},i} v^{r\beta}_{-\mathbf{k},i}) = \delta_{\alpha\beta} \, .$$

**– Fock space for** $\nu_1$**,** $\nu_2$**:**

$$\mathcal{H}_{1,2} = \left\{ \alpha^\dagger_{1,2} \, , \, \beta^\dagger_{1,2} \, , \, |0\rangle_{1,2} \right\} .$$

• **The vacuum $|0\rangle_{1,2}$ is not invariant under the action of the generator $G_\theta(t)$:**

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\;|0\rangle_{1,2} = e^{-\theta\left(S_+(t)-S_-(t)\right)}\;|0\rangle_{1,2}$$

**The vacuum $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state*.**

• **Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)**

$$\lim_{V\to\infty} {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\, e^{V\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

**with**

$$|V_\mathbf{k}|^2 \equiv \sum_{r,s}|\,v^{r\dagger}_{-\mathbf{k},1}u^s_{\mathbf{k},2}\,|^2\;\;, \qquad 0 \le |V_\mathbf{k}|^2 \le \frac{1}{2}$$

*A. Perelomov, *Generalized Coherent States and Their Applications*, (Springer V., 1986)

**Condensate structure of $|0\rangle_{e,\mu}$ (use $\epsilon^r = (-1)^r$ )**

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1-\sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_{\mathbf{k}}||U_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_{\mathbf{k}}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

**- 4 kinds of particle-antiparticle pairs with zero momentum and spin.**

**- Time dependence:**

$$|0\rangle_{e,\mu} \equiv |0(0)\rangle_{e,\mu} = e^{-iHt}|0(t)\rangle_{e,\mu}$$

**- Condensation density:**

$${}_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},1}|0(t)\rangle_{e,\mu} = \sin^2\theta\,|V_{\mathbf{k}}|^2$$

**vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing). Same result for $\alpha_2$, $\beta_1$, $\beta_2$.**

## The flavor fields can be expanded as:

$$\nu_e(x) \;=\; \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \left[u^r_{\mathbf{k},1}(t)\,\alpha^r_{\mathbf{k},e}(t) \,+\, v^r_{-\mathbf{k},1}(t)\,\beta^{r\dagger}_{-\mathbf{k},e}(t)\right]$$

$$\nu_\mu(x) \;=\; \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \left[u^r_{\mathbf{k},2}(t)\,\alpha^r_{\mathbf{k},\mu}(t) \,+\, v^r_{-\mathbf{k},2}(t)\,\beta^{r\dagger}_{-\mathbf{k},\mu}(t)\right]$$

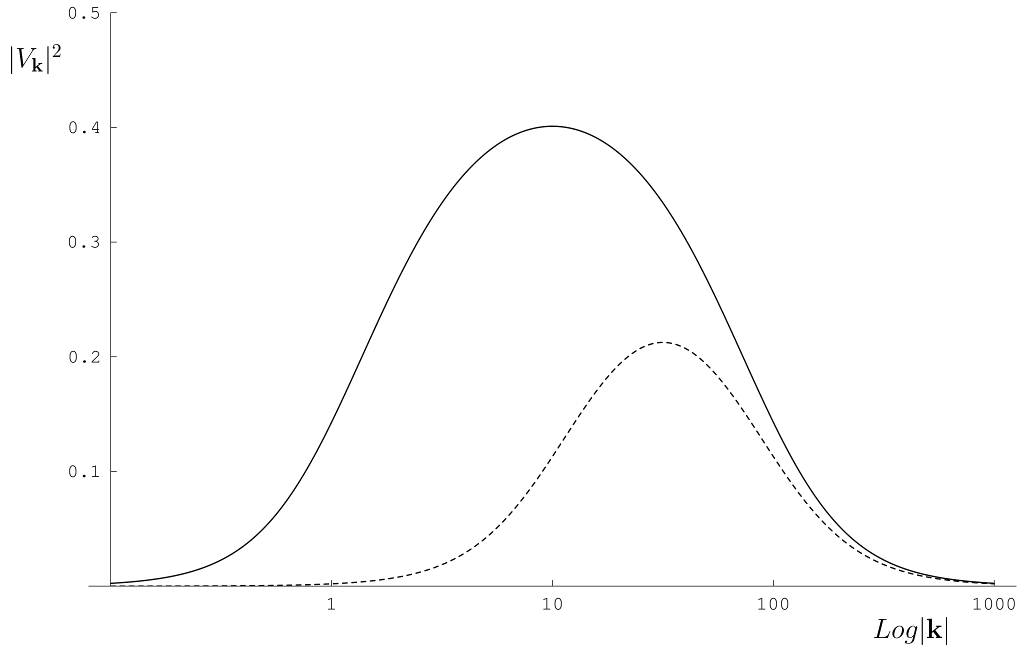
## Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \;=\; u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\; e^{i(\omega_{k,2}-\omega_{k,1})t} \qquad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\; u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\; e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

$$|V_{\mathbf{k}}|^2 \;\equiv\; \sum_{r,s} |\; v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\; |^2 = \frac{k^2\left[(\omega_{k,2}+m_2) - (\omega_{k,1}+m_1)\right]^2}{4\;\omega_{k,1}\omega_{k,2}(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}$$

# Condensation density for mixed fermions




**Solid line:** $m_1 = 1$, $m_2 = 100$; **Dashed line:** $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ **when** $m_1 = m_2$ **and/or** $\theta = 0$.

- **Max. at** $k = \sqrt{m_1 m_2}$ **with** $V_{max} \to \frac{1}{2}$ **for** $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.

- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ **for** $k \gg \sqrt{m_1 m_2}$.

**We have described flavor oscillations as a consequence of the interaction with the gauge field. This may be described as acting as a sort of refractive medium.**

**An interesting analogy is the one of some scenarios in which, for the case of photons, the vacuum has been thought to act as a refractive medium in consequence of quantum gravity fluctuations*.**

**For simplicity, let us use the Pontecorvo formalism. Our argument does not depend on such a simplification. To start, assume degeneracy $\omega_1 = \omega_2$ ($m_1 = m_2$) and that, in obvious notation, time evolution "in the vacuum" is given by**

$$\begin{pmatrix} |\nu_e(t)\rangle \\ |\nu_\mu(t)\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \begin{pmatrix} |\nu_e(0)\rangle \\ |\nu_\mu(0)\rangle \end{pmatrix},$$

**Let $\omega = 2\pi\,\nu$, and the propagation speed in the vacuum $v_0 = \lambda\,\nu$. Suppose then that the propagation occurs in a medium presenting**

*J.R.Ellis, N.E.Mavromatos and D.V.Nanopoulos, Phys. Lett. (2008)

different refraction indexes, $n_1$ and $n_2$ for $\nu_e$ and $\nu_\mu$, respectively, i.e. the propagation over a given path of length $L$ occurs in different times, $t_1$ and $t_2$, respectively:

$$t_1 = \frac{L}{v_1} = \frac{L\,n_1}{v_0} = t\,n_1 \; ; \qquad t_2 = \frac{L}{v_2} = \frac{L\,n_2}{v_0} = t\,n_2 \; ,$$

with $v_1$ and $v_2$ the propagation speeds in the medium, respectively, and $t = \frac{L}{v_0}$.

Time evolution is then described by the phase factors $e^{-i\omega\, t_1} = e^{-i\omega_1 t}$ and $e^{-i\omega\, t_2} = e^{-i\omega_2 t}$ for the two particles, respectively, where $\omega\, t_i = \omega \frac{L}{v_0}\, n_i = 2\,\pi\,\nu\,t\,n_i = 2\,\pi\,\nu_i\,t = \omega_i\,t$, $i = 1, 2$, has been used, together with $\lambda_i\,\nu = v_i$, $\lambda_i\,\nu_i = v_0$ and $n_i = \frac{v_0}{v_i} = \frac{\nu_i}{\nu}$. Thus,

$$\begin{pmatrix} |\nu_e(t)\rangle \\ |\nu_\mu(t)\rangle \end{pmatrix} = e^{-i\omega_1 t} \begin{pmatrix} \cos\theta & e^{-i(\omega_2-\omega_1)t}\sin\theta \\ -e^{-i(\omega_2-\omega_1)t}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix}, \qquad (5)$$

which is the time evolution we started with our original Lagrangian and motion equations.

**In conclusion, the effect of time evolution through the refractive medium is equivalent to the effect of the background gauge field** $A_0^{(1)} = \frac{1}{2}(\omega_2 - \omega_1)\ \cos 2\theta = \frac{1}{2}\omega(n_2 - n_1)\ \cos 2\theta$**, for** $\theta \neq \frac{\pi}{4} + \frac{n\pi}{2}$**, which indeed disappears when propagation occurs in the vacuum with** $n_1 = n_2 = n_0 = 1$ **(i.e.** $\omega_1 = \omega = \omega_2$**).**