Celestial OPEs from the worldsheet

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Celestial Amplitudes & Flat Space Holography

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work with W. Bu, E. Casali & A. Sharma [2109.xxxxx]

Motivation

Celestial holography: lots of hints about existence of celestial CFT (CCFT)

These hints are usually:

- kinematic/imposed by symmetries
- imported 'for free' from momentum space (take favorite momentum-space expression, Mellin transform it)

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Can we generate anything *dynamically*?

Important example: OPE coefficients

$O^a_{\pm,\Delta}(z,\bar{z})$: $\pm$ helicity (outgoing) gluon of weight Δ at $(z,\bar{z}) \in S^2$

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OPE limit on $S^2 \leftrightarrow$ collinear limits in momentum space

Allows determination of some explicit CFT data!

[Fan-Fotopoulos-Taylor, Pate-Raclariu-Strominger-Yuan, Himwich-Pate-Singh]

$$O^{a}_{+,\Delta_i}\, O^{b}_{+,\Delta_j} \sim \frac{f^{abc}}{z_{ij}}\, B(\Delta_i - 1, \Delta_j - 1)\, O^{c}_{+,\Delta_i+\Delta_j-1}(z_j,\bar{z}_j)$$

$$O^{a}_{-,\Delta_i}\, O^{b}_{+,\Delta_j} \sim \frac{f^{abc}}{z_{ij}}\, B(\Delta_i + 1, \Delta_j - 1)\, O^{c}_{-,\Delta_i+\Delta_j-1}(z_j,\bar{z}_j) + \frac{f^{abc}}{\bar{z}_{ij}}\, B(\Delta_i - 1, \Delta_j + 1)\, O^{c}_{+,\Delta_i+\Delta_j-1}(z_j,\bar{z}_j)$$

Today:

Generate these OPE coefficients using a CFT

- Use worldsheet description of asymptotically flat gauge theory/gravity ('ambitwistor strings')
- Conformal primary wavefunctions $\leftrightarrow$ vertex operators in worldsheet theory
- OPE limit on worldsheet $\leftrightarrow$ OPE limit on celestial sphere
- Correct coefficients generated by worldsheet CFT!

Ambitwistor space

Let $\mathbb{PA} = \{\text{complexified null geodesics in } \mathbb{M}\}/\text{scale}$
$\mathbb{PA} \cong T^*\mathscr{I}_{\mathbb{C}}/\text{scale}$: point on $\mathscr{I}_{\mathbb{C}}$ + null direction [LeBrun, Geyer-Lipstein-Mason]

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In 4d, parametrized by $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$ and $W_A = (\tilde\lambda_{\dot\alpha}, \tilde\mu^\alpha)$ on $\mathbb{CP}^3 \times \mathbb{CP}^3$ subject to

$$Z \cdot W = [\mu\, \tilde\lambda] + \langle \tilde\mu\, \lambda \rangle = 0$$

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Coordinates on $\mathscr{I}_{\mathbb{C}}$: $(u, \lambda, \tilde\lambda) \sim (r\tilde r u, r\lambda, \tilde r \tilde\lambda)$ [Sparling, Eastwood-Tod]

$$[\mu\, \tilde\lambda] = u = \langle \lambda\, \tilde\mu \rangle$$

Ambitwistor string

Governs holomorphic maps $\Sigma \to \mathbb{PA}$ [Mason-Skinner, Geyer-Lipstein-Mason]

Let Z^A, W_A take values in $K_\Sigma^{1/2}$

$$S = \frac{1}{2\pi} \int_\Sigma W \cdot \bar{\partial} Z - Z \cdot \bar{\partial} W + a\, Z \cdot W + S_{\mathrm{matter}}$$

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We'll take S_{matter} a worldsheet current alg. for $\mathrm{SU}(N)$, level k

Quantization

Set $a = 0$, fix conformal gauge (i.e., $\bar{\partial} = \mathrm{d}\bar{\sigma}\partial_{\bar{\sigma}}$)

Anomaly-free for appropriately chosen target SUSY + (N, k) (not terribly important @ genus zero)

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Fact: model describes tree-level 4d Yang-Mills theory when $\Sigma \cong \mathbb{CP}^1$ and $k \to 0$ [Berkovits-Witten, Geyer-Lipstein-Mason, TA-Casali-Nekovar]

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Only worldsheet OPEs:

$$Z^A(\sigma)\,W_B(\sigma') \sim \frac{\delta^A_B}{\sigma - \sigma'}\,, \qquad j^{\mathrm{a}}(\sigma)\,j^{\mathrm{b}}(\sigma') \sim \frac{f^{\mathrm{abc}}\,j^{\mathrm{c}}(\sigma')}{\sigma - \sigma'}$$

Vertex operators

pos/neg helicity gluons represented by:

$$U_+^{\mathsf{a}} = \int_\Sigma j^{\mathsf{a}}(\sigma)\, a(Z(\sigma))\,, \qquad U_-^{\mathsf{a}} = \int_\Sigma j^{\mathsf{a}}(\sigma)\, \tilde{a}(W(\sigma))$$

for $a \in H^{0,1}(\mathbb{PT}, \mathcal{O})$, $\tilde{a} \in H^{0,1}(\mathbb{PT}^*, \mathcal{O})$

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Gluon conformal primaries of $\{\Delta, (z, \bar{z})\}$: [TA-Mason-Sharma]

$$a(Z) = (-\mathrm{i})^{\Delta-1}\,\Gamma(\Delta-1)\int_{\mathbb{C}^*}\frac{\mathrm{d}s}{s^\Delta}\,\frac{\bar{\delta}^2(z_\alpha - s\,\lambda_\alpha)}{[\mu\,\bar{z}]^{\Delta-1}}$$

$$\tilde{a}(W) = (-\mathrm{i})^{\Delta-1}\,\Gamma(\Delta-1)\int_{\mathbb{C}^*}\frac{\mathrm{d}\tilde{s}}{\tilde{s}^\Delta}\,\frac{\bar{\delta}^2(\bar{z}_{\dot\alpha} - \tilde{s}\,\tilde{\lambda}_{\dot\alpha})}{\langle\tilde{\mu}\,z\rangle^{\Delta-1}}$$

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Some unpacking:

- $z_\alpha = (-z, 1)$, $\bar{z}_{\dot\alpha} = (\bar{z}, 1)$, homog. coords on $S^2 \cong \mathbb{CP}^1$ celestial sphere
- s an (affine) scaling parameter, ensuring appropriate homogeneity
- poles in $\Delta = 1, 0, -1, \ldots \leftrightarrow$ conformally soft gluons

[Donnay-Puhm-Strominger, Pate-Raclariu-Strominger, Guevara]

Basic Idea

Look at worldsheet OPE between $U_{\pm}^{a}$ insertions

What we know:

- Correlation functions of these vertex operators give tree-level S-matrix in conf. primary basis [TA-Mason-Sharma, Casali-Sharma]
- Correlator inherits correct soft/collinear behaviour from momentum space [Geyer, Lipstein]

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But still a non-trivial calculation!

- Looking at 'raw' OPE – not inside correlation function/path integral
- No reason why OPE limit on worldsheet should correspond to OPE limit on celestial sphere

Same helicity

Only $j - j$ OPE contributes to this case:

$$U^{\mathsf{a}}_{+,\,\Delta_i}\, U^{\mathsf{b}}_{+,\,\Delta_j} \sim \int\limits_{\Sigma_i\times\Sigma_j\times(\mathbb{C}^*)^2} \frac{\mathrm{d}s_i\,\mathrm{d}s_j}{s_i^{\Delta_i}\, s_j^{\Delta_j}}\, \frac{\bar{\delta}^2(z_i - s_i\,\lambda(\sigma_i))}{[\mu(\sigma_i)\,\bar{z}_i]^{\Delta_i-1}} \times \frac{\bar{\delta}^2(z_j - s_j\,\lambda(\sigma_j))}{[\mu(\sigma_j)\,\bar{z}_j]^{\Delta_j-1}}\, \mathrm{d}\sigma_i\, \frac{f^{\mathsf{abc}}\, j^c(\sigma_j)}{\sigma_{ij}}$$

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Observe:

- Delta functions $\Rightarrow \sigma_{ij} \to 0$ corresponds to $z_{ij} \to 0$
- So OPE on worldsheet implies (holomorphic) OPE on celestial sphere

To isolate OPE, use

$$\bar{\delta}^2(z_i - s_i\,\lambda(\sigma_i)) := \frac{1}{(2\pi \mathrm{i})^2} \bigwedge_{\alpha=1,2} \bar{\partial}\left(\frac{1}{z_{i\,\alpha} - s_i\,\lambda_\alpha(\sigma_i)}\right)$$

to:

- perform s_i integral
- integrate-by-parts wrt σ_i
- perform σ_i integral against $\bar{\delta}(\sigma_{ij})$

Result

$$\frac{f^{\mathsf{abc}}}{z_{ij}} \int_{\Sigma_j \times \mathbb{C}^*} \frac{\mathrm{d}s_j}{s_j^{\Delta_j - 1}} \frac{\langle \xi\, \lambda(\sigma_j)\rangle^{\Delta_i}}{\langle \xi\, z_i\rangle^{\Delta_i}} \frac{j^{\mathsf{c}}(\sigma_j)}{[\mu(\sigma_j)\, \bar{z}_i]^{\Delta_i - 1}} \frac{\bar{\delta}^2(z_j - s_j\, \lambda(\sigma_j))}{[\mu(\sigma_j)\, \bar{z}_j]^{\Delta_j - 1}}$$

ξ^α (almost) arbitrary reference spinor – arises from s_i integral

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Mellin transform $[\mu(\sigma_j)\, \bar{z}_i]^{1-\Delta_i}$, $[\mu(\sigma_j)\, \bar{z}_j]^{1-\Delta_j}$ plus auspicious choice of $\xi \Rightarrow$

$$\boxed{\frac{f^{abc}}{z_{ij}} B(\Delta_i - 1, \Delta_j - 1)\, U^c_{+,\Delta_i + \Delta_j - 1}}$$

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Gives correct celestial OPE coefficient!

Mixed helicity

OPE structure much more complicated

$$U^{\mathrm{a}}_{+,\Delta_i}\, U^{\mathrm{b}}_{-,\Delta_j} \sim \int_{\Sigma_i\times\Sigma_j\times(\mathbb{C}^*)^2\times(\mathbb{R}_+)^2} \frac{\mathrm{d}t_i\,\mathrm{d}s_i\,\mathrm{d}t_j\,\mathrm{d}\tilde{s}_j}{t_i^{2-\Delta_i}\, s_i\, t_j^{2-\Delta_j}\, \tilde{s}_j}\, \mathrm{d}\sigma_i\, \frac{f^{\mathrm{abc}}\, j^{\mathrm{c}}(\sigma_j)}{\sigma_{ij}}$$

$$\times\, \bar{\delta}^2\left(z_i - s_i\,\lambda(\sigma_i) + \frac{t_j\,\tilde{s}_j\,s_i\,z_j}{\sigma_{ij}}\right)\, \bar{\delta}^2\left(\bar{z}_j - \tilde{s}_j\,\tilde{\lambda}(\sigma_j) + \frac{t_i\,s_i\,\tilde{s}_j\,\bar{z}_i}{\sigma_{ij}}\right)$$

$$\times\, \exp\left(\mathrm{i}\, t_i s_i\, [\mu(\sigma_i)\,\bar{z}_i] + \mathrm{i}\, t_j \tilde{s}_j\, \langle\tilde{\mu}(\sigma)\, z_j\rangle\right)$$

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$$\times \exp\left(\mathrm{i}\,t_i s_i\,[\mu(\sigma_i)\,\bar{z}_i] + \mathrm{i}\,t_j\tilde{s}_j\,\langle\tilde{\mu}(\sigma)\,z_j\rangle\right)$$

Looks like a mess...and is!

- try to perform s_i, σ_i integrals as before $\to 0$ or ∞
- because worldsheet OPE probes region where $s_i, \tilde{s}_j$ not good affine coordinates! (cf., [Ohmori])

Twistor transform

Solution: Change variables to different (legit) affine patch

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Two choices, implemented by twistor/dual twistor transform:

[Penrose-MacCallum, Mason-Skinner]

$$\mathbf{T}[f(Z)] := \int \mathrm{d}^4 Z \, \mathrm{e}^{-\mathrm{i}\, Z \cdot W} \, f(Z) \,,$$

$$\mathbf{T}^*[g(W)] := \int \mathrm{d}^4 W \, \mathrm{e}^{\mathrm{i} Z \cdot W} \, g(W)$$

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Can act directly on wavefunctions, e.g.:

$$\mathbf{T}[a(Z)] \in H^{0,1}(\mathbb{PT}^*, \mathcal{O}(-4)) \,, \quad \mathbf{T}^*[\tilde{a}(W)] \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-4))$$

Outline calculation

Perform the following operations to $U^{\mathrm{a}}_{+,\Delta_i}\, U^{\mathrm{b}}_{-,\Delta_j}$:

- Apply $\mathbf{T}$ to worldsheet OPE
- Perform σ_i and scale parameter integrals against delta functions
- Apply $\mathbf{T}^{-1}$

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Result:

$$\frac{f^{\mathrm{abc}}}{\bar{z}_{ij}}\, B(\Delta_i - 1, \Delta_j + 1)\, U^{\mathrm{c}}_{+,\Delta_i+\Delta_j-1}$$

with delta functions enforcing $\sigma_{ij} \to 0 \Leftrightarrow \bar{z}_{ij} \to 0$

Also need to include other patch, given by using $\mathbf{T}^*$

Total answer given by sum of both patches (i.e., covering relevant portion of moduli space):

$$\frac{f^{\mathrm{abc}}}{z_{ij}} B(\Delta_i + 1, \Delta_j - 1)\, U^{\mathrm{c}}_{-,\Delta_i+\Delta_j-1} + \frac{f^{\mathrm{abc}}}{\bar{z}_{ij}} B(\Delta_i - 1, \Delta_j + 1)\, U^{\mathrm{c}}_{+,\Delta_i+\Delta_j-1}$$

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Correct celestial OPE recovered again!

What else can we do?

Following similar steps:

- mixed in/out configurations
- generate correct graviton-graviton celestial OPE – more subtle story due to worldsheet SUSY
- generate correct gluon-graviton celestial OPE – surprising, since we don't know 4d ambitwistor string for EYM
- use instead momentum eigenstate vertex operators: worldsheet OPE $\rightarrow$ collinear splitting functions
- subleading contributions to worldsheet OPE $\leftrightarrow$ *some* conformal descendant contributions to celestial OPE (cf., [Banerjee-Ghosh-Gonzo, Banerjee-Ghosh-Paul, Ebert-Sharma-Wang,...])

Where to next?

Using worldsheet theory, can we make predictions that can't simply be deduced from symmetries or by Mellin transforming momentum-space results?

Can't get enough CCFT?

Celestial Sphere: holography, CFT and amplitudes
Higgs Centre for Theoretical Physics
13-15 September 2021

Register at: https://indico.ph.ed.ac.uk/event/84/