Soft modes at $\mathcal{I}$ and the BMS Algebra

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Partly based on recent work with Miguel Campiglia

arxiv : 2106.14717

Motivation

- Super-translations (ST) and Super-rotations (SR) are the most well understood asymptotic symmetries of the (perturbative) Gravitational S matrix. {*A. Strominger 2013, (T. He, S. Lysov, P. Mitra, A. Strominger, 2014), (D. Kapec, S. Lysov, S. Pasterski, A. Strominger 2014), ...*}

- It is by now well understood that super-translations are exact symmetries of the Quantum S matrix in 4 dimensions.
- In any given asymptotically flat space-time, super-translations are spontaneously broken and the associated Goldstone modes are the soft gravitons.

- One can also classify the vacuum states as eigen-states of super-translation charge and the conservation law between $\mathcal{I}^-$ and $\mathcal{I}^+$ mandates transition between different vacuua. This transition is quantified by Weinberg soft factor.

Motivation

- Situation with super rotations is far more intricate.

- Unlike ST, It has not been proved that SR is a symmetry of classical gravitational scattering. (*K. Prabhu, 2019)*

- ST charge at $\mathcal{I}^{\pm}$ is unique but there is a two parameter ambiguity in deriving SR charge from phase space methods. $\{$*G. Compere, A. Fiorucci, R. Ruzziconi, 2018) (A. Elhashash, D. Nichols, 2021)* $\}$

- **Q1** : Can we identify the corresponding Goldstone modes and classify the space of vacuua once SR are included? This is far more intricate as SR do not close among themselves and can not be analysed without including ST.

Motivation

- **Q2** : If we choose the simplest prescription for defining the SR charge, then in the momentum space basis at null infinity, the algebra generated by ST and SR charges does not close and in fact has a field dependent extension. This result is consistent with double soft graviton theorem. So is the symmetry anomalous? { *(G. Barnich 2014), (J. Distler, R. Flauger, B. Horn 2018) (Anupam A. H, A. Kundu, K. Ray, 2017)* }

- But recent works in celestial holography show that this extended BMS (EBMS) algebra closes upto a possible central extension. *{(S. Steiberger, T. Taylor, 2020), (S. Banerjee, S. Ghosh and P. Paul, 2020) }*
- In fact, S. Steiberger and T. Taylor used conformal double soft theorems to derive the closed EBMS algebra !

In this talk

- We will try to at least partially answer both (**Q1**, **Q2**) from the perspective of momentum space soft theorems and asymptotic quantisation at $\mathcal{I}^{\pm}$.

- For the analysis of these questions from Celestial perspective see **arxiv : 2108.11969 by L. Donnay and R. Ruzziconi**

- All our external states are gravitons. Inclusion of massive particles in this analysis is non-trivial and requires a nuanced understanding of the asymptotic Fock spaces in super-rotated backgrounds. (*Anupam A H, M. Campiglia, A. Kundu in progress*)
- In this talk, super-rotations are generated by holomorphic (or anti-holomorphic) vector fields (Two copies of the Witt algebra)

Radiative data at Null infinity

- Space of asymptotically flat geometries can be parametrized by modes of gravitational field at $\mathcal{I}^{\pm}$. In particular at $\mathcal{I}^{+}$ this data includes the shear field $C_{zz}(u, z, \bar{z}), C_{\bar{z}\bar{z}}(u, z, \bar{z})$ and metric on the sphere $q_{z\bar{z}}$.

- The S^2 metric information is (upto area preserving diffeomorphisms) in the conformal factor $q_{z\bar{z}} = e^{2\psi(z,\bar{z})}\gamma_{z\bar{z}}$

- Equivalently, it is in (trace-free part of) the Geroch Tensor $T_{zz}(z, \bar{z}) = 2(D_z\psi\, D_z\psi + D_z^2\,\psi)$
- So the complete information is in the pair $(C_{zz}, T_{zz}, C_{\bar{z}\bar{z}}, T_{\bar{z}\bar{z}})$.
- $(C_{zz}, C_{\bar{z}\bar{z}})$ **is the renormalized Shear field.**

Radiative data at Null Infinity

Phase space for Gravitational Scattering

$$\Gamma_0 = \{ T_{zz}, C_{zz}(u,\hat{x}), \ldots \}$$
$$C_{zz}(u,z,\bar{z}) = -2C^0_{zz}(z,\bar{z}) + \sigma_{zz}(u,z,\bar{z})$$
$$\sigma_{zz}(u,\hat{x}) \overset{u \to \pm\infty}{=} \sigma^{\pm}_{zz}(\hat{x}) + O(\frac{1}{|u|^{1+\epsilon}})$$

- $N_{zz}(u,z,\bar{z}) = -\partial_u \sigma_{zz}(u,z,\bar{z})$ is the News tensor.
- For a class of Asymptotically flat geometries in the "neighbourhood" of Minkowski space,

Christodolou, Klainermann spaces

$$C^0_{AB} = 2D_A D_B C$$
$$\sigma^+_{AB} - \sigma^-_{AB} =: -D_A D_B \mathcal{N}(\hat{x})$$

- As $\sigma^{+}_{zz} - \sigma^{-}_{zz} = \int du\, N_{zz}$, $\mathcal{N}$ is called the soft news mode.

- For generic gravitational scattering

$$\sigma_{AB}(u, \hat{x}) \overset{u \to \pm\infty}{=} \sigma^{\pm}_{AB} + \frac{1}{|u|} \tag{1}$$

(L. Blanchet, T. Damour), (B. Sahoo, A.Sen)

- But EBMS is well defined only with slightly stronger fall-offs $O(\frac{1}{|u|^{1+\epsilon}})$.

- This subtlety is intricately related to the 1-loop exact universal $\ln \omega$ soft theorem derived by B.Sahoo and A.Sen (**Miguel's talk**)

Phase space at $\mathcal{I}^+$

- Ashtekar and Strubel showed that in Γ_0, one has the following Poisson brackets.

$$\{\partial_u \sigma_{zz}(u,\hat{x}), \sigma_{\overline{w}\,\overline{w}}(u',\hat{y})\} \sim \delta(u-u')\,\delta^2(z,w)$$

- However the "corner modes" $C(z,\overline{z}), \mathcal{N}(z,\overline{z})$ and Geroch tensor T_{AB} were left out.

- He, Lysov, Mitra and Strominger (HLMS) showed how to include the boundary modes in the phase space analysis.

- Warning : T_{AB} **is not the finite energy Stress tensor !**

Extending the phase space to include corner modes

- The hint of how to include the corner modes comes by comparing the action of super-translations $f(z, \bar{z})$ on the radiative data from space-time and phase space perspective.

$$\delta_f \, \sigma_{AB} \; = \; f \, \partial_u \sigma_{AB}$$
$$\delta_f \, C \; = f \quad \delta_f \, T_{AB} = 0$$

$$Q_f \sim - \int_{S^2} f \, D^4 \mathcal{N} + \frac{1}{2} \int_{\mathcal{I}^+} f(z, \bar{z}) \, \partial_u \sigma_{AB} \partial_u \sigma^{AB}$$

- ST does not shift T_{AB}, and

- Looking at the Red terms, we expect $D^4 \mathcal{N}$ to be conjugate to C.

Phase space at Null infinity with $T_{AB} = 0$

- Consider "Auxiliary independent phase spaces"
  $\Gamma_{\rm hard} := \{\, \sigma_{AB}(u, z, \bar{z}) \,\}$ and $\Gamma_{\rm soft} := \{ C(z, \bar{z}), \mathcal{N}(z, \bar{z}) \,\}$

- σ_{zz} is conjugate to $\partial_u \sigma_{\bar{z}\bar{z}}$ and $D_z^2 C$ is conjugate to $D_z^2 \mathcal{N}$ on the celestial sphere.

- We now impose the constraints on $\Gamma_{\rm hard} \times \Gamma_{\rm soft}$

Phase space at Null infinity

HLMS constraints

$$\sigma^{+}_{zz} + \sigma^{-}_{zz} = 0$$
$$D^2_z \mathcal{N} = \sigma^{+}_{zz} - \sigma^{-}_{zz}$$

- These are second class constraints. HLMS showed that solving them leads us to a more refined definition of the phase space structure at null infinity which becomes crucial for analysing super-translation symmetries of the S-matrix.

$\Gamma_{\rm HLMS}$

$$\{\sigma_{zz}(u,z,\bar{z}),\sigma_{\overline{ww}}(u',w,\overline{w})\}_\star \sim \tfrac{1}{2}\theta(u-u')\,\delta^2(z,w)\gamma_{z\bar{z}}$$
$$\{D_z^2 C, D_{\overline{w}}^2\mathcal{N}\}_\star \sim \delta^2(z,w)$$
$$\{C(z,\bar{z}),N_{ww}\}_\star,\ \{\mathcal{N}(z,\bar{z}),N_{ww}\}_\star = 0$$

- We will refer to the extended phase space at $\mathcal{I}^+$ as $\Gamma^+_{\rm HLMS}$. (Similar construction at $\mathcal{I}^-$.)

- It has an additional conjugate pair of constant shear mode and the soft News $(C(z,\bar{z}),\mathcal{N}(z,\bar{z}))$ which commutes with $N_{AB}(u,z,\bar{z})$.

- We can now quantize $\Gamma^\pm_{\rm HLMS}$ and $Q_f^{\mathcal{I}^\pm}$.

ST Ward identity and the space of Vacuua

- As shown by HLMS, the Ward identity $\langle\Psi|[Q_f^{\mathcal{I}}, S]|\Phi\rangle = 0$ is equivalent to Weinberg soft theorem.

- One can e.g. diagonalise the soft News operators such that $\hat{\mathcal{N}}(z,\bar{z})|\Psi, N\rangle = N(z,\bar{z})|\Psi, N\rangle$

$$(N(z,\bar{z}) - N'(z,\bar{z}))\,\langle p_1,\ldots,p_n; N|S|p_{n+1},\ldots,p_m;\ N'\rangle = S^{(0)}_{\mathrm{Weinberg}}\,\langle\, p_1,\ldots,p_n; N|S|p_{n+1},\ldots,p_m;\ N'\rangle$$

- So one has an infinite dimensional space of vacuua parametrized by Eigen-states of the ST charge.

- How do SR symmetries act on them?

SR charges on Γ

From the space-time action we know that for a holomorphic $Y = Y^z \partial_z$,

$$\begin{aligned} \delta_Y \sigma_{zz} &= L_Y \sigma_{zz} - \frac{1}{2}(D \cdot Y) u \partial_u \sigma_{zz} \\ \delta_Y T_{zz} &= D_z^3 Y^z \\ \delta_Y C &= L_Y C(z, \bar{z}) - \frac{1}{2}(D \cdot Y) C(z, \bar{z}) \end{aligned}$$

- The Covariant Phase Space analysis can be used to derive SR charges at $\mathcal{I}^+$. **Flanagan, Nichols**

SR charge

$$Q_Y = \int_{\mathcal{I}^+} \left[N^{zz} \delta_Y \sigma_{zz} + u N^{zz} \delta_Y T_{zz} \right] + c.c. + \ldots$$

SR charge and Ward identity

- . . . in the above equation indicate that this definition is ambiguous upto addition of corner terms at boundaries of $\mathcal{I}^+$.
- But we can ask, what is the consequence of using the “bulk term” Q_Y as a symmetry generator for the (tree-level) S matrix.
- It was shown by He, Lysov, Pasterski and Strominger in 2014 that $[Q_Y, S] = 0$ is equivalent to the sub-leading soft graviton theorem.
- However a complete understanding of SR symmetries in gravitational scattering is lacking. **Hawking, Perry, Strominger, 2017**

Subtleties with Super Rotations

- Although $\delta_Y T_{zz} = D_z^3 Y^z$, it is not clear how to derive this result using the commutation relations on phase space Γ_{HLMS}. (Naive analysis leads to $\{\, Q_Y, T_{zz}(z, \bar{z})\} \,=\, 0$).

- Hence we do not have a faithful representation of SR on Γ_{HLMS}.

- Quatization of Q_f leads to eigen-states of soft news $|N\rangle$. But action of $\hat{Q}_Y$ on these states is not clear.

- In fact we do not even have a closed algebra !

2 cocycle in EBMS algebra

- In a beautiful paper, Flauger, Distler and Horn used consecutive double soft theorem to show that the algebra generated by Q_f, Q_Y does not close and has a 2 co-cycle anomaly.

$$\mathcal{M}_{n+2}(p_1, \ldots, p_n, [k_1^s, k_2^{sl}]) = S([k_1^s, k_2^{sl}])\, \mathcal{M}_n$$

$$\Downarrow$$

$$[Q_f, Q_Y] = Q_{L_Y f - \frac{1}{2}(D\cdot Y) f} + K(f, V)$$

$$K(f, V) = -2 \int_{S^2} f(L_Y T_{zz} - D_z^3 Y^z) D_{\bar{z}}^2 \mathcal{N} + c.c.$$

2 cocycle in EBMS algebra

- In classical theory the 2 co-cycle was already derived by G. Barnich in 2017.

- But in the case of symmetry algebra for the gravitational scattering, presence of 2 co-cycle is in tension with the conservation laws thanks to Jacobi .

$$[[Q_f, Q_Y], S] - [[Q_f, S], Q_Y] + [[Q_Y, S], Q_f] = 0$$

- Thus quantized SR charge Q_Y (ambiguous due to possibility of corner terms) is infected with three issues. (1) Existence of the 2-cocycle in the algebra, (2) T_{zz} is a c-number in the phase space and so commutes with Q_Y and (3) the action on soft news eigenstates is not defined.

How to fix the corner terms

- In 2017, S. Pasterski, A. Strominger and S. Zhiboedov proposed a specific prescription for the corner terms in Q_Y in the classical theory such that the associated conservation law (between $\mathcal{I}^+$ and $\mathcal{I}^-$) for the new SR charge Q_Y^{new} turned into the prediction of Spin memory effect.

- These charges were investigated in more detail by $\{$*(E. Flanagan, D. Nichols 2014), (G. Compere, A. Fiorucci, R. Ruzziconi, 2018), (M. Campiglia, J. Perazza, 2020)* $\ldots\}$ and the latter work showed that Q_Y^{new} and Q_f form a closed *classical* classical algebra with no 2 co-cycle or central extension!

- So we will use this definition of SR charge and see where it leads us in quantum theory.

New charge and super-rotation Modes

$$Q_Y^{\text{new}} = \int_{S^2} \left(-\mathcal{D}^4 C \delta_Y \mathcal{N} + \int_{-\infty}^{\infty} \mathbf{duuN^{zz}} \delta_Y T_{zz}\right) + \int_{\mathcal{I}^+} N^{zz}\, \delta_Y\, \sigma_{zz} + \text{c.c.}$$

where

$$\mathcal{D}^4 C \sim (D^4 - 2D_z^2 T^{zz} + T_{zz} T^{zz}) C$$

- A mode corresponding to $\int_{-\infty}^{\infty} u\, N_{AB}$ should be conjugate to T_{AB}.
- In energy space $\int_{-\infty}^{\infty} uN_{AB} = \lim_{\omega \to 0} \partial_\omega \omega N_{AB}(\omega)$ and hence corresponds to sub-leading soft graviton insertion in the Fock space.

New Charge and super-rotation modes

- What is the extension of $\Gamma^{+}_{\mathrm{HLMS}}$ on which Q_Y acts faithfully ? We follow precisely the same strategy as HLMS.
- Introduce a new conjugate pair $(T_{AB}, \mathcal{N}^1_{AB})$ and start with $\mathcal{D}^4\mathcal{N}$ (instead of $D^4\mathcal{N}$) conjugate to C.
- Let us now consider an auxiliary phase space, $\Gamma = \Gamma_{\mathrm{soft}} \times \Gamma_{hard}$ where Γ_{soft} is enhanced by addition of T_{AB} and it's conjugate. These modes are not independent of σ_{AB} and hence we need to impose constraints.

Phase space with super-rotation modes

The constraints

$$\sigma^{+}_{zz} + \sigma^{-}_{zz} = 0$$
$$\mathcal{D}^{2}_{z}\mathcal{N} = \sigma^{+}_{zz} - \sigma^{-}_{zz}$$
$$D_{\bar{z}} T_{zz} = 0$$
$$\mathcal{N}^{1}_{AB} = \int du\, u\, N_{AB}$$

- We will refer to $\mathcal{N}^{1}_{AB}$ as the sub-leading soft News operator.
- The constraints are second class and solving them leads to the *Physical* phase space of the theory which has super-rotation as well as super-translation soft modes.

Symplectic structure of the Radiative data

- In addition to the commutation relations in Γ_{HLMS}, we have new brackets/commutators.

- Let $\hat{O}_w = \partial_w \partial_{\overline{w}}^{-2} \partial_w$ be a holomorphic projection.

$$\{\mathcal{N}^1_{zz}, N_{\overline{w}\,\overline{w}}(u)\}_\star = -\hat{O}_{\overline{z}} \delta^2(z-w)$$

$$\{\mathcal{N}^1_{zz}, T_{\overline{w}\,\overline{w}}\}_\star = -2\,(1-\hat{O}_{\overline{z}})\,\mathcal{D}^2_z\,\delta^2(z-w)$$

$$\{\mathcal{N}^1_{zz}, C(w,\overline{w})\} \neq 0$$

$$\{\mathcal{N}^1_{zz}, \mathcal{N}(w,\overline{w})\}_\star = 0$$

Some Consequences

- Essentially, $\mathcal{N}(z, \bar{z})$ is conjugate to $C(z, \bar{z})$ and $\mathcal{N}^1_{zz}$ is conjugate to T_{ww}.
- But the hard and soft sectors do not quite decouple as $\mathcal{N}^1_{zz}$ does not commute with finite energy News tensor $N_{AB}(u, z, \bar{z})$.
- We can now quantize the physical phase space, Q_V^{new} and analyse the Ward identities.
- An additional subtlety : $\mathcal{N}^1_{zz}$ has a non-trivial action on hard states.

Main Results

- All our results are rigorously derived only for holomorphic super-rotations which have at most one pole on the celestial sphere. For more general meromorphic vector fields, detailed analysis remains to be done.

- Maximal set of commuting operators in the soft sector are $\mathcal{N}(z, \bar{z})$ and T_{AB} and so the space of vacuua is enhanced.

State Space

$$\hat{T}_{zz} \,|\, (p_1, \ldots, p_n) \mathcal{N}, T \,\rangle \;=\; T_{zz} \,|\, (p_1, \ldots, p_n) \mathcal{N}, T \,\rangle$$

- On these class of states (with Vacuum parametrized by soft News and Geroch Tensor) we can evaluate the Ward identity of Q_Y^{new}.

Main Results

$$\langle\, \mathrm{out}\, \mathcal{N},\, T = 0 \,|\, [Q_Y^{\mathrm{new}}, S] \,|\, \mathrm{in}, \mathcal{N}'\; T = 0 \,\rangle = 0$$

$$\Downarrow$$

$$\langle\, \mathrm{out}\, \mathcal{N}, \delta_Y T \,|\, S |\, \mathrm{in}, \mathcal{N}'\; T = 0 \,\rangle =$$
$$S^{(1)} \langle\, \mathrm{out}\, \mathcal{N},\, T = 0 \,|\, S |\, \mathrm{in}, \mathcal{N}'\; T = 0 \,\rangle$$

- The "super-rotation modes" are shifted under the action of Q_Y^{new} and correspond to insertion of a sub-leading soft graviton.

And finally

$$[\, \hat{Q}_V^{\mathrm{new}}, \hat{Q}_f \,] \;=\; \hat{Q}_{L_V f - \frac{1}{2}(D\cdot V) f}$$

Outlook

- SR symmetry leads to a finer classification of vacuua parametrized by the Geroch Tensor T_{AB}.
- This result compliments a beautiful analysis by K. Nguyen and J. Salzer on effective action of super-rotation modes.
- The constraints crucial to derive the physical phase space are far more involved if we consider the generalised BMS symmetries where super-rotations are smooth diffeomorphisms. **However, intricate relation between the two descriptions in the conformal primary basis should shed light on soft modes conjugate to generalised BMS symmetry.** *L. Donnay, S. Pasterski and A. Puhm, (2020)*

Outlook

- Can one prove the conservation law for Q_Y^{new} classically?
- In QED, the dressing of asymptotic states lead to the modification of asymptotic charges and the Ward identities are consistent with the infra-red finite, universal and (one-loop exact) Sahoo-Sen soft theorem. What is the fate of super-rotations? **T. He, D. Kapec, A. Raclariu and A. Strominger (2017)** and **S. Bhatkar (2020)**

Thank you for listening !