Invariants of multi-Higgs doublet models

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Overview

Introduction

Model building — we look for signals and constraints.

- But the theory has too many parameters.
- More particles are great, but more spurious parameters are not.
- How to distinguish models?

Can we find the relevant physical parameters that measure a given property?

Physical parameters (invariants) hold most of the physical information!

Invariant theory

What is it?

Invariant theory was developed by many unknown-to-physics mathematicians (Hilbert, Noether, Klein, Cayley, Weyl).

Invariant theory

It concerns the systematic categorization of invariants under the action of groups.

Invariant theory in physics

It concerns what we must call physical quantities!

How does it work?

The road to all physical parameters

$V \xrightarrow{\text{non-linear}} \mathbb{C}[V] \xrightarrow{G} \mathbb{C}[V]^G \rightarrow$ Physical parameters

1. We start with a theory with certain parameters (V).
2. Observables will be polynomials of parameters ($\mathbb{C}[V]$).
3. A group action enforces some polynomials to be invariants ($\mathbb{C}[V]^G$).
4. These polynomials of invariants are in fact physical parameters.

The dimension of $\mathbb{C}[V]^G$ is the number of invariants needed to define the whole theory! It is a generating set.

How does it work, really?

Aside all complications, the method is straightforward:

From $V_{\text{Scalar}} = {\Phi^{\dagger}}^{i} Y_{i}^{\ j} \Phi_{j} + {\Phi^{\dagger}}^{i} {\Phi^{\dagger}}^{j} Z_{ij}^{\ \ kl} \Phi_{k} \Phi_{l}$

1. Find how the tensors Y and Z transform under a reparametrization group (basis).
2. Break them into irreducible representations (building blocks).
3. From it, compute the Hilbert series, as it gives us the generating set of invariants.
4. Use the building blocks to build the generating set.

Now all of the physical information is contained in the generating set.

But why?

Apparent features of theories may be present in the Lagrangian but not in the physical parameters — Ivanov and Silva on the 3HDM with CP4 [1].

We may write hundreds of CP conservation conditions without attaining sufficiency. We do not know the generating set.

With invariant theory we know the degree of certain invariants. A specific physical parameter may be a polynomial of order N, where N is large.

There may be other applications.

Great, but is it possible to do it?

Sometimes...

The Hilbert series

The Hilbert series tells us how many and of what degree are the invariants and generating set. It can be calculated from a complex integration.

Hilbert series

$$H(t) = \oint_{|z_1|=1} \cdots \oint_{|z_i|=1} d\mu_G \, \mathrm{PE}[t, z, \mathsf{r}_1 \oplus \cdots \oplus \mathsf{r}_j] \tag{1}$$

Plethystic exponential

$$\mathrm{PE}[t, z, \mathsf{r}_1 \oplus \cdots \oplus \mathsf{r}_j] = \exp\left(\sum_{k\geq 1} \frac{t^k}{k} \chi_{\mathsf{r}_1 \oplus \cdots \oplus \mathsf{r}_j}(z_1^k, \cdots, z_i^k) \right) \tag{2}$$

Invariant theory — 2HDM

The Hilbert series — 2HDM

Y and Z break into $1 \oplus 3$ and $1 \oplus 1 \oplus 3 \oplus 5$, respectively.

Hilbert series of the 2HDM

$$H(t) = \oint_{z=1} \left(dz \, \frac{1 - z^2}{z} \right) \frac{z^{10}}{(1-t)^6 \left(z^2 - t\right)^3 \left(1 - tz^2\right)^3 \left(z^4 - t\right)\left(1 - tz^4\right)} \tag{3}$$

Traces

$$\chi_1(z) = 1\,, \quad \chi_3(z) = \frac{1}{z^2} + 1 + z^2\,, \quad \chi_5(z) = \frac{1}{z^4} + \frac{1}{z^2} + 1 + z^2 + z^4 \tag{4}$$

The Hilbert series — 2HDM

Hilbert series of the 2HDM

$$H(t) = \frac{1 + t^3 + 4t^4 + 2t^5 + 4t^6 + t^7 + t^{10}}{(1-t)^3(1-t^2)^4(1-t^3)^3(1-t^4)} \tag{5}$$

The denominator tell us that the generating set is:

- Three linear invariants (that can always be factorized).
- Four degree 2 invariants.
- Three degree 3 invariants.
- One degree 4 invariant.

In the 2HDM

1. Y and Z break into $1 \oplus 3$ and $1 \oplus 1 \oplus 3 \oplus 5$, respectively. This can be attained with projection operators.
2. The Hilbert series has 11 invariants — generating set.
3. Now we can build them by making tensor products and finding the singlet.

This method was followed by Trautner [2]. With it, he showed four necessary and sufficient conditions for CP conservation.

Although the conditions were known, this is not true for an arbitrary NHDM.

Invariant theory — NHDM

Group structure of the NHDM

The Y and Z tensors always break into [4] (Bento, 2021):

N	Y	Z	No. parameters
2	$1 \oplus 3$	$2(1) \oplus 3 \oplus 5$	14
3	$1 \oplus 8$	$2(1) \oplus 2(8) \oplus 27$	54
4	$1 \oplus 15$	$2(1) \oplus 2(15) \oplus 20 \oplus 84$	152
5	$1 \oplus 24$	$2(1) \oplus 2(24) \oplus 75 \oplus 200$	350
6	$1 \oplus 35$	$2(1) \oplus 2(35) \oplus 189 \oplus 405$	702
7	$1 \oplus 48$	$2(1) \oplus 2(48) \oplus 392 \oplus 735$	1274
...	...	...	...
N	$1 \oplus (\mathrm{N}^2 - 1)$	$2(1) \oplus 2\left(\mathrm{N}^2 - 1\right) \oplus \mathrm{a}_N \oplus \mathrm{b}_N$	$\frac{N^2(N^2+3)}{2}$

Table: Representation decomposition of the NHDM where a_N and b_N are given by $a_N = \frac{N^2(N+1)(N-3)}{4}$ and $b_N = \frac{N^2(N-1)(N+3)}{4}$.

Hilbert series

Contrary to the case of the 2HDM, in the 3HDM the Hilbert series is a double contour integration. These are known to be specially difficult.

How difficult? It can produce thousands of terms and in every each one of them we need to worry about branch cuts.

The solution? Number partitions and some weird algebra.

Guoce Xin's method

Let I be multiple contour integral. Then it can be solved by successive partial fraction expansions. It works efficiently in Maple.

Hilbert series

The result however is horrendous. The denominator is

$$\left(1-t\right)^3\left(1-t^2\right)^7\left(1-t^3\right)^8\left(1-t^4\right)^6\times$$
$$\left(1-t^5\right)^9\left(1-t^6\right)^3\left(1-t^7\right)^5\left(1-t^9\right)\left(1-t^{12}\right)^4 \tag{6}$$

- The numerator involves coefficients with 18 digits.
- Nevertheless, now we know that there are 46 invariants (physical parameters) in the theory, with specified degrees.

Properties — No. of invariants

We don't need to compute the Hilbert series to know how many invariants we need.

Number of physical parameters

The number of invariants is given by $N_p = N - \dim G + \dim G_v$, where $\dim G_v$ is almost always zero.

Number of physical parameters in NHDM

The number of invariants in NHDM is given by $N_p = (N^4 + N^2 + 2)/2$.

Conclusions

There are conclusions

3HDM — physical parameters

There are 46 physical parameters in the 3HDM, and we can build a generating set.

Minimal number of physical parameters

Because the Lagrangian must decompose under $\mathrm{SU}(N)$ with guaranteed 3 singlets, no symmetry can get rid of these 3 physical parameters.

Questions?

References

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