Conformal Bootstrap Review Talk

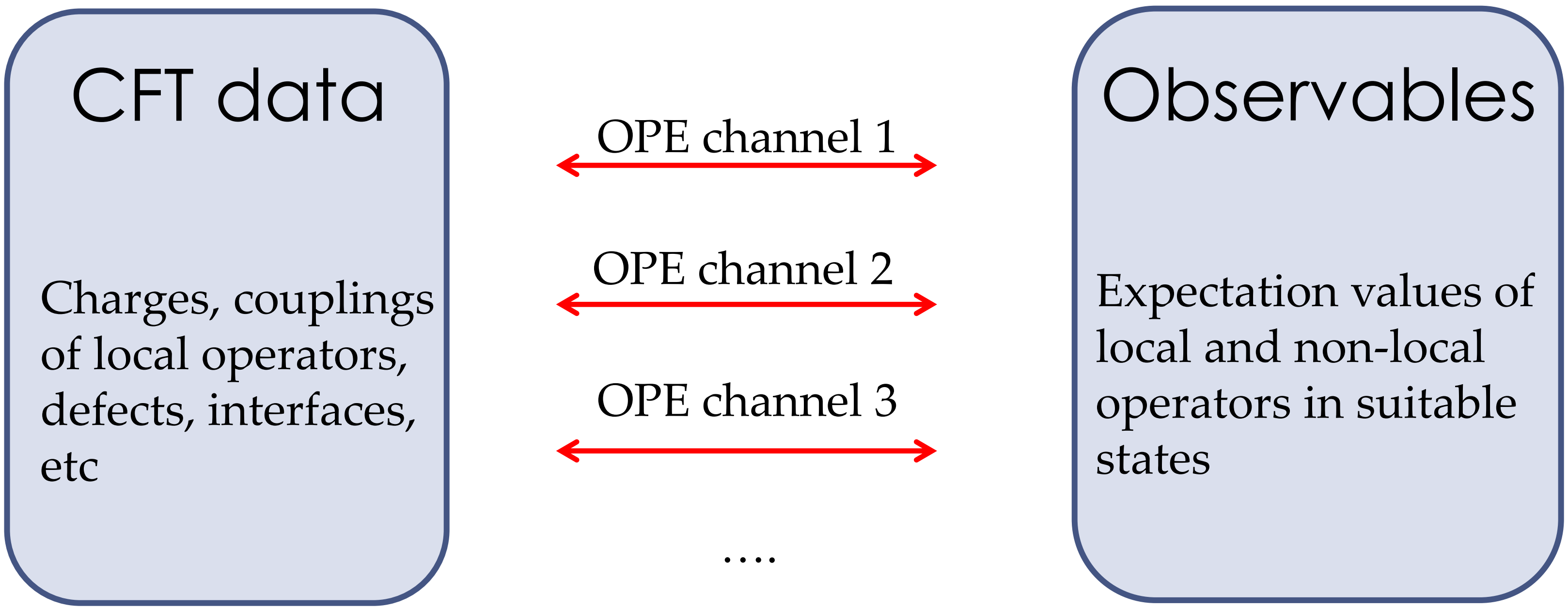
Miguel F. Paulos

Workshop on Celestial Amplitudes and Flat Space holography 2021

Conformal Bootstrap

A set of ideas, methods and tools to constrain and determine
CFT *data* and *observables*.

Bootstrap philosophy



Constraints arise from equivalency of different OPE channel decompositions, formulated as *crossing equations*

Bootstrap philosophy

- The Conformal Bootstrap involves the study of both kinematics and dynamics. Or, the dialog of the two.

Kinematics

Defining observables, formulating crossing equations, unitarity constraints, computing conformal blocks

Dynamics

Extracting information and constraints, or even solving for the CFT data using crossing equations and kinematic restrictions.

One informs the other!

Bootstrap philosophy

- Most mileage has been obtained by studying specific observables: four-point vacuum correlators of local operators. Focus on these. E.g.

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{G}(z,\bar{z})}{x_{13}^{2\Delta_\phi} x_{24}^{2\Delta_\phi}}$$

- In this case unitarity and the OPE allows us to formulate constraints involving *positive* quantities. Positivity is crucial in the current formulation of the conformal bootstrap and greatly reduces its (non-perturbative) scope.

- Some formulations of the bootstrap allow us to do away with positivity, but they are underdeveloped, comment at the end. Essential for studying non-unitary theories, defects, ….

- Almost no work has been done understanding higher point correlators (a mistake?).

Dynamics from kinematics

Basic kinematics

- Local operators characterized by quantum numbers in the product of (super)conformal group with global symmetry. Two point functions uniquely defined.

- Three point functions fixed by kinematics up to a *finite* set of couplings: OPE coefficients.

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_{\mu_1 \ldots \mu_\ell} \rangle \supset \lambda_{12\mathcal{O}_{\Delta,\ell}}$$

$$\langle T_{\mu_1\nu_1} T_{\mu_2\nu_2} T_{\mu_3\nu_3} \rangle \supset c, a, t_4$$

- Four point correlators can be written as finite set of functions of cross-ratios

$$\langle \mathcal{O}_{\mu_1} \mathcal{O}_{\mu_2} \mathcal{O}_{\mu_3} \mathcal{O}_{\mu_4} \rangle = \sum_a \mathcal{T}^a_{\mu_1\mu_2\mu_3\mu_4} \mathcal{G}^a(z, \bar{z})$$

Conformal blocks

- Conformal correlators admit expansions in terms of conformal blocks, which capture contributions of primaries and their descendants in the OPE

$$\mathcal{G}^a(z,\bar{z}) = \sum_{\mathcal{O}\in\mathcal{O}_1\times\mathcal{O}_2} \sum_{m,n} \lambda^{(m)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}} \lambda^{(n)}_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}} \underbrace{G^{a;m,n}_{1234;\mathcal{O}}(z,\bar{z})}_{\text{Conformal block}}$$

- The (efficient) computation of conformal blocks is a non-trivial problem, especially for generic spin representations or supersymmetry.

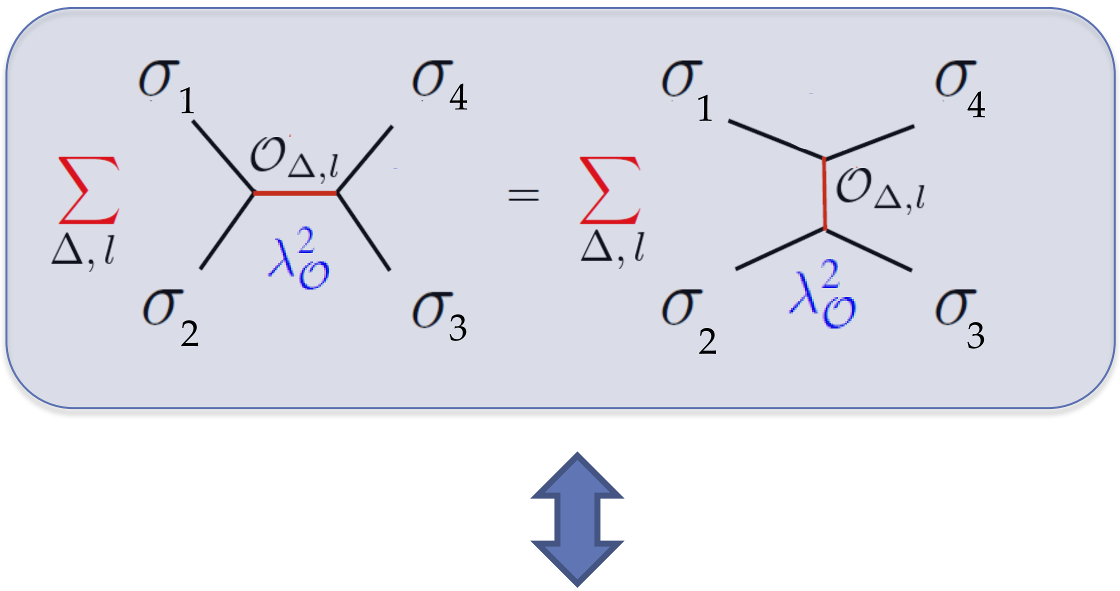
- In 3d problem is completely solved, with efficient computational package available. In 4d ``seed'' blocks are also known (these allow us to obtain all other ones).

- In general spacetime dimension problem is *in principle* solved in terms of weight-shifting operators, though in practice significant amounts of work are necessary on a case-by-case basis.

Castedo, Cuomo, Dolan, Echeverri, Ekhidir, Erramilli, Iliesu, Landry, Karateev,Kravchuk, Osborn, Poland, Rychkov, Serone, Simmons-Duffin,…

Crossing equations

- Crossing symmetry combined with OPE:



$$\sum_{\ell=0,2,\dots}\ \sum_{\Delta\geq\Delta_u(\ell)} \lambda^2_{\Delta,\ell} F_{\Delta,\ell}(z,\bar z) = -F_{0,0}$$

$$F_{\Delta,\ell}(z,\bar z) = \frac{G_{\Delta,\ell}(z,\bar z)}{(z\bar z)^{\Delta_\phi}} - \frac{G_{\Delta,\ell}(1-z,1-\bar z)}{[(1-z)(1-\bar z)]^{\Delta_\phi}}$$

Crossing equations

- For a collection of correlators, each crossing equation is a *quadratic* constraint on the OPE coefficients. For scalar correlators have:

$$\boldsymbol{\lambda}_0 \cdot \boldsymbol{\mathcal{F}}^p_{0,0}(z,\bar{z}) \cdot \boldsymbol{\lambda}_0 + \sum_{\Delta,\ell\in\mathcal{S}} \boldsymbol{\lambda}_{\Delta,\ell} \cdot \boldsymbol{\mathcal{F}}^p_{\Delta,\ell}(z) \cdot \boldsymbol{\lambda}_{\Delta,\ell} = 0\,, \qquad p = 1,\ldots,P$$

Matrices involving conformal blocks

$$\boldsymbol{\lambda}_{\Delta,\ell} = \left(\lambda_{\phi_1\phi_1\mathcal{O}_{\Delta,\ell}}\,, \lambda_{\phi_1\phi_2\mathcal{O}_{\Delta,\ell}}\,, \ldots, \lambda_{\phi_k\phi_k\mathcal{O}_{\Delta,\ell}}\right)$$

Extracting information

$$F_{0,0}(z,\bar{z}) + \sum_{\Delta,\ell \in S} \lambda^2_{\Delta,\ell} F_{\Delta,\ell}(z,\bar{z}) \stackrel{?}{=} 0$$

- Make assumptions on possible intermediate states and try to get a contradiction.
- Such contradictions are possible thanks to positivity of coefficients.
- They are made explicit by the construction of *linear functionals* with good positivity properties.

Feasibility problem

- Does there exist a linear functional ω satisfying:

$$\omega\left[F_{\Delta,\ell}\right] \geq 0\,, \qquad \text{for all} \quad (\Delta,\ell) \in \mathcal{S}$$
$$\omega\left[F_{0,0}\right] > 0$$

If there is, then no solution to the crossing equation exists with quantum numbers in *S:*

$$\omega\left[F_{0,0} + \sum_{\Delta,\ell\in\mathcal{S}} \lambda^2_{\Delta,\ell} F_{\Delta,\ell}\right] = \omega\left[F_{0,0}\right] + \sum_{\Delta,\ell\in\mathcal{S}} \lambda^2_{\Delta,\ell}\omega\left[F_{\Delta,\ell}\right] > 0$$

Rattazzi, Rychkov, Tonni, Vichi

Feasibility problem

- Do there exist linear functionals ω^p satisfying:

$$\sum_{p=1}^{P} \omega^p \left[\boldsymbol{\mathcal{F}}^p_{\Delta,\ell} \right] \succeq 0 \,, \qquad \text{for all} \quad (\Delta, \ell) \in \mathcal{S}$$

$$\sum_{p=1}^{P} \omega^p [\boldsymbol{\lambda}_0 \cdot \boldsymbol{\mathcal{F}}^p_{0,0} \cdot \boldsymbol{\lambda}_0] > 0$$

If there are, then no solution to the crossing equations exist with quantum numbers in S

$$\sum_{p=1}^{P} \omega^p \left[\boldsymbol{\lambda}_0 \cdot \boldsymbol{\mathcal{F}}^p_0(z) \cdot \boldsymbol{\lambda}_0 + \sum_{\Delta,\ell \in \mathcal{S}} \boldsymbol{\lambda}_{\Delta,\ell} \cdot \boldsymbol{\mathcal{F}}^p_{\Delta,\ell}(z) \cdot \boldsymbol{\lambda}_{\Delta,\ell} \right]$$

$$= \boldsymbol{\lambda}_0 \cdot \left(\sum_{p=1}^{P} \omega^p [\boldsymbol{\mathcal{F}}^p_{0,0}] \right) \cdot \boldsymbol{\lambda}_0 + \sum_{\Delta,\ell \in S} \boldsymbol{\lambda}_{\Delta,\ell} \cdot \left(\sum_{p=1}^{P} \omega^p [\boldsymbol{\mathcal{F}}^p_{\Delta,\ell}] \right) \cdot \boldsymbol{\lambda}_{\Delta,\ell} > 0$$

Feasibility problem

- The general feasibility problem is a *semidefinite program*. An optimization problem with positive semidefinite constraints.

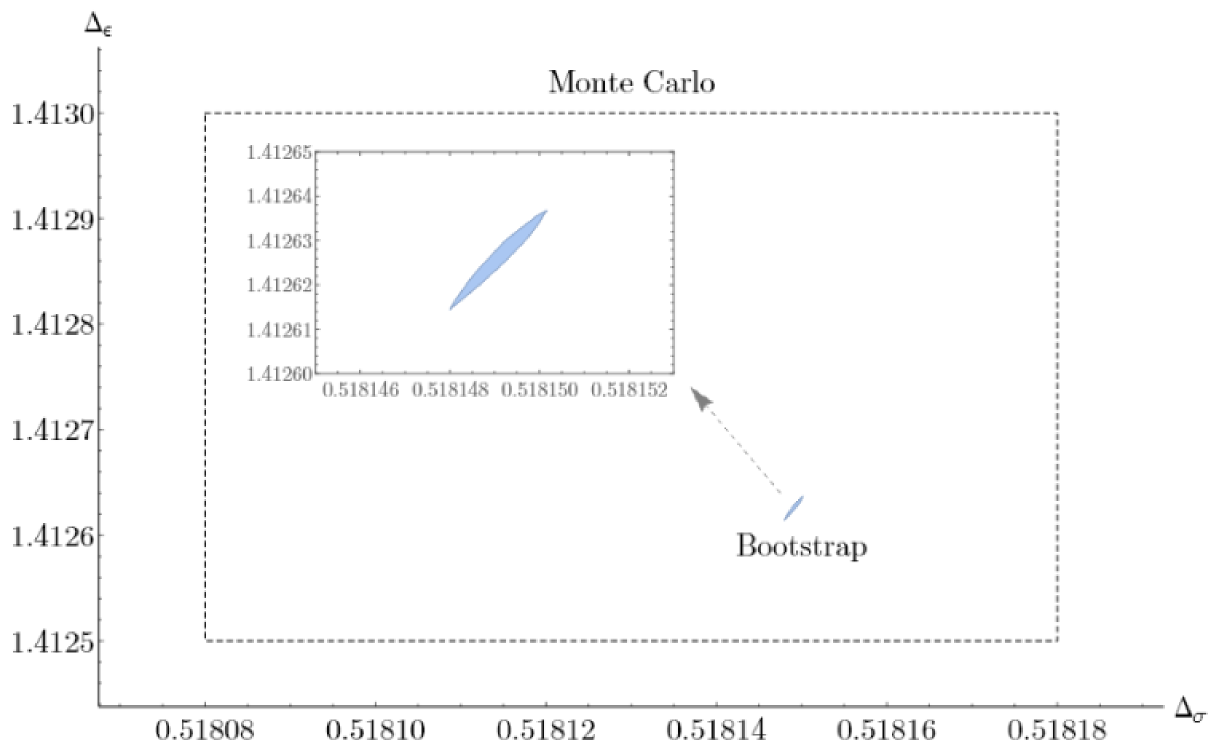
- Solving such problems analytically is *hard*. Standard algorithms exist for solving such problems numerically. They require however searching for functionals in a *finite dimensional* space.

- This amounts to restricting attention to a truncated set of crossing constraints: e.g. Taylor expanding in the cross-ratios to a finite order.

- The state-of-the-art solver of such problems adapted to the bootstrap is SDPB, with a small ecosystem of packages built around it.

Landry, Simmons-Duffin,…
https://gitlab.com/bootstrapcollaboration/

Example: 3d Ising



Includes constraints from correlators of spin and energy fields.

El-Showk, MP, Kos, Poland, Rychkov, Simmons-Duffin, Vichi

Example: 3d Ising

- Old results by now, could be substantially improved:
  - Higher order truncations
  - More correlators

- By sitting at the edge of the allowed region, or by optimizing some quantity inside the island, can obtain an extremal spectrum for the 3d Ising CFT (*extremal functional method)*.

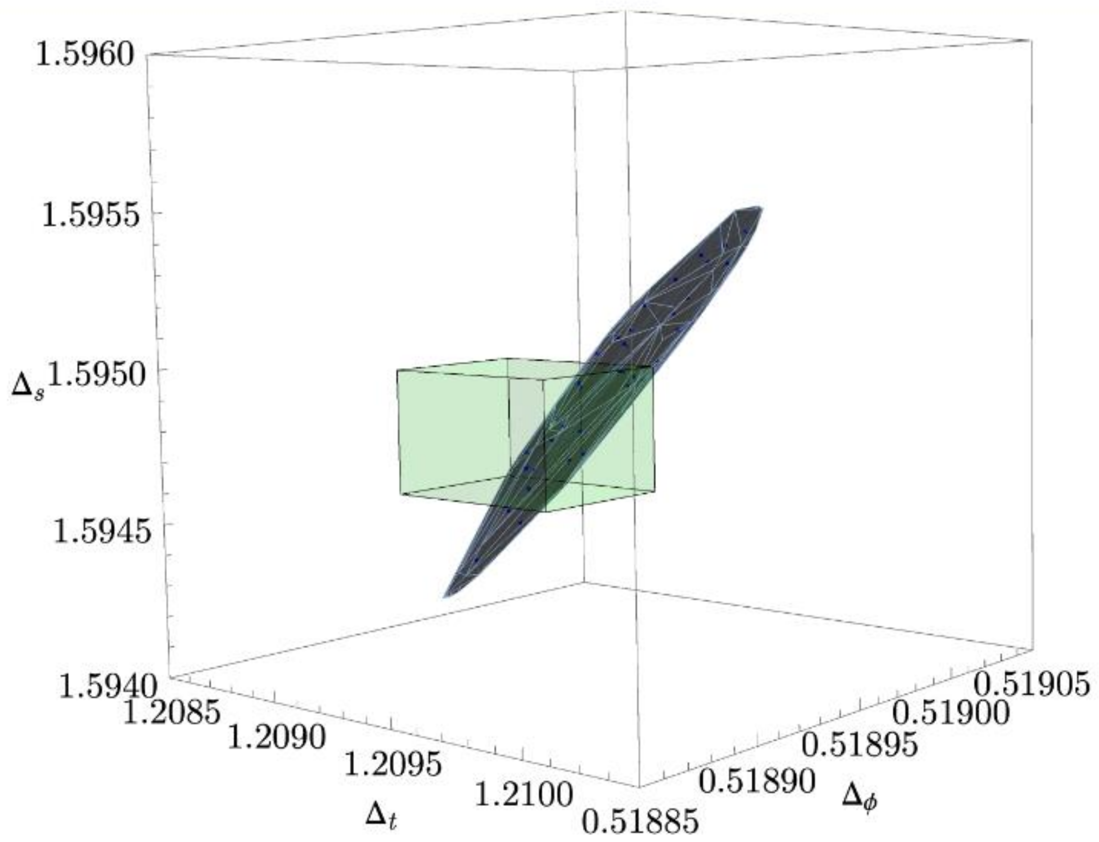
El-Showk, MP

- It is unkown whether the island ever shrinks to a point given a finite number of correlators, but probably not if 3d Ising is ``chaotic'' CFT.

- We expect this spectrum to systematically give better and better approximations to the true Ising spectrum as we include constraints from more and more correlators (which also makes the island shrink).

- It seems more progress can be made by adding constraints from more correlators rather than pushing to higher order truncations. But this adds other difficulties…

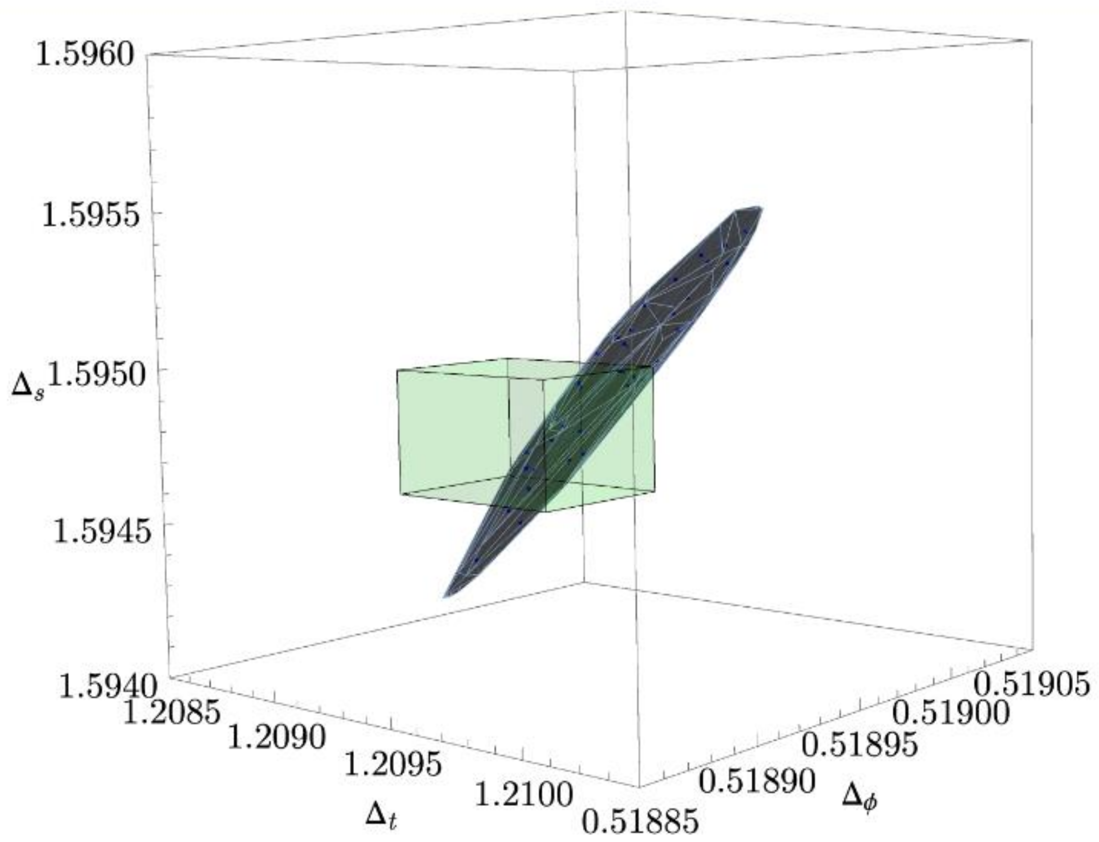
Example: O(3) model



$$\phi_i, \quad s = \phi_i \phi^i, \quad t = \phi_{\{i}\phi_{j\}}$$

Chester, Landry, Liu, Poland, Simmons-Duffin, Vichi, Su

Example: O(3) model



Inside allowed region establish the leading rank-4 scalar must be *relevant*:

$$\Delta_{t_4} < 2.99056$$

Thus O(3) model is unstable to cubic anisotropy!

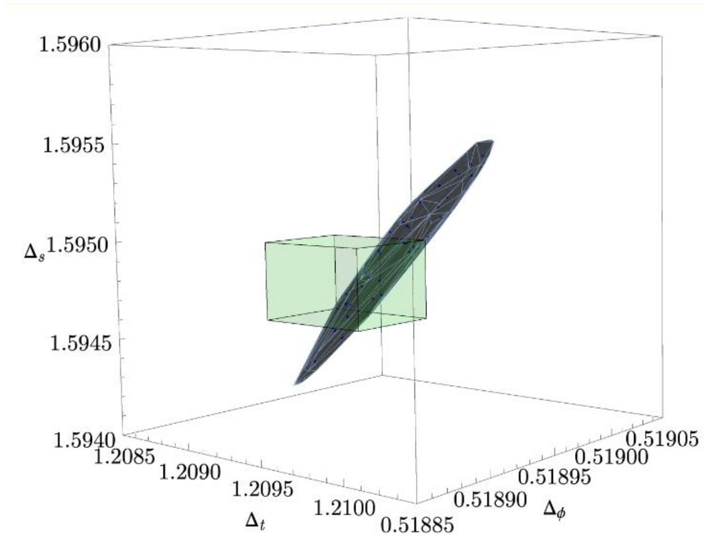
$$\phi_i, \quad s = \phi_i \phi^i , \quad t = \phi_{\{i}\phi_{j\}}$$

Chester, Landry, Liu, Poland, Simmons-Duffin, Vichi, Su

Example: O(3) model

- Tight bounds require natural assumptions, which are however hard to impose numerically.

- For instance, in general there could be multiple copies of the same operator (i.e. same quantum numbers) but differing couplings to external states. Demanding this does not occur requires imposing single copy and *scanning* over orientations in OPE space.



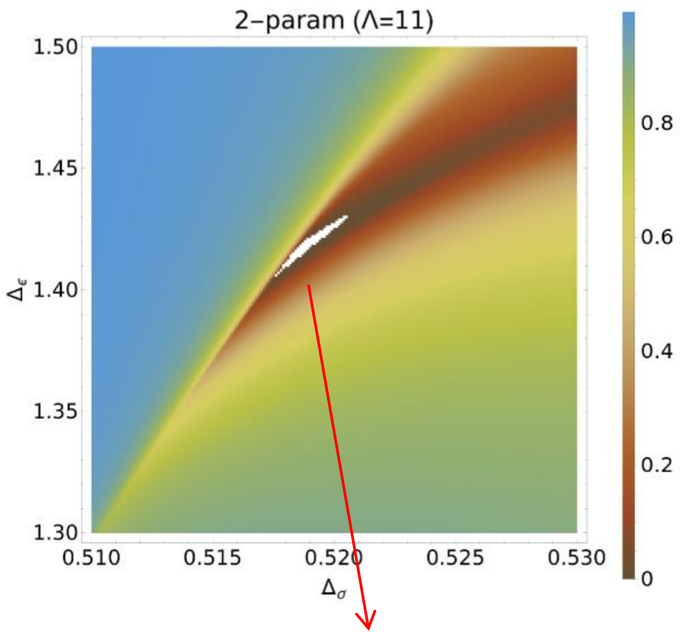
- In the present case we have to scan over

$$\{\Delta_\phi, \Delta_s, \Delta_t\} \qquad \left\{\frac{\lambda_{sss}}{\lambda_{\phi\phi s}}, \frac{\lambda_{tts}}{\lambda_{\phi\phi s}}, \frac{\lambda_{\phi\phi t}}{\lambda_{\phi\phi s}}, \frac{\lambda_{ttt}}{\lambda_{\phi\phi s}}\right\}$$

Chester, Landry, Liu, Poland, Simmons-Duffin, Vichi, Su

Navigating the bootstrap

- This requires clever algorithms for searching in high dimensional spaces
- A very nice recent proposal is to modify the feasibility problem such that it is always feasible, but with a special choice of objective function.
- This continuous function, called the navigator, is positive/negative when the unmodified problem is infeasible/feasible.
- Finding the boundaries of the allowed region amounts to finding the zeros of this function which can be done efficiently even for high dimensional search spaces.



Ising island in white

Reehorst, Rychkov, Simmons-Duffin, Sirois, Su, van Rees

Kinematics from dynamics

Generalized Free fields

- Let us go back to the simplest case of a single scalar correlator:

$$\sum_{\ell=0,2,\dots} \sum_{\Delta \geq \Delta_u(\ell)} \lambda^2_{\Delta,\ell} F_{\Delta,\ell}(z,\bar{z}) = -F_{0,0}$$

- Infinite variables for infinite constraints. But, admits very simple solution

$$\mathcal{G}(z,\bar{z}) = \frac{1}{(z\bar{z})^{\Delta_\phi}} + \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_\phi}} + 1 \longrightarrow \textit{Wick contractions}$$

$$= G_0(z,\bar{z}) + \sum_{n=0}^{\infty} \sum_{\ell=0,2,4,\dots} a_{n,\ell} G_{\Delta_{n,\ell},\ell}(z,\bar{z})$$

- What does this teach us?

$$\Delta_{n,\ell} = 2\Delta_\phi + 2n + \ell$$

$$\phi \times \phi = 1 + \sum_{n,\ell} \phi \Box^n \partial^\ell \phi$$

El-Showk, Papadodimas

Generalized Free fields

- Hard to check that it even is a solution of crossing equation. Can we rewrite it such that it is obviously so?

- Idea: crossing equation has different representations corresponding to different functional *bases*:

  - Point evaluation
  - Taylor expansion
  - Smearing against some kernel…

- Each basis is obtained by acting with a complete set of *linear functionals* on the crossing equation. Equivalently, representing crossing vectors in different bases.

- Idea: Construct set of functionals such that GFF is *obviously* a solution!

Line restriction

- Restrict attention to the line

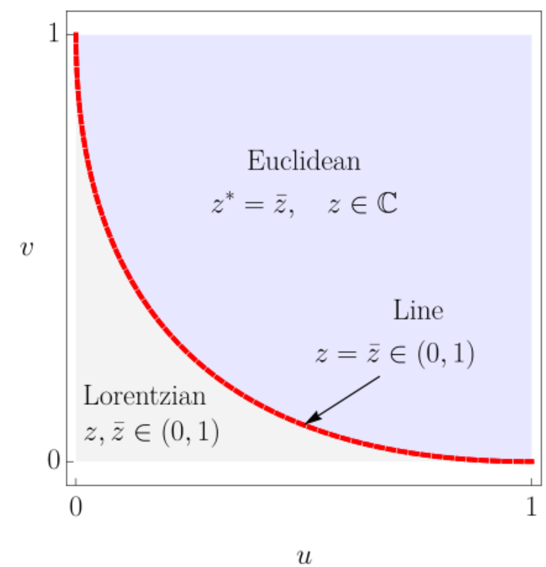
$$\mathcal{G}(w) \equiv \mathcal{G}(w, w)$$

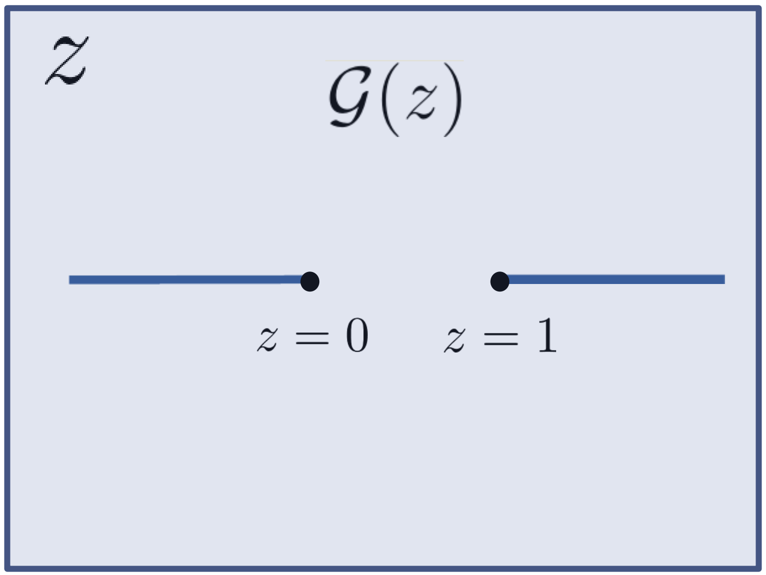
- Use only constraints of SL(2,R)

$$\mathcal{G}(w) = \sum_{\Delta \geq 0} a_\Delta^{1d} G_\Delta(w|\Delta_\phi)$$

$$G_\Delta(w|\Delta_\phi) = w^{\Delta - 2\Delta_\phi} \; {}_2F_1(\Delta, \Delta, 2\Delta, w)$$

- *Note: upon restriction get piecewise analytic function, focus on branch analytic in (0,1)*





Analytic Functionals

- It is possible to find a reformulation of the crossing constraints:

$$\sum_{\Delta} a_\Delta F_\Delta(z) = 0 \qquad \Leftrightarrow \qquad \sum_{\Delta} a_\Delta \omega(\Delta) = 0$$

$$\text{for all} \quad \omega \in \mathcal{S}_B$$

- Where the functional set $\mathcal{S}_B = \{\alpha_n^B, \beta_n^B, \quad n \in \mathbb{Z}_{\geq 0}\}$

satisfies the *duality conditions*:

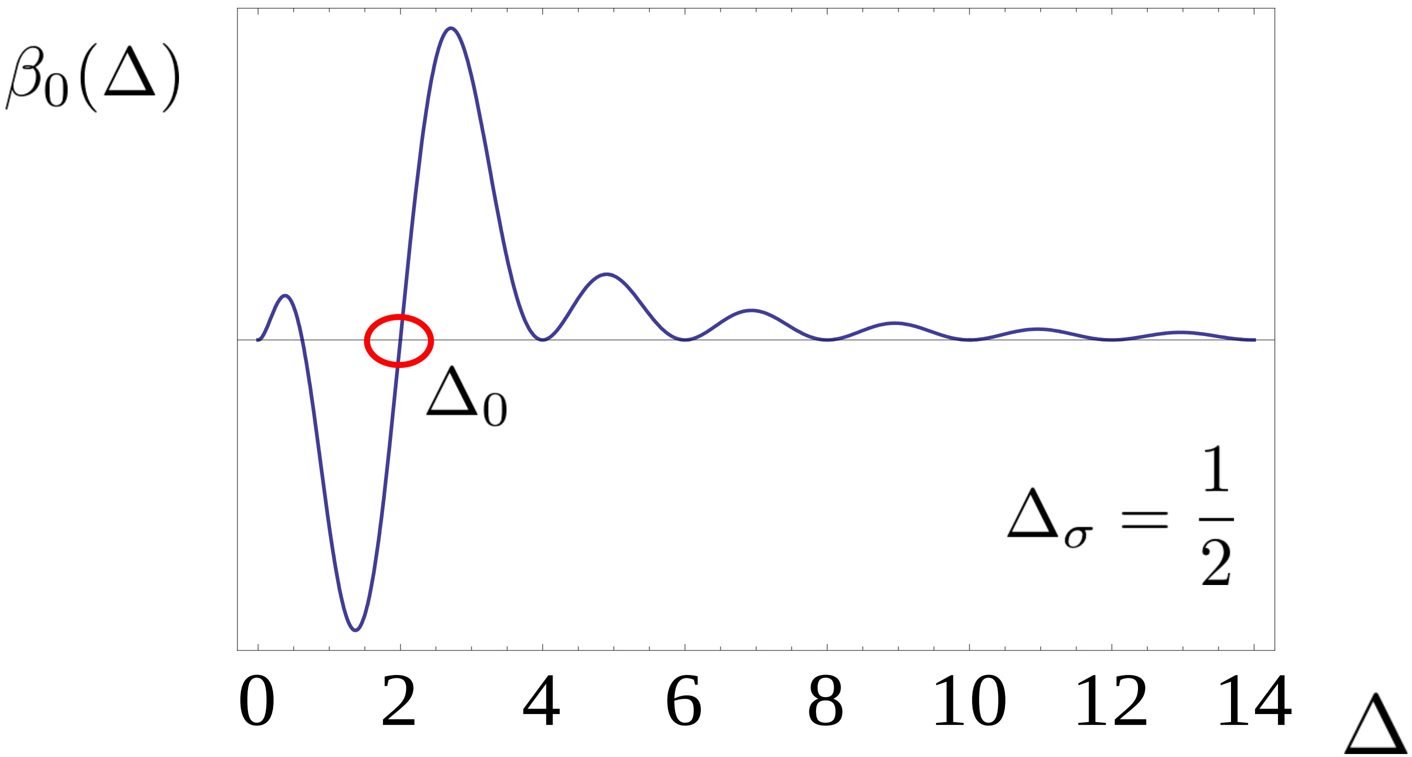
$$\begin{aligned} \alpha_n^B(\Delta_m^B) &= \delta_{n,m}, & \partial_\Delta \alpha_n^B(\Delta_m^B) &= -c_n \delta_{m0}, \\ \beta_n^B(\Delta_m^B) &= 0, & \partial_\Delta \beta_n^B(\Delta_m^B) &= \delta_{n,m} - d_n \delta_{m0} \end{aligned}$$

$$\Delta_n^B = 2\Delta_\phi + 2n$$

(*for some known constants c,d*)

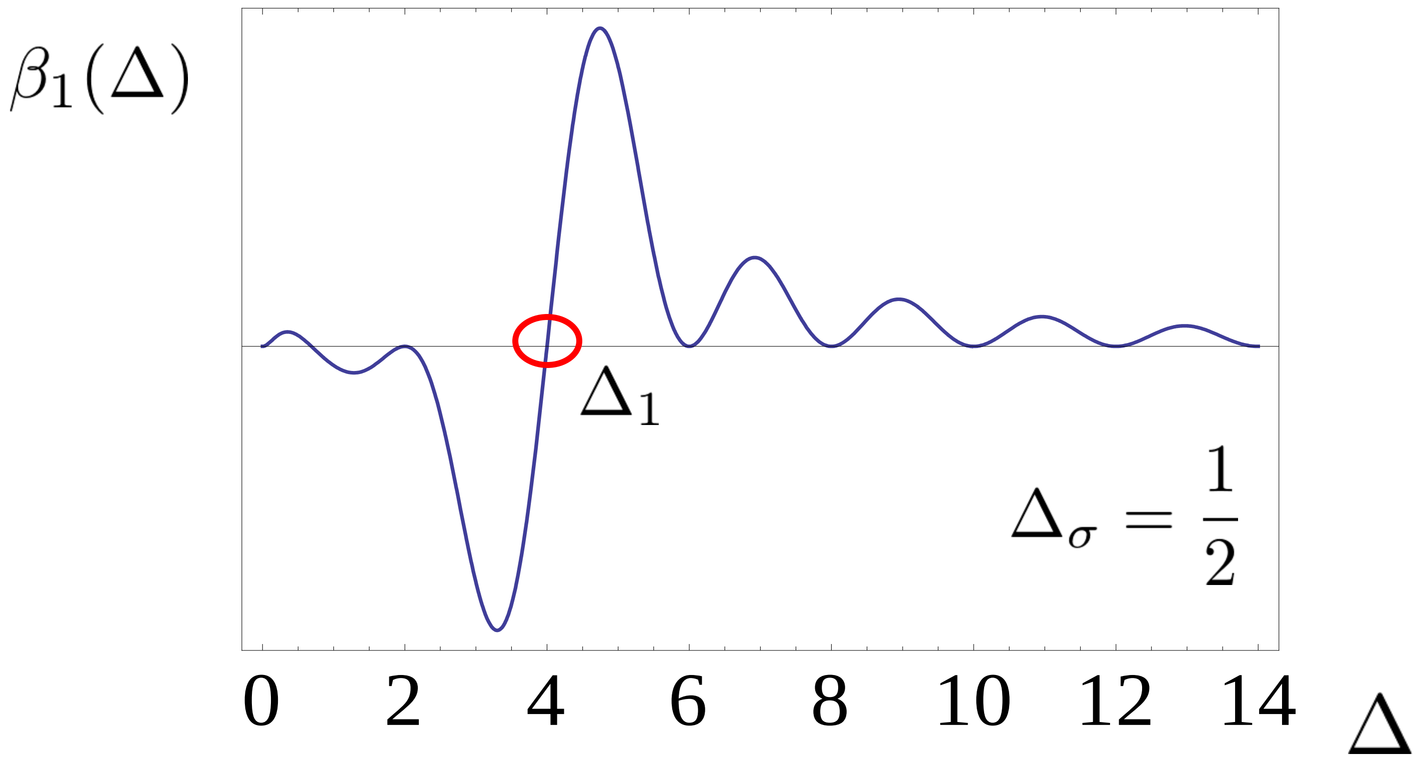
Mazac, MP

Analytic Functionals



$$\Delta_n = 1 + 2\Delta_\sigma + 2n, \quad n \in \mathbb{N}$$

Analytic Functionals



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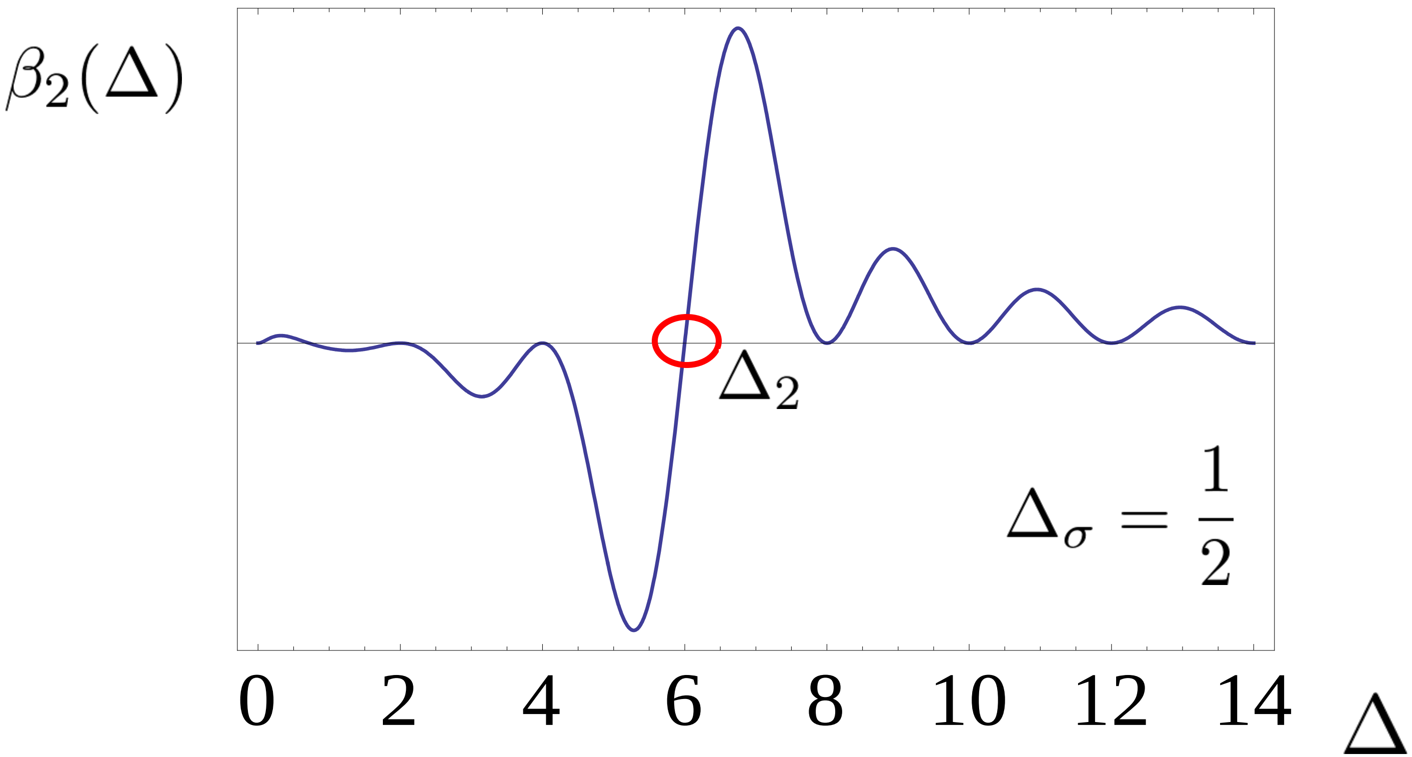

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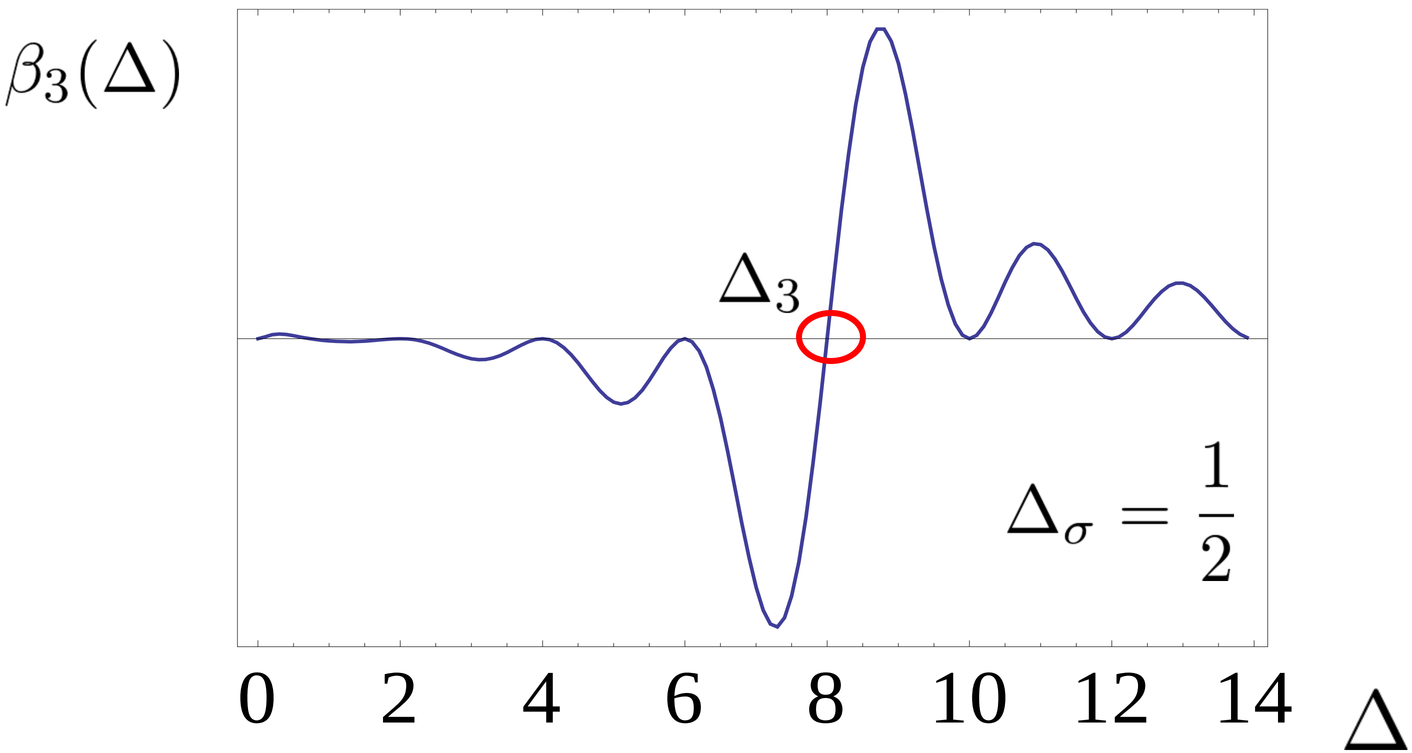
Analytic Functionals



$$\Delta_n = 1 + 2\Delta_\sigma + 2n, \quad n \in \mathbb{N}$$

Analytic Functionals

- Applying to GFF we get:

$$F_0 + \sum_{n=0}^{\infty} a_n F_{\Delta_n^B} = 0 \qquad \Leftrightarrow \qquad \begin{cases} \sum_n a_n \beta_n^B(\Delta_n^B) = 0\,, \\ \sum_n a_n \alpha_n^B(\Delta_n^B) = 0 \end{cases}$$

- Using duality conditions these collapse to:

$$a_n = -\alpha_n^B(0)\,, \qquad \beta_n^B(0) = 0$$

Define *the OPE data*

Can be verified by direct computation

Analytic Functionals

- Functional sum rules imply bounds, thanks to positivity of the functional actions.

- Functionals *know* about the full GFF solution. In particular, they automatically encode the correct high energy asymptotics.

- Since these asymptotics are universal on average, they allow us to decouple IR from UV in the crossing equation. This implies this is a nice basis to do numerics, since it will lead to very fast convergence.

MP, Zan

Analytic Functionals

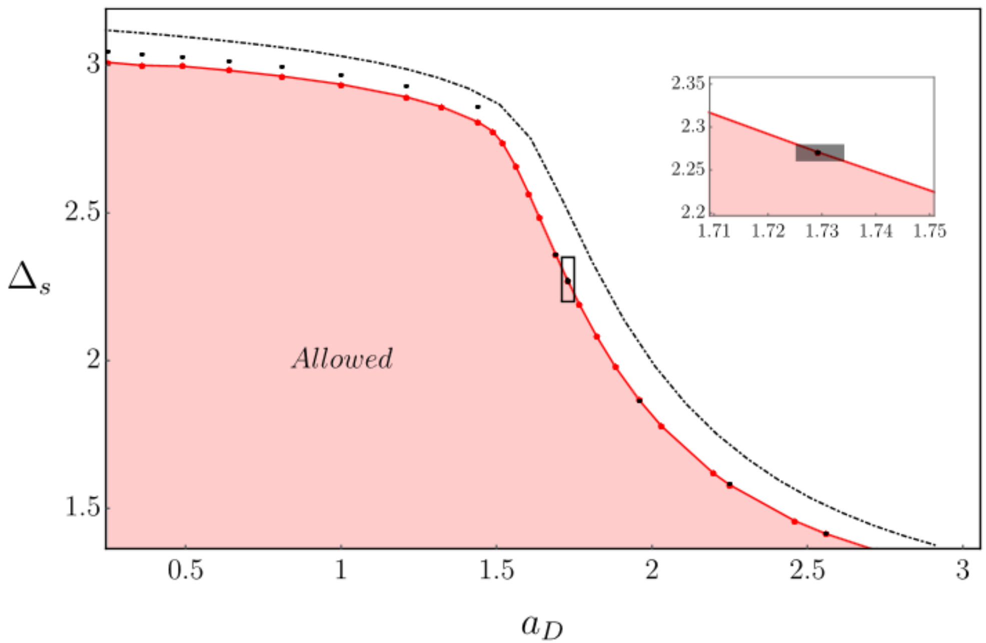


Figure 8: Maximizing the singlet gap. The red dots represent the bound obtained using GFF functionals with $n_{\max} = 6$ (40 components), black dots correspond to $n_{\max} = 2$ (16 components). The dashed-dotted black line represents the same bound using a derivative basis with 60 components [25]. The inset shows a zoomed version of the plot around the region where the twist defect is supposed to lie, with the box representing $\Delta_s = 2.27(1)$.

3d Ising twist defect (d=1 CFT with O(2))

Ghosh, Kaviraj MP

Analytic Functionals

- The statement of universality can be rigorously shown from the functionals themselves as bounds on the OPE density at large scaling dimension.

$$\limsup_{n\to\infty} \sum_{|\Delta-\Delta_n|\leq 1} \frac{4\sin^2\left[\frac{\pi}{2}\left(\Delta-\Delta_n\right)\right]}{\pi^2(\Delta-\Delta_n)^2}\left(\frac{a_\Delta}{a_\Delta^{\text{free}}}\right) \leq 1$$

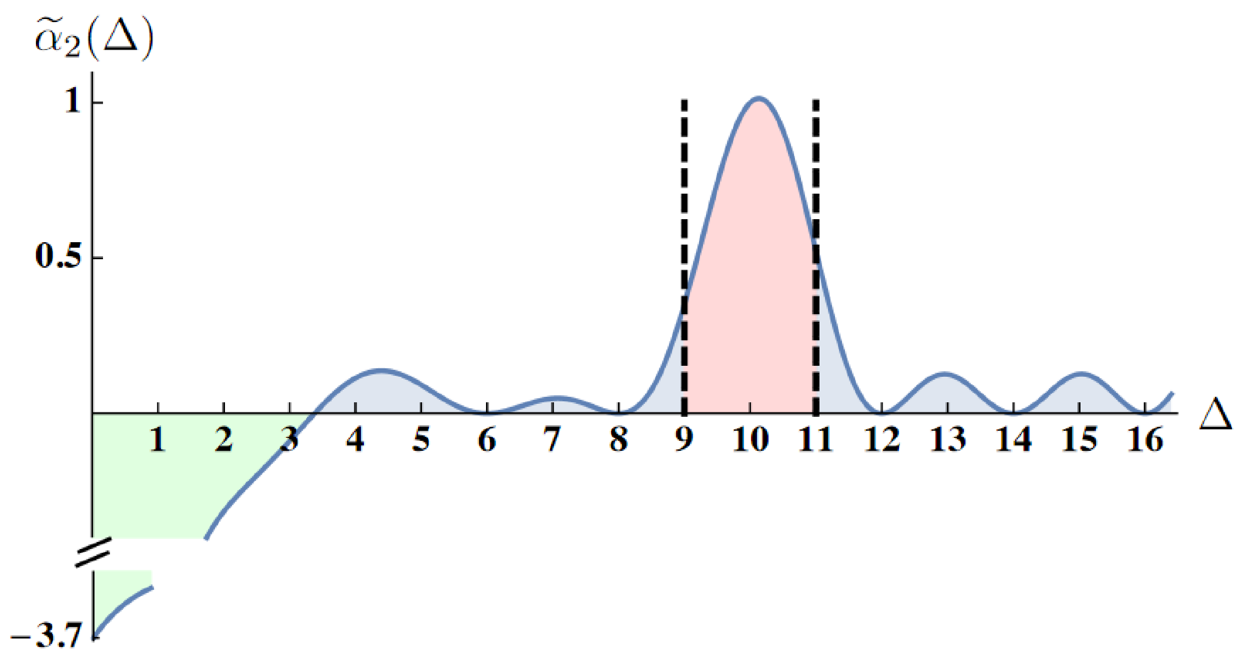
$$\liminf_{n\to\infty} \sum_{|\Delta-\Delta_n|\leq 2} \frac{16\sin^2\left[\frac{\pi}{2}(\Delta-\Delta_n)\right]}{\pi^2(\Delta-\Delta_n)^2(\Delta-\Delta_{n-1})(\Delta_{n+1}-\Delta)}\left(\frac{a_\Delta}{a_\Delta^{\text{free}}}\right) \geq 1$$

Mazac, MP

Bounds from positivity

- These bounds follow almost directly from the sum rules associated to the functionals.

$$\sum_{\Delta} a_{\Delta}\omega(\Delta) = 0$$



Higher dimensions

- It is possible to find similar bases in higher dimensions

$$\alpha_{n,J}(\Delta_{m,\ell},\ell)=\delta_{n,m}\delta_{J,\ell}\,, \qquad \partial_\Delta\alpha_{n,J}(\Delta_{m,\ell},\ell)=0$$
$$\beta_{n,J}(\Delta_{m,\ell},\ell)=0\,, \qquad \partial_\Delta\beta_{n,J}(\Delta_{m,\ell},\ell)=\delta_{n,m}\delta_{J,\ell}$$

- Basis now also includes *odd* spins – extra constraints not directly dual to GFF data.

- Positivity properties not as good due to negative regions for small twists, hard to use as basis for numerics.

- Other basis are possible (better?), by reshuffling odd-spin functionals among even spin ones. Explicit proposal manifesting s,t,u crossing but seemingly at the cost of locality. Also, other non-gff basis constructed in d=2.

Caron-Huot, Gopakumar, Mazac, MP, Rastelli, Simmons-Duffin, Sen Sinha, Zahed

Higher dimensions

- Functional sum rules are equivalent to positive dispersion relation for (subtracted) CFT correlators:

$$\overline{\mathcal{G}}(z,\bar{z}) = \int \mathrm{d}w\mathrm{d}\bar{w}\, K(z,\bar{z},w,\bar{w})\mathrm{dDisc}\,\overline{\mathcal{G}}(w,\bar{w})$$

- Obtained from Lorentzian inversion formula

Carmi, Caron-Huot, Mazac, MP, Rastelli, Simmons-Duffin,

- Formulates the correlator in terms of independent data: dDisc + subtractions. dDisc kills double trace operators, formula insensitive to GFF asymptotics!

- Equivalent to dispersion relation in Mellin space.

Gopakumar, Penedones, Silva, Zhiboedov, Sinha, Zahed

- Nice positivity properties: K is positive. Negativity from subtractions.

Polyakov bootstrap

- Expanding in conformal blocks leads to a representation of the correlator in terms of crossing symmetric Polyakov-Regge blocks

$$\mathcal{P}_{\Delta,\ell}(z,\bar{z}) = \int \mathrm{d}w\mathrm{d}\bar{w}\, K(z,\bar{z},w,\bar{w})\mathrm{dDisc}\, G_{\Delta,\ell}(w,\bar{w})$$

$$\mathcal{G}(z,\bar{z}) = \sum_{\Delta,\ell} \lambda^2_{\Delta,\ell}\, \mathcal{P}_{\Delta,\ell}(z,\bar{z})$$

Carmi, Caron-Huot, Mazac, MP, Rastelli, Simmons-Duffin

- Polyakov-Regge blocks are computable as crossing-symmetric sums of Witten exchange diagrams + contact terms. *But above is universal statement for any CFT.*

- Functional sum rules arise by demanding consistency of OPE (i.e. decoupling of non-physical double trace blocks).

Polyakov, Dey, Gopakumar, Ghosh, Sen, Sinha, Zahed, MP, Mazac

Perturbation theory

- Both functionals and Polyakov blocks have double zeros at dimensions of (generalized) free fields.

- This allows for systematic computations around (generalized) free fields with small perturbations, e.g. Wilson-Fisher fixed point. Systematic understanding of lightcone bootstrap expansions.

Alday, Bissi, Dey, Johansson,Ghosh, Gopakumar, Hansen, Kaviraj, Lukowsky, Penedones, Silva, Sinha, Zhiboedov,…

- Intuition: dispersion relation encodes correlator in terms of its ``imaginary'' part. Method is then similar to using unitarity to learn about higher loops from lower loops in scattering amplitude context.

- By the same logic, it is possible to use positivity properties of functional sum rules to bound possible EFT coefficients in AdS.

Caron-Huot, Haldar, Mazac, Rastelli, Simmons-Duffin, Gopakumar, Raman, Sinha, Zahed,

Bounding observables

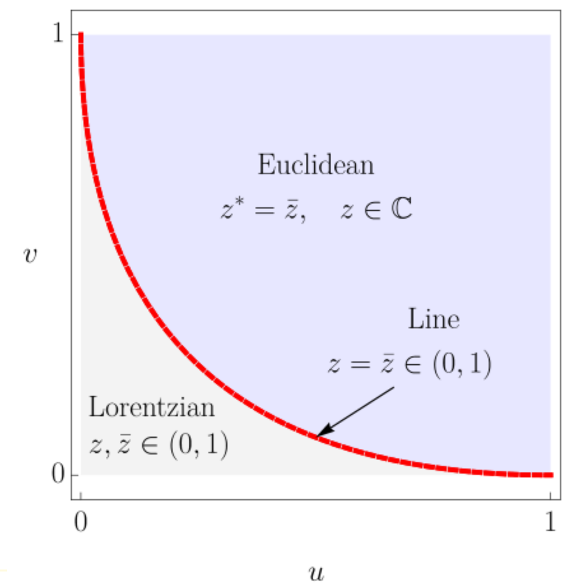
MP, Zheng

Correlator space

- Interested in 4-pt functions

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{G}(z,\bar{z})}{x_{13}^{2\Delta_\phi} x_{24}^{2\Delta_\phi}}$$

- Correlator space:



$$\mathfrak{G}^{(d)}_{\Delta_\phi} := \left\{ Identical\ scalar\ correlators\ \mathcal{G} = \langle\phi\phi\phi\phi\rangle\ \ of\ CFT_d \right\}$$

$$\mathfrak{G}^{(d')}_{\Delta_\phi} \subset \mathfrak{G}^{(d)}_{\Delta_\phi}, \qquad 1 < d < d'$$

$$\mathcal{G}\Big|_{z=\bar{z}} \in \mathfrak{G}^{(d=1)}_{\Delta_\phi} \qquad \text{for any} \quad \mathcal{G} \in \mathfrak{G}^{(d)}_{\Delta_\phi}$$

d-dimensional CFT is also a (d-1) CFT, bounds have to fit inside each other

Correlator space

$$\mathfrak{G}^{(d)}_{\Delta_\phi} := \left\{ \textit{Identical scalar correlators } \mathcal{G} = \langle \phi\phi\phi\phi \rangle \textit{ of } CFT_d \right\}$$

- Notable points:

$$\left\{ \begin{array}{l} \mathcal{G}_{\text{gapmax}} := \underset{\mathcal{G}\in\mathfrak{G}}{\arg\max} \quad \Delta_g^{\mathcal{G}} \\ \mathcal{G}_{\text{opemax}}(\bullet|\Delta_g) := \underset{\mathcal{G}\in\mathfrak{G}:\ \Delta_{\mathcal{G}}^{\text{gap}}\geq\Delta_g}{\arg\max} \quad a_{\Delta_g,0} \end{array} \right.$$

→ *Leading (scalar) dimension in correlator*

→ *OPE coefficient squared*

- Extremal values:

$$\left\{ \begin{array}{l} \mathcal{G}_{\min;w,\bar{w}} = \arg\min \mathcal{G}(w,\bar{w}) \\ \mathcal{G}_{\max;w,\bar{w}}(\bullet|\Delta_g) := \underset{\mathcal{G}\in\mathfrak{G}:\Delta_g^{\mathcal{G}}\geq\Delta_g}{\arg\max} \ \mathcal{G}(w,\bar{w}) \end{array} \right.$$

→ *Note: distinct choices of point lead to distinct correlators*

Bounds from optimization

Crossing equation:

Block – cross channel block

$$F_{0,0}(w,\bar{w}) + \sum_{\Delta,\ell\in\mathcal{S}} a_{\Delta,\ell} F_{\Delta,\ell}(w,\bar{w}|\Delta_\phi) = 0$$

Assumptions on spectrum

Construct linear functionals satisfying:

$$\underline{\Omega}_{w,\bar{w}}(\Delta,\ell) \geq -G_{\Delta,\ell}(w,\bar{w}|\Delta_\phi)$$

$$\overline{\Omega}_{w,\bar{w}}(\Delta,\ell) \geq G_{\Delta,\ell}(w,\bar{w}|\Delta_\phi)$$

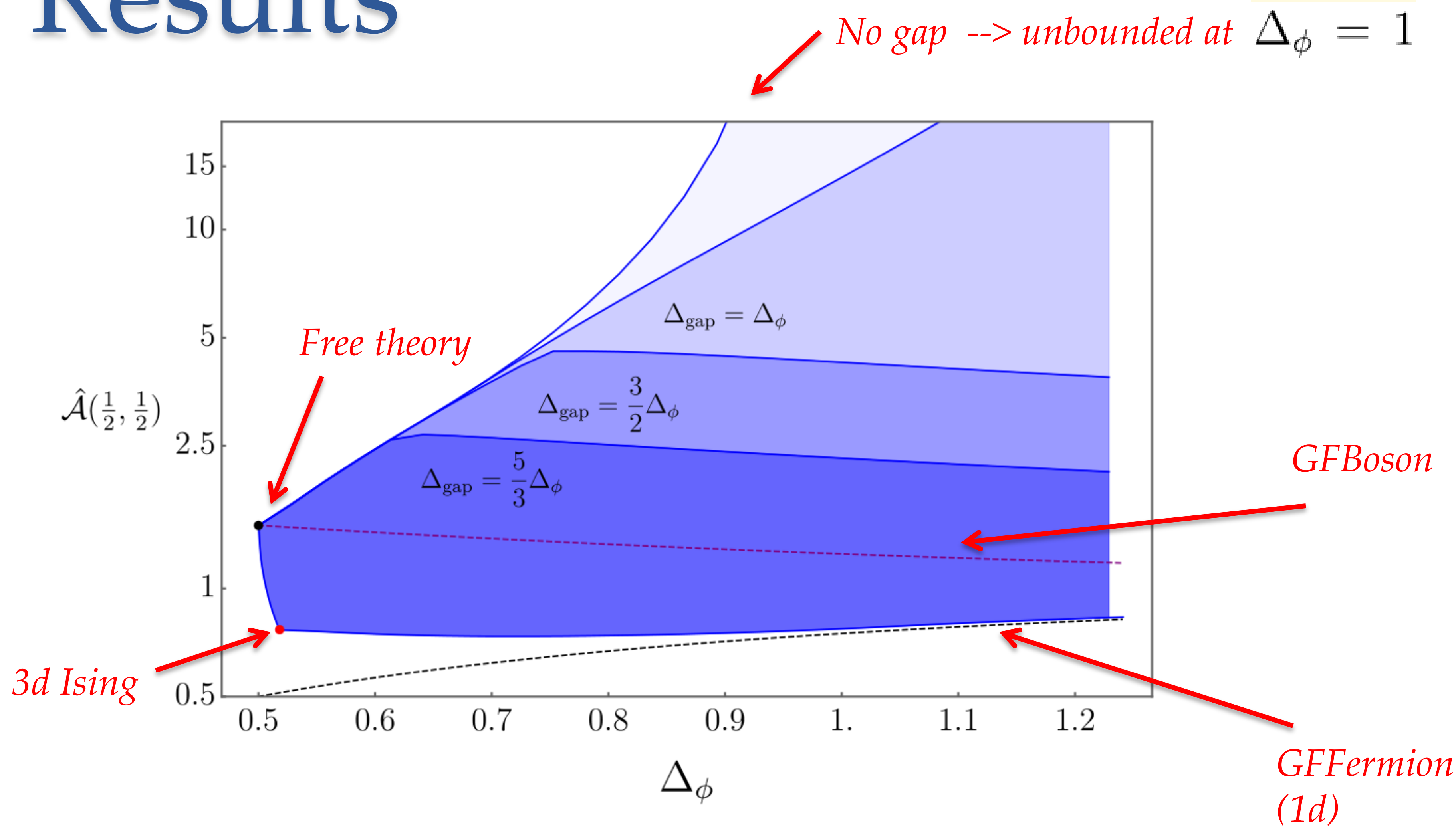
$$\text{for all} \quad \Delta,\ell\in\mathcal{S}.$$

Acting on crossing equation, using positivity of OPE:

$$\frac{1}{(w\bar{w})^{\Delta_\phi}} + \underline{\Omega}_{w,\bar{w}}(0,0) \leq \mathcal{G}(w,\bar{w}) \leq \frac{1}{(w\bar{w})^{\Delta_\phi}} - \overline{\Omega}_{w,\bar{w}}(0,0)$$

Maximize action on identity subject to constraints above

Results



$$\hat{\mathcal{A}}(z, \bar{z}) = (z\bar{z})^{2\Delta_\phi}\mathcal{G}(z, \bar{z}) - 1$$

$$\mathcal{G}^B(w) \equiv \frac{1}{w^{2\Delta_\phi}} + \frac{1}{(1-w)^{2\Delta_\phi}} + 1$$

$$\mathcal{G}^F(w) \equiv \frac{1}{w^{2\Delta_\phi}} + \frac{1}{(1-w)^{2\Delta_\phi}} - 1$$

Results

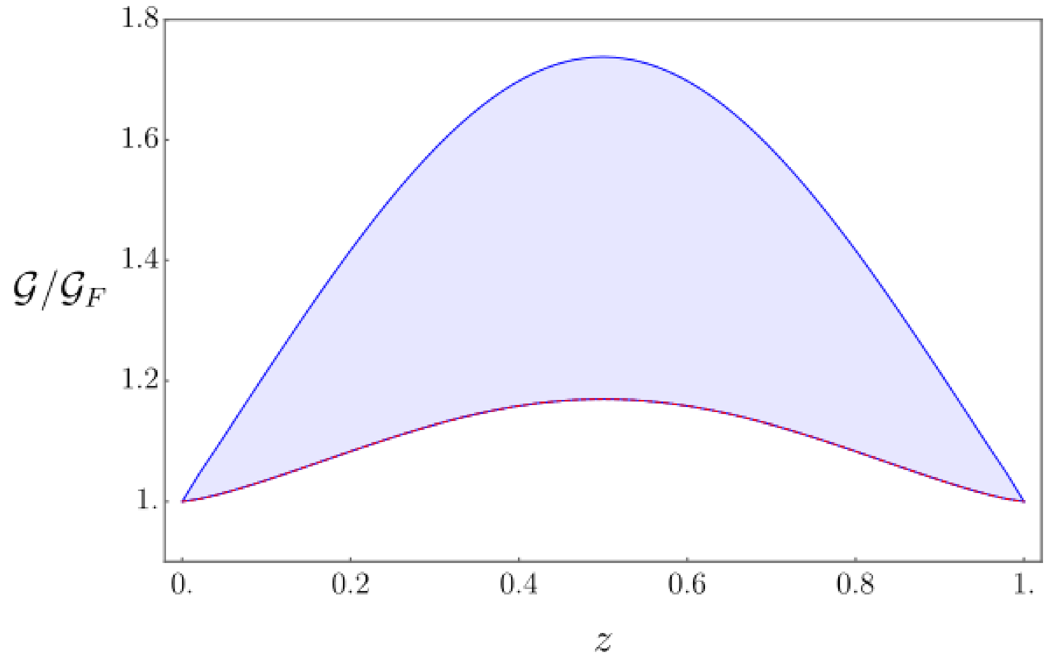


Figure 1: Upper and lower bounds on 3d CFT correlators $\mathcal{G}(z,z)$ normalized by $\mathcal{G}^F(z) = -1 + z^{-2\Delta_\phi} + (1-z)^{-2\Delta_\phi}$. Here $\Delta_\phi = \Delta_\sigma^{\text{Ising}} \sim 0.518149$. The 3d Ising spin four-point function saturates the lower bound.

Outlook

Bootstrap: near future

- Effective solutions of D=1 crossing
  - No spin to deal with.
  - Efficient functional basis, fast convergence – push to large numbers of correlators.
  - Several nice candidates: long-range Ising, line defects, 2d QFTs, ….
- Better numerics in higher-D
  - Better functional bases in higher-D? (positivity)
  - Better algorithms for SDP
  - Larger number of correlators, 4d stress-tensors, …
  - *Effective* solutions of models such as 3d Ising (essentially a reality already).

Non-positive bootstrap

- Many interesting situations involve OPE expansions where positivity conditions are not available. No positivity constraints, no bounds! E.g. non-unitary CFTs allow for $\lambda^2_{\Delta,\ell} < 0$

- One possibility is to note that in our solution of GFF with functionals, positivity was an outcome not an input. Crucial point is to construct complete bases with nice properties.

- Bases associated to sparsest possible solutions – *extremal CFTs*. Demanding sparsity it is possible to find special solutions to crossing independently of positivity.

Questions, questions

- What is the bootstrap? A sequence of approximations in sparsity of CFT spectrum? ``Average'' CFTs? How can chaotic UV spectra be eventually reproduced?

- What are functionals? Why are they positive? Can we construct bases dual to interacting CFTs analytically? Are there ``integrable'' CFTs?

- Is there a way to bootstrap higher-point correlators to access lots of data in one go? E.g. in 1d knowing a generic 8 pt function encodes the full set of 4 pt functions. Higher point conformal blocks? Kinematics?