Earth as a baseline for measuring CP violating phase in neutrino oscillations in matter

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Based on 2005.07719 (A. Ioannisian, S. Pokorski, J. Rosiek, M. Ryczkowski)

September 2, 2021

Neutrino oscillations in vacuum

$$P(\nu_\alpha \to \nu_\beta) = \underbrace{\left|\langle \nu_\beta \,|\nu_{L\alpha}(x,t)\rangle\right|^2}_{|S_{\beta\alpha}|^2} = \left| U^* e^{-i\mathcal{H}^d x} U \right|^2$$

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PMNS (Pontecorvo–Maki–Nakagawa–Sakata) lepton mixing matrix & vacuum Hamiltonian:

$$U = O_{23} U_\delta O_{13} O_{12}, \quad \mathcal{H} = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m_\odot^2}{2E} & 0 \\ 0 & 0 & \frac{\Delta m_a^2}{2E} \end{pmatrix} U^\dagger = U \mathcal{H}^d U^\dagger$$

$$O_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad O_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \quad O_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \quad U_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}$$

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- $\Delta m_\odot^2 = m_2^2 - m_1^2$, $\Delta m_a^2 = m_3^2 - m_1^2$ (well constrained),
- $s \equiv \sin\theta$, $c \equiv \cos\theta$, θ_{12}, θ_{13}, θ_{23} - mixing angles (well constrained),
- δ - **CP violating** phase (weakly constrained).

Neutrino oscillations in matter

Hamiltonian in matter, assuming constant matter density (following A. Ioannisian & S. Pokorski 1801.10488):

$$\mathcal{H}_m = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m^2_\odot}{2E} & 0 \\ 0 & 0 & \frac{\Delta m^2_a}{2E} \end{pmatrix} U^\dagger + \begin{pmatrix} V & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = U_m \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m^2_{21}}{2E} & 0 \\ 0 & 0 & \frac{\Delta m^2_{31}}{2E} \end{pmatrix} U_m^\dagger \equiv U_m \mathcal{H}_m^d U_m^\dagger$$

with PMNS matrix in matter and interaction potential:

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and effective parameters:

$$\theta_{12} \to \theta^m_{12}, \quad \Delta m^2_\odot \to \Delta m^2_{21}, \quad \theta_{13} \to \theta^m_{13}, \quad \Delta m^2_a \to \Delta m^2_{31}, \quad \theta^m_{23} \equiv \theta_{23}, \quad \delta^m \equiv \delta$$

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- SM sources of CP violation - weak interactions:
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Question: **What is the value of δ?**

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Answer: **Neutrino oscillations!**

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$$\delta^m \equiv \delta \to \text{measurement of } \delta^m \to \textbf{CP violation}$$

New approach to measure δ in neutrino oscillations

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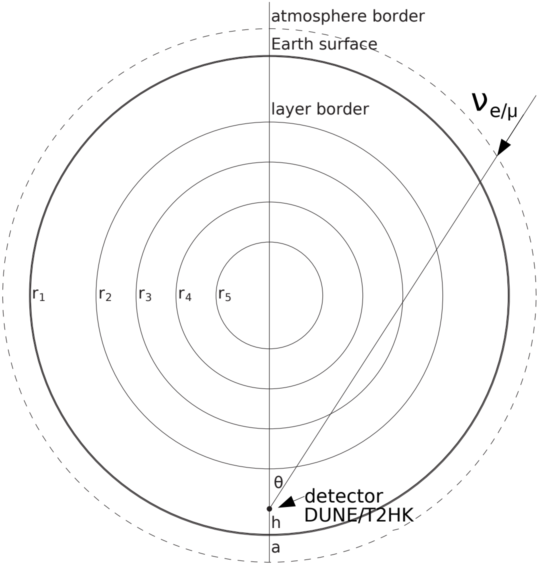
Oscillations of sub-GeV atmospheric neutrinos traversing the Earth

2005.07719: "Analytical description of CP violation in oscillations of atmospheric neutrinos traversing the Earth"
A. Ioannisian, S. Pokorski, J. Rosiek, M. Ryczkowski

Atmospheric neutrinos traversing the Earth: setup



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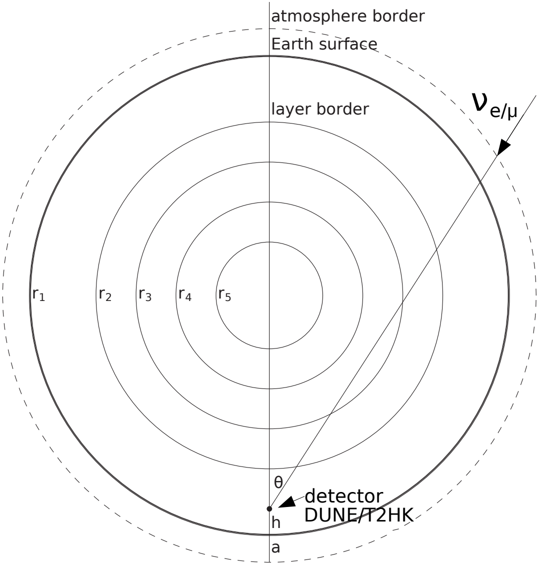

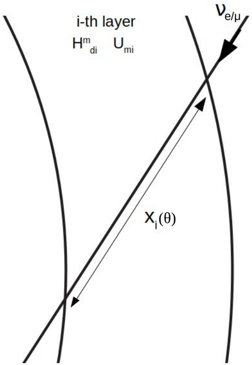

Atmospheric neutrinos traversing the Earth

Exact S-matrix for neutrinos traversing n Earth's layers (normal mass ordering):

$$S^m = T\Pi_i U^*_{mi} e^{-i\mathcal{H}^d_{mi}} U_{mi} = e^{i\xi} U_a T\Pi_i \left(O^m_{i13} O^m_{i12} \mathcal{E}_i O^{mT}_{i12} O^{mT}_{i13} \right) U^\dagger_a$$

$$\mathcal{E} = \text{diag}\left(e^{i\left(\frac{\Delta m^i_{21}}{4E}\right)x_i}, e^{-i\left(\frac{\Delta m^i_{21}}{4E}\right)x_i}, e^{-i\left(\frac{\Delta m^i_{31}+\Delta m^i_{32}}{4E}\right)x_i} \right), \quad U_a = U_{23} U_\delta, \quad \xi - \text{overall phase}$$

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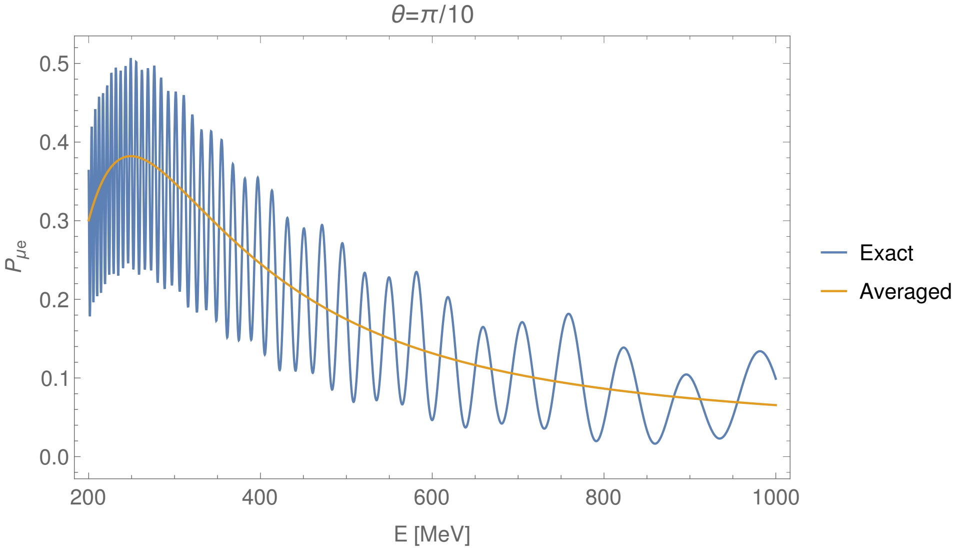
Solution: average probabilities!

Averaged vs exact probabilities



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How to do it?

Averaging probabilities 1

1. Approximate exact S^m-matrix:

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2. Separate product (and S^m) into the form:

$$T \Pi_i \left(O^m_{i12} \mathcal{E}_i O^{mT}_{i12} \right) = \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{21} & X_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^m \approx \underbrace{U_0 \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{21} & X_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} U_0^\dagger}_{A} + \Pi_i \left(\mathcal{E}_i \right)_{33} \underbrace{U_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} U_0^\dagger}_{B}$$

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4. Average out quickly oscillating term $\equiv$ average probability:

$$\bar{P}^m(E,\theta)_{\alpha\beta} = |A_{\beta\alpha}|^2 + \overbrace{2\Re[A^*_{\beta\alpha} B_{\beta\alpha} \Pi_i \left(\mathcal{E}_i\right)_{33}]}^{0} + |B_{\beta\alpha}|^2$$

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- $\cdots$ or analytical approximation for $\phi_X(E,\theta)$ and $\alpha_X(E,\theta)$ for n-layers.

Analytical approximation for $\phi_X(E,\theta)$ and $\alpha_X(E,\theta)$ for k-layers

1. Expand product in $S^m = U_0 \, T\Pi_i \left(O^m_{i12}\mathcal{E}_i O^{mT}_{i12}\right) U_0^\dagger$ in terms of small parameter:

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Analytical approximation for $\phi_X(E,\theta)$ and $\alpha_X(E,\theta)$ for k-layers

1. Expand product in $S^m = U_0\, T\Pi_i \left(O^m_{i12}\mathcal{E}_i O^{mT}_{i12}\right) U_0^\dagger$ in terms of small parameter:

$$\epsilon_i = \sin 2\theta^m_{i12} \propto \frac{1}{EV_i}$$

2. Keep terms linear in ϵ,

Result: remarkably compact formulas!

$$\phi_X = \nu_1 + \nu_2 + \ldots + \frac{1}{2}\nu_k, \quad \nu_i \approx V_i \cos^2 2\theta_{13} x_i(\theta)$$

$$\sin\alpha_X = (\epsilon_k - \epsilon_{k-1})\sin\frac{\nu_k}{2} + (\epsilon_{k-1} - \epsilon_{k-2})\sin\left(\nu_{k-1} + \frac{\nu_k}{2}\right) + \ldots$$
$$+ (\epsilon_2 - \epsilon_1)\sin\left(\nu_2 + \nu_3 + \ldots + \frac{\nu_k}{2}\right) + \epsilon_1 \sin\left(\nu_1 + \nu_2 + \ldots \frac{\nu_k}{2}\right)$$

Similar approach works for antineutrinos!

Features of $\phi_X(E,\theta)$ and $\alpha_X(E,\theta)$

- $\phi_X(E,\theta) = \phi_X(\theta)$
- $\sin\alpha_X(E,\theta) = f(\theta)/E$

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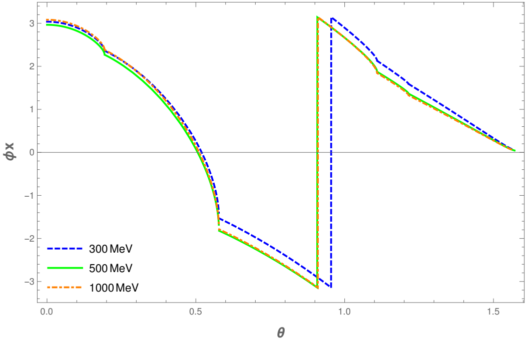
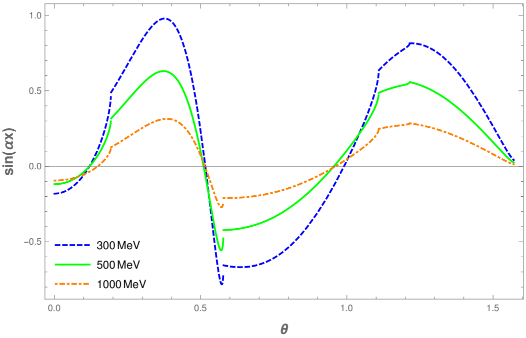
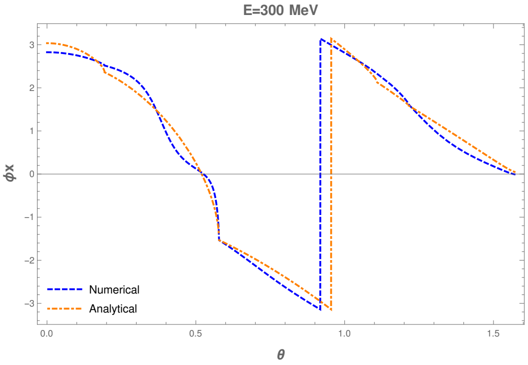
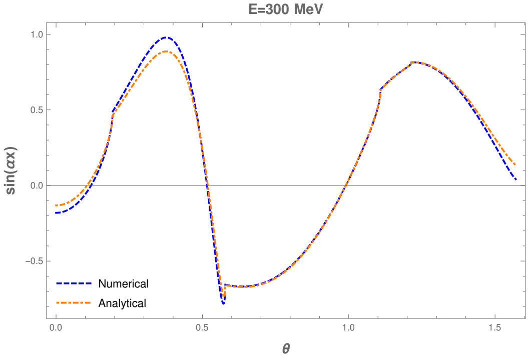




Numerical fits vs analytical approximation





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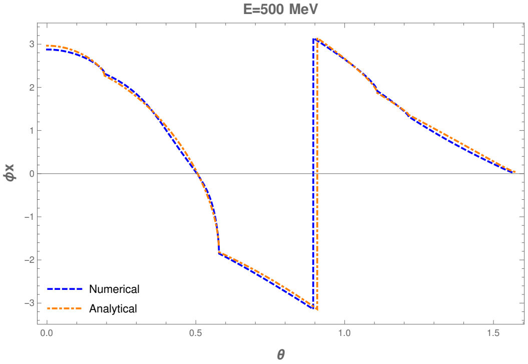

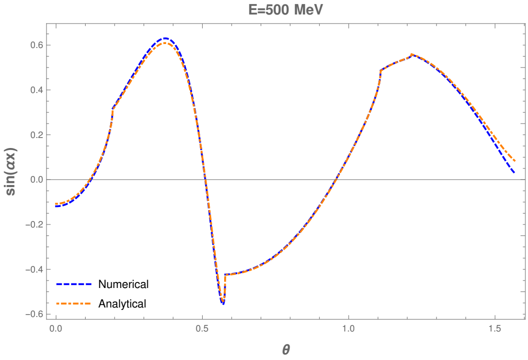

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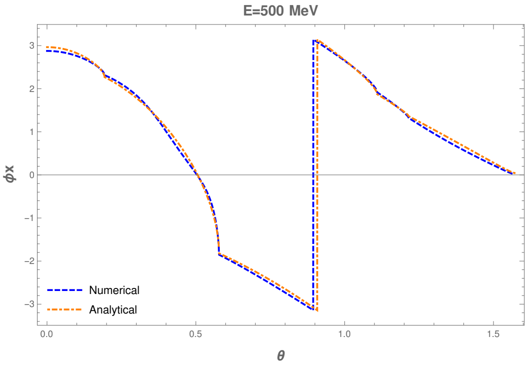

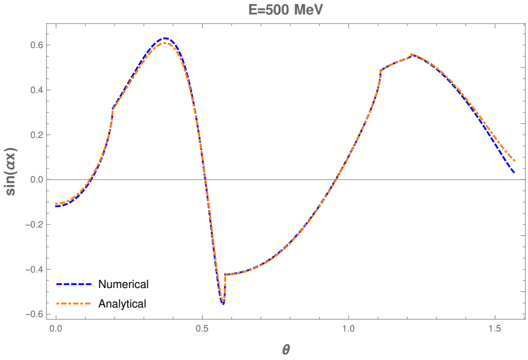

Works for $E > 300$ MeV!

Behavior of probabilities

Result No.1 - **analytical formulas for averaged oscillation probabilities:**

$$\bar{P}^m(E,\theta)_{\alpha\beta} = \bar{P}^m(\phi_X(E,\theta), \alpha_X(E,\theta))$$

$$\text{e.g.}\quad \bar{P}^m_{\mu e} \approx 0.024 + 0.450\sin^2\alpha_X - 0.0724\sin 2\alpha_X \underbrace{\sin(\delta+\phi_X)}_{\delta\text{ dependence}}$$

Analytical understanding of $\bar{P}^m_{\alpha\beta} \equiv$ **better chances for** δ **detection**!

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Noticing $N_{\nu_\mu} = 2N_{\nu_e}$ $\rightarrow$ quantity that gives number of neutrinos observed by detectors:

$$\bar{P}^m_e = \bar{P}^m_{ee} + 2\bar{P}^m_{\mu e} \approx 1.00 - \underbrace{0.94 \sin^2\alpha_X}_{\propto 1/E^2} - 0.143 \underbrace{\sin 2\alpha_X \sin(\delta + \phi_X)}_{\propto g(\theta)/E}$$

Optimal azimuthal angles

Result No. 2 - **azimuthal angles optimized for δ detection:**

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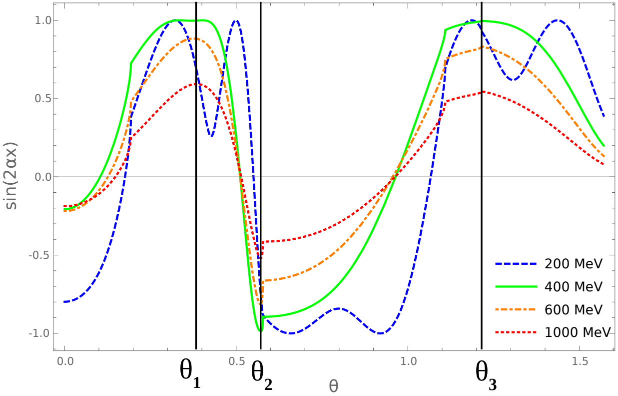
$\theta_{1,2,3}$ (vertical lines) - angles that maximize effects of δ on $\bar{P}_e^m$:



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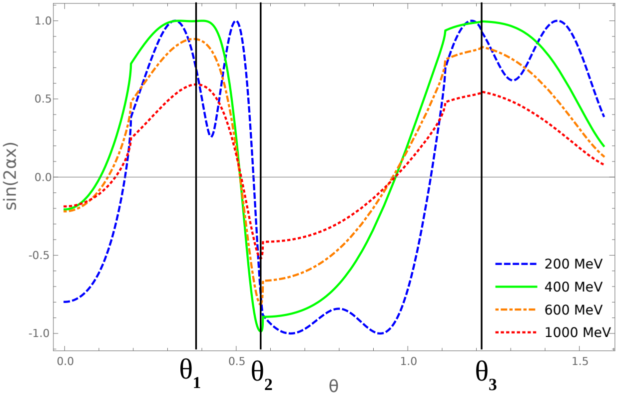

$$\theta_1 = 0.12\pi, \quad \theta_2 = 0.18\pi, \quad \theta_3 = 0.39\pi$$

Optimal observable $\Delta \bar{P}_e^m$

Result No. 3 - **Observable optimized for δ measurement $\equiv$ strongest δ dependence:**

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$$\Delta \bar{P}_e^m \left(E_1, E_2, \theta, \delta\right) = \frac{E_1^2}{E_2^2} \bar{P}_e^m \left(E_1, \theta\right) - \bar{P}_e^m \left(E_2, \theta\right) - \left(1 - \frac{\sin 2\theta_{13} \cos 2\theta_{23}}{2}\right)\left(\frac{E_1^2}{E_2^2} - 1\right)$$

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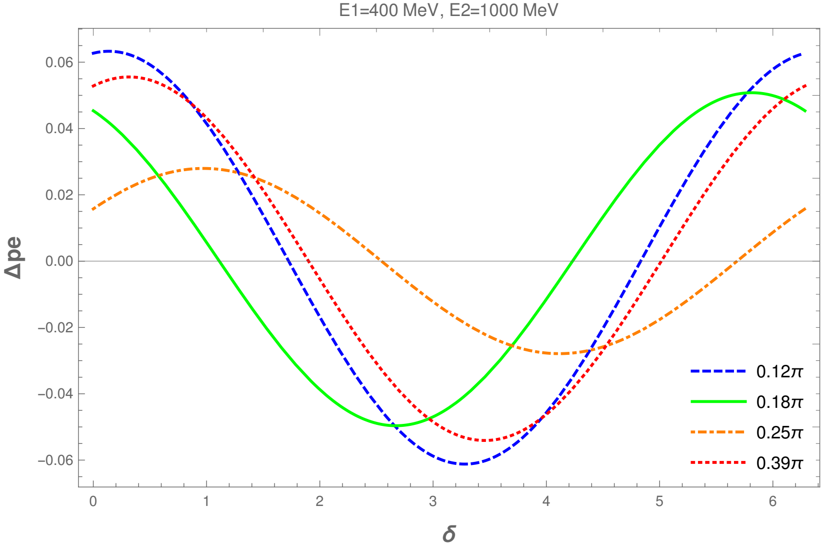
$$\Delta \bar{P}_e^m (E_1, E_2, \theta, \delta) \approx -0.14 \left(\frac{E_1^2}{E_2^2} \sin 2\alpha_X(E_1) - \sin 2\alpha_X(E_2)\right) \sin(\delta + \phi_X)$$

$$\Delta \bar{P}_e^m (E_1, E_2, \theta, \delta) \propto \sin(\delta + \phi_X)$$

Optimal observable $\Delta \bar{P}_e^m$



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What's next?

1. More realistic analysis including experiment characteristics and simulations (DUNE and T2HK),
2. Apply similar approach to other celestial bodies (e.g. stars, neutron stars).

Thank you!

Additional slides

Neutrino oscillations in vacuum

- $m_{\odot}^2 = m_2^2 - m_1^2$, $m_a^2 = m_3^2 - m_1^2$
- Normal Mass Ordering (NO) with $m_1 < m_2 < m_3$
- Inverted Mass Ordering (IO) with $m_3 < m_1 < m_2$

Quantity	Value (NO)	Value (IO)
δ_{CP}	$\left(218^{+38}_{-27}\right)^{\circ}$	$\left(281^{+23}_{-27}\right)^{\circ}$
θ_{12}	$\left(34.5^{+1.2}_{-1.0}\right)^{\circ}$	$\left(34.5^{+1.2}_{-1.0}\right)^{\circ}$
θ_{23}	$\left(47.7^{+1.2}_{-1.7}\right)^{\circ}$	$\left(47.9^{+1.0}_{-1.7}\right)^{\circ}$
θ_{13}	$\left(8.45^{+0.16}_{-0.14}\right)^{\circ}$	$\left(8.53^{+0.14}_{-0.15}\right)^{\circ}$
$\Delta m_{\odot}^2$	$7.55^{+0.20}_{-0.16} \times 10^{-5}\mathrm{eV}^2$	$7.55^{+0.20}_{-0.16} \times 10^{-5}\mathrm{eV}^2$
Δm_a^2	$+2.50 \pm 0.03 \times 10^{-3}\mathrm{eV}^2$	$-2.42^{+0.03}_{-0.04} \times 10^{-3}\mathrm{eV}^2$

Effective parameters

$$\sin 2\theta_{13}^m = \frac{\sin 2\theta_{13}}{\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}}}, \quad \Delta m_{ee}^2 = c_{12}^2 \Delta m_a^2 + s_{12}^2 \left(\Delta m_a^2 - \Delta m_\odot^2 \right)$$

$$\sin 2\theta'_{13} = \frac{\epsilon_a \sin 2\theta_{13}}{\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}}}, \quad \epsilon_a = \frac{2EV}{\Delta m_{ee}^2} \tag{1}$$

$$\sin 2\theta_{12}^m = \frac{\cos \theta'_{13} \sin 2\theta_{12}}{\sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2 \theta'_{13} \sin^2 2\theta_{12}}}, \quad \epsilon_\odot = \frac{2EV}{\Delta m_\odot^2} \left(\cos^2 \left(\theta_{13} + \theta'_{13} \right) + \frac{\sin^2 \theta'_{13}}{\epsilon_a} \right)$$

$$\mathcal{H}_2 - \mathcal{H}_1 \equiv \frac{\Delta m_{21}^2}{2E} = \frac{\Delta m_\odot^2}{2E} \sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2 \theta'_{13} \sin^2 2\theta_{12}} \tag{2}$$

$$\begin{aligned} \mathcal{H}_3 - \mathcal{H}_1 \equiv \frac{\Delta m_{31}^2}{2E} = \frac{3}{4} \frac{\Delta m_{ee}^2}{2E} \sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}} + \\ \frac{1}{4} \left[\frac{\Delta m_{ee}^2}{2E} + V \right] + \frac{1}{4E} \left(\Delta m_{21}^2 - \Delta m_\odot^2 \cos 2\theta_{12} \right) \end{aligned} \tag{3}$$

Averaging probabilities 1

1. Take exact S^m-matrix:

$$S^m = e^{i\xi} U_a T\Pi_i \left(O^m_{i13} O^m_{i12} \mathcal{E}_i O^{mT}_{i12} O^{mT}_{i13} \right) U_a^\dagger = \ldots \mathcal{E}_i O^{mT}_{i12} O^{mT}_{i13} O^{mT}_{(i+1)13} O^{mT}_{(i+1)12} \mathcal{E}_{i+1} \ldots$$

2. Simplify $O^{mT}_{i13} O^{mT}_{(i+1)13}$ products (works for realistic Earth densities):

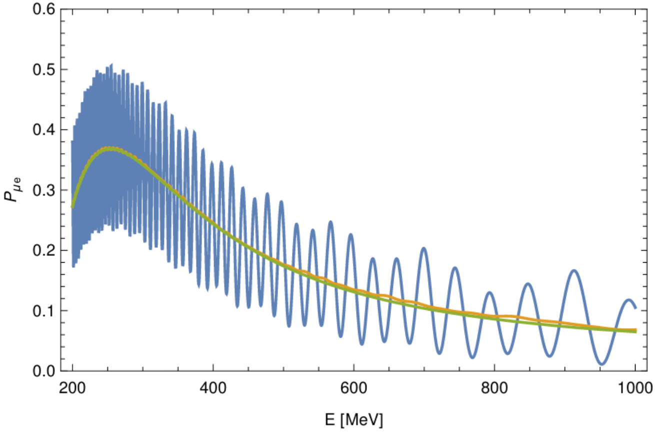
$$O^{mT}_{i13} O^{mT}_{(i+1)13} = \begin{pmatrix} \cos(\theta^m_{i13} - \theta^m_{(i+1)i13}) & 0 & \sin(\theta^m_{i13} - \theta^m_{(i+1)i13}) \\ 0 & 1 & 0 \\ -\sin(\theta^m_{i13} - \theta^m_{(i+1)i13}) & 0 & \cos(\theta^m_{i13} - \theta^m_{(i+1)i13}) \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(10^{-2})$$

$$S^m \approx O^m_{13-first} T\Pi_i \underbrace{\left(O^m_{i12} \mathcal{E}_i O^{mT}_{i12} \right)}_{\text{2x2 matrix}} O^{mT}_{13-last}$$

3. Assume $O^m_{13-first} = O^m_{13-last} = O_{13}$ & obtain simplified S^m matrix:

$$S^m \approx U_0 \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{21} & X_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} U_0^\dagger + \Pi_i \left(\mathcal{E}_i \right)_{33} U_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} U_0^\dagger \equiv A + \Pi_i \left(\mathcal{E}_i \right)_{33} B, \quad U_0 = O_{23} U_\delta O_{13}$$

Numerical vs analytical averaging



- $\theta = \pi/10$
- Exact $P_{\alpha\beta}$ - blue line
- Numerical averaging $\hat{P}_{\alpha\beta}$ - orange line
- Analytical averaging $\bar{P}^m_{\alpha\beta}(E,\theta)$ - green line

Numerical averaging

$$\hat{P}_{\alpha\beta}(E,\theta) = \frac{1}{4\Delta E}\int_{E-2\Delta E}^{E-2\Delta E} P_{\alpha\beta}\left(E'\right) dE' d\theta$$

Averaging over 4 periods ΔE of "fast" oscillation in energy:

$$\Delta E = \frac{4\pi E}{\Delta m_a^2 L(\theta)}$$

Finite resolutions

$$\bar{P}_{\alpha\beta}(E,\theta) = \frac{1}{\Delta E \Delta\theta} \int_{E-\frac{\Delta E}{2}}^{E+\frac{\Delta E}{2}} \int_{\theta-\frac{\Delta\theta}{2}}^{\theta+\frac{\Delta\theta}{2}} P_{\alpha\beta}\left(E',\theta'\right) dE' d\theta'$$

$$= \frac{1}{\Delta\theta} \int_{\theta-\frac{\Delta\theta}{2}}^{\theta+\frac{\Delta\theta}{2}} P_{\alpha\beta}\left(E,\theta'\right) d\theta' + \mathcal{O}\left(\frac{\Delta E^2}{E^2}\right)$$