A Short Glimpse into the Loop Vertex Expansion

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Motvation : Divergence of perturbative expansions

Perturbative expansion in QFT over **Feynman graphs**

$$\log Z = \int [\mathcal{D}\phi] \exp - \int \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 \right\}$$

$$\text{"} = \text{"} \sum_{G \text{ Feynman graph}} \mathcal{A}(G)\, g^{\#\text{vertices}}$$

The perturbative expansion is a **divergent** power series (otherwise Z defined for $\mathrm{Re}(g) < 0$, $g = 0$ boundary of analyticity domain).

Perturbative expansion only valid as an **asymptotic series** for $g \to 0$ but does not allow for a definition of a QFT.

Origins of the divergence : $\sum_{G \text{ order } n} \mathcal{A}(G) \sim n!$

- **too many graphs** of given order (instantons)
- **too large graph amplitudes** at given order (renormalons)

Construction of QFT from its perturbative expansion usually addressed using **Borel summation**.

Combinatorial approach : Loop Vertex Expansion

Basic idea (V. Rivasseau, `arxiv 0706.1224`) : **expand the partition function over forests** (= not necessarily connected graphs without loops) over instead of graphs and **logarithm expanded over trees** (connected components)

$$Z=\sum_{F\ \text{forest}}\mathcal{A}_F(g)\qquad\Leftrightarrow\qquad\log Z=\sum_{T\ \text{tree}}\mathcal{A}_T(g)$$

Convergence of the expansion possible because of power law growth (solving the "too many graphs" issue)

$$\#\begin{pmatrix}\text{trees of}\\ \text{order }n\end{pmatrix}\underset{n\to+\infty}{\sim}\kappa^n\quad\text{vs}\quad\#\begin{pmatrix}\text{graphs of}\\ \text{order }n\end{pmatrix}\underset{n\to+\infty}{\sim}n!$$

and power law bounds on tree amplitudes $|\mathcal{A}_T(g)|\leq C^n|g|^n$

Usual perturbative expansion recovered by further expanding $\mathcal{A}_T(g)$ in powers of g (addition of loops to T)

Open question in QFT but interesting results for random matrices.

Random Matrices

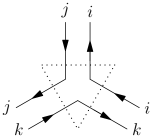
Topological **ribbon graph expansion** of matrix integral

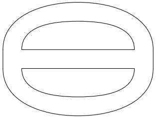
$$\frac{1}{N^2}\log\int DM \exp -N\left\{\mathrm{Tr}\, M^2 + g\,\mathrm{Tr}\, M^{2p}\right\} = \sum_{G \text{ ribbon graph}} \mathcal{A}_G\, g^{\#(\text{vertices})} N^{\chi(G)}$$

with $\chi = 2 - \text{genus} = \#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces})$

Ribbon Feynman graph (double line) dual to trianagulations

$$\mathrm{Tr}\, M^3 = \sum_{i,j,k} M_{ij} M_{jk} M_{ki} \rightarrow$$





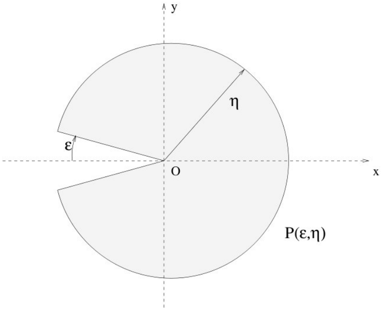
Multiple occurence in physics as random Hamiltonians (spectra of heavy nuclei, JT gravity in the Schwarzian limit, ..) or topological expansion (large N QCD, 2d gravity, ...).

Main result : Uniform analyticity in a "Pacman" domain

For any $\epsilon > 0$ there exists $\eta > 0$ such that the LVE expansion

$$\frac{1}{N^2} \log \int DM \exp -N\Big\{\mathrm{Tr}\, M^2 + g \mathrm{Tr}\, M^{2p}\Big\} = \sum_{T \text{tree}} \mathcal{A}_T(g, N)$$

defines an analytic function of $g \in \{0 < |g| < \eta, |\arg \lambda| < \pi - \epsilon\}$.
and is bounded by a constant independent of N.



See arxiv 1712.05670 and 1910.13261(Rivasseau, Sazonov, and K.)

Forest Formula (Abdesselam, Brydges, Kennedy, Rivasseau)

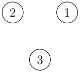
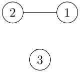
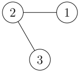
ϕ function of $\frac{n(n-1)}{2}$ variables $x_{ij} \in [0,1]$(edges between n vertices)

$$\phi(1,\ldots,1) = \sum_{\substack{F \text{ forest} \\ \text{on } n \text{ vertices}}} \int_0^1 \prod_{(i,j)\in F} du_{ij} \left(\frac{\partial^{\#(\text{edges in } F)}\phi}{\prod_{(i,j)\in F} \partial x_{ij}} \right) (v_{ij}) \ ,$$

where v_{ij} is the infimum of u_{kl} along the path from from i to j in F if it exists and 0 otherwise

- $n = 2$: 2 forests ,

$$\phi(1) = \phi(0) + \int_0^1 du_{12} \left(\frac{\partial \phi}{\partial x_{12}} \right) (u_{12})$$

- $n = 3$: , , ... , ...

$$\phi(1,1,1) = \phi(0,0,0) + \int_{[0,1]} du_{12} \left(\frac{\partial \phi}{\partial x_{12}} \right) (u_{12},0,0) + \text{perm.}$$

$$+ \int_{[0,1]^2} du_{12} du_{23} \left(\frac{\partial^2 \phi}{\partial x_{12} \partial x_{23}} \right) (u_{12}, u_{23}, \inf(u_{12}, u_{23})) + \text{perm.}$$

Tree expansion of the matrix integral

Partial expansion of the potential and introduction of n copies of A

$$\int DA \exp -\left\{ N \operatorname{Tr} A^2 + V(A) \right\}$$

$$= \int DA \exp -\left\{ N \operatorname{Tr} A^2 \right\} \left(\sum_n (-1)^n \frac{\left[V(A)\right]^n}{n!} \right)$$

$$= \sum_n \frac{(-1)^n}{n!} \int DA_1 \cdots DA_n \exp -\left\{ N \sum_{1 \le i,j \le n} C_{ij}^{-1} \operatorname{Tr}(A_i A_j) \right\}$$

$$V(A_1) \cdots V(A_1) \underbrace{\Big|_{C_{ij}=1}}_{\text{sets } A_i = A_j}$$

$$= \sum_{F \text{ forest}} \mathcal{A}_F$$

Conclusion from the forest formula with $x_{ij} = C_{ij}$ and sum over trees from logarithm (connected parts)

Bounds from a change of variables

Change of variable $A = M\sqrt{1 + gM^{2p-1}}$ in the partition function

$$\int DM \exp -N\Big\{\mathrm{Tr}\, M^2 + g\,\mathrm{Tr}\, M^{2p}\Big\} =$$

$$\int DA \exp -\Big\{N \mathrm{Tr}\, A^2 + V_{eff}(A)\Big\}$$

with effective potential computed from the Jacobian

$$V_{eff}(A) = \mathrm{Tr}_{\otimes} \log \frac{\partial M}{\partial A} =$$

$$\mathrm{Tr}_{\otimes} \log \left\{ \frac{A\sqrt{T(-gA^{p-2})} \otimes 1 - 1 \otimes A\sqrt{T(-gA^{p-2})}}{A \otimes 1 - 1 \otimes A} \right\}$$

with T Fuss-Catalan function such that $T(z) = 1 + zT^p(z)$.

Derivative of log = resolvent and analytic properties of $T(z)$ lead to useful bounds on tree amplitudes establishing the theorem.

Towards a similar approach in Quantum Field Theory

Change of variables from **Morse-Palais lemma** : reduction of a functional around a critical point in Hilbert space to a quadratic form $S[\phi] = \langle \chi(\phi), \chi(\phi) \rangle$

$$\int \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 \right\} = \int \left\{ \frac{1}{2}(\partial\chi)^2 + \frac{m^2}{2}\chi^2 \right\}$$

leading to the non local effective potential (Jacobian)

$$V_{\text{eff}}[\chi] = \log\det\frac{\delta\phi}{\delta\chi} = \text{Tr}\,\log\frac{\delta\phi}{\delta\chi}$$

Difficulty : find suitable **cut-off independent bounds**.

Matrix model with kinetic term (Grosse-Wulkenhaar model)

$$\int DM \exp -\left\{ \text{Tr}\, KM^2 + g\text{Tr}\, M^4 \right\}$$

2d case by V. Rivasseau and Z.T. Wang `arxiv1805.06365`.