EISA

European Institute for Sciences and Their Applications

Workshop on the Standard Model and Beyond

AUGUST 29 - SEPTEMBER 8, 2021

THE HIGH ENERGY BEHAVIOUR OF THE SCATTERING AMPLITUDES IN NON-RENORMALIZABLE THEORIES

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Motivation:

- The Standard Model is renormalizable
- Gravity is not renormalizable

Non-renormalizable theories are not accepted due to:

- UV divergences are not under control - infinite number of new types of divergences
- The amplitudes increase with energy (in PT) and violate unitarity

However:

- R-operation equally works for NR theories and leads to local counter terms
- Due to locality all higher order divergences are related to the lower ones

■ These properties allow one to write down the RG equations for the scattering amplitudes which sum up the leading divergences (logarithms) and to find out the high energy behaviour

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Based on: Phys. Lett. B734 (2014), arXiv:1404.6998 [hep-th]
JHEP 1511 (2015) 059, arXiv:1508.05570 [hep-th]
JHEP 1612 (2016) 154, arXiv:1610.05549v2 [hep-th]
Phys.Rev. D95 (2017) no.4, 045006 arXiv:1603.05501 [hep-th]
Phys.Rev. D97 (2018) no.12, 125008, arXiv:1712.04348 [hep-th],
Phys.Lett. B786 (2018) 327-331, arXiv:1804.08387 [hep-th]
Phys.Lett.B 797 (2019) 134801, arXiv:1904.08690 [hep-th]
Труды Мат. Инст. им. В.А. Стеклова, 2020, т. 308, с. 1–8

In collaboration with L.Bork, A.Borlakov, D.Tolkachev and D.Vlasenko

Renormalization

Bogolyubov Laboratory of Theoretical Physics

Bogolyubov-Parasiuk Theorem: In any local quantum field theory to get the UV finite S-matrix one has to introduce local counter terms to the Lagrangian in each order of perturbation theory - R-operation

$$\mathcal{L} \Rightarrow \mathcal{L} + \Delta\mathcal{L}$$

In renormalizable case this is equivalent to the operation of multiplication by a renormalization constant Z

Consider 2->2 scattering amplitude on shell

$$\Gamma_4(s,t,u) = \Gamma_4^{tree}\bar{\Gamma}_4(s,t,u) \qquad \bar{\Gamma}_4 = 1 + \lambda ... + \lambda^2 ...$$

Renormalization (dimensional regularization)

BPHZ R-operation

$$\bar{\Gamma}_4 = Z_4(\lambda)\bar{\Gamma}_4^{bare}|_{\lambda_{bare} -> \lambda Z_4},$$
$$\lambda_{bare} = \mu^\epsilon Z_4(\lambda)\lambda$$

$$RG = (1-K)R'G$$

$$Z = 1 - \sum_i KR'G_i$$

Renormalization

In <u>non-renormalizable</u> case the BP theorem is still valid and the counter terms are also local (at maximum are polynomial over momenta)

Kazakov,18

- <u>Multiplication</u> operation is replaced by acting of an <u>operator</u> $Z \to \hat{Z}$

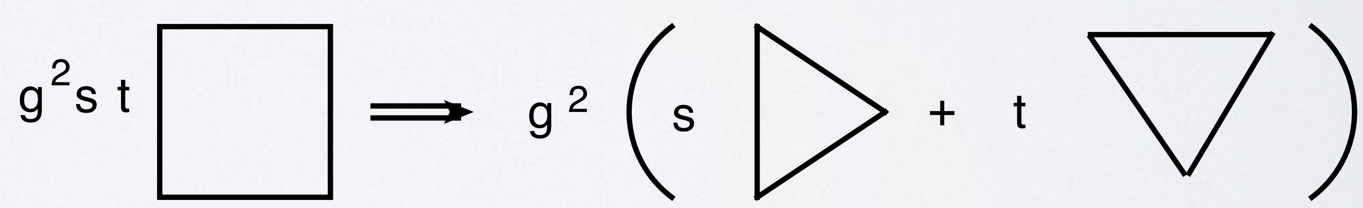
$\hat{Z}$ is a function (polynomial) of momenta (s,t,u for the 4-point case)

- When acting on the diagram the $\hat{Z}$ factor has to inserted inside the diagram and integrated over the internal loop

Example (taken from D=8 YM theory)

Exactly follows the BPHZ R-operation

$$\hat{Z} = 1 + g^2 \frac{st}{\epsilon} \qquad \Rightarrow$$



Either s or t are to be inserted into the loop and integrated

BPHZ R-operation

$$\mathcal{R}'G_n = \frac{A_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + \ldots + \frac{A_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^n}$$

$$+ \quad \text{lower pole terms}$$

$A_k^{(n)}(\mu^2)^{k\epsilon}$ terms appear after subtraction of (n-k) loop counter terms

Statement: $R'G_n$ is local, i.e. terms like $\log^k \mu^2/\epsilon^m$ should cancel for any k and m

Consequence: $A_n^{(n)} = (-1)^{n+1}\frac{A_1^{(n)}}{n}$

$$KR'G_n = \sum_{k=1}^{n}\left(\frac{A_k^{(n)}}{\epsilon^n}\right) \equiv \frac{A_n^{(n)'}}{\epsilon^n} \qquad A_n^{(n)'} = (-1)^{n+1}A_n^{(n)} = \frac{A_1^{(n)}}{n}.$$

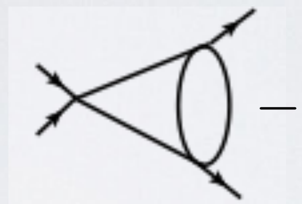
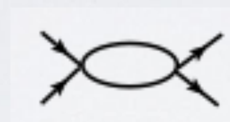
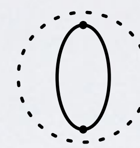
$A_1^{(n)}$ is the contribution to the leading pole in n-loops from the diagrams appearing in due corse of R-operation after subtraction of (n-1) loop counter terms

The leading divergences are governed by 1 loop diagrams!

Two loop example

ϕ^4_4

 $= \left(\frac{A_2^{(2)}}{\epsilon^2} + \frac{A_1^{(2)}}{\epsilon}\right)(\frac{\mu^2}{s})^{2\epsilon}$

$\mathcal{R}'$  $=$  $-$   $= \left(\frac{A_2^{(2)}}{\epsilon^2} + \frac{A_1^{(2)}}{\epsilon}\right)(\frac{\mu^2}{s})^{2\epsilon} - \frac{A_1^{(1)}}{\epsilon}(\frac{\mu^2}{s})^{\epsilon}\frac{A_1^{(1)}}{\epsilon}$

$$= \frac{A_2^{(2)}}{\epsilon^2} - \frac{(A_1^{(1)})^2}{\epsilon^2} + 2\frac{A_2^{(2)}}{\epsilon}\log(\mu^2/s) - \frac{(A_1^{(1)})^2}{\epsilon}\log(\mu^2/s) = -\frac{1}{2}\frac{(A_1^{(1)})^2}{\epsilon^2} + \ldots$$

non-local terms to be cancelled

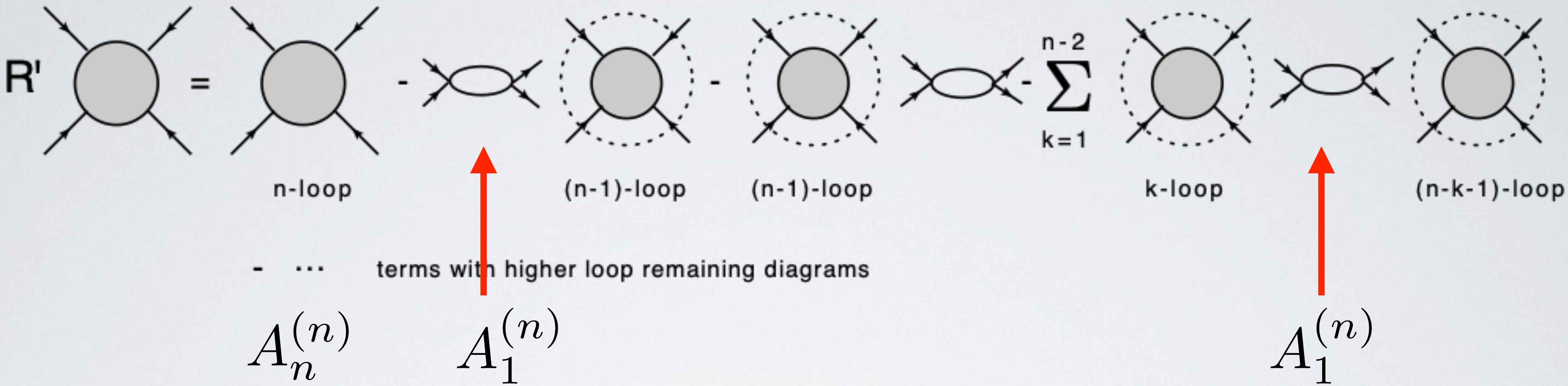
Leading divergence is given by the one-loop term $A_2^{(2)} = \frac{1}{2}(A_1^{(1)})^2$

ϕ^4_D

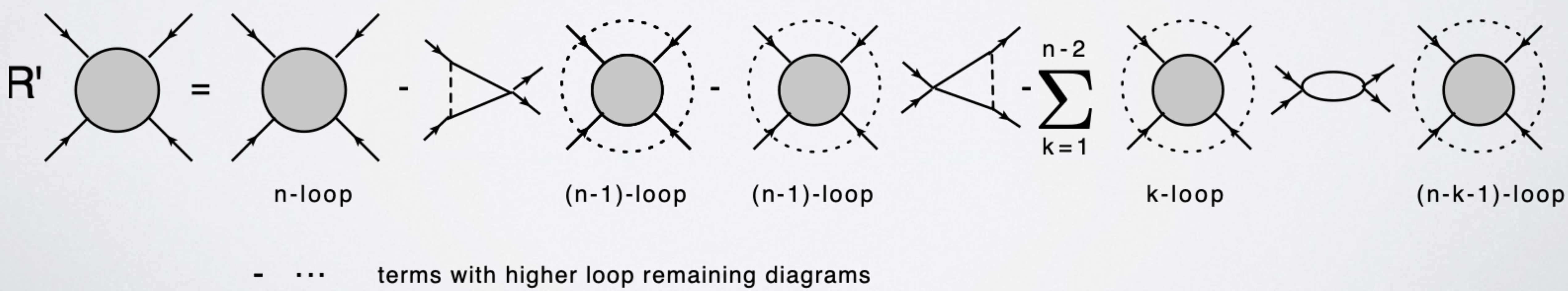
- These statements are universal and are valid in non-renormalizable theories as well.
- The only difference is that the counter term $A_1^{(1)}$ depends on kinematics and has to be integrated through the remaining one-loop graph.
- As a result $A_2^{(2)}$ is not the square of $A_1^{(1)}$ anymore but is the integrated square (see below).
- This last statement is the general feature of any QFT irrespective of renormalizability

Leading divergences

Quartic vertices

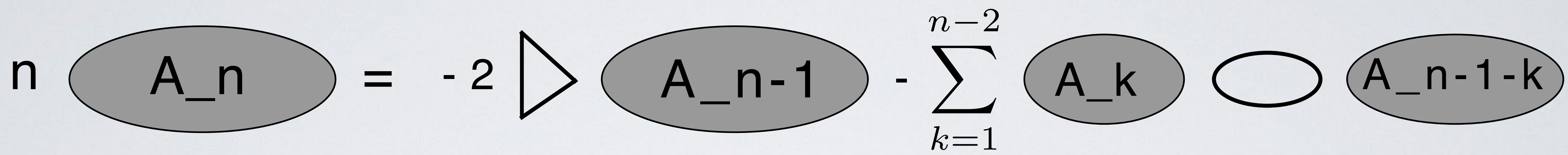


Cubic vertices



The Recurrence Relation

Kazakov,20



- This is the general recurrence relation that reflects the locality of the counter terms in any theory
- In renormalizable theories A_n is a constant and this relation is reduced to the algebraic one
- In non-renormalizable theories A_n depends on kinematics and one has to integrate through the one loop diagrams

Taking the sum $\sum_n A_n(-z)^n = A(z)$ one can transform the recurrence relation into integro-diff equation

$$\frac{d}{dz}A(z) = b_0\{-1 - 2\int_\triangle A(z) - \int_\bigcirc A^2(z)\}$$

$$\frac{d}{dz} = \frac{d}{d\log\mu^2}$$

This is the generalized RG equation valid in any (even non-renormalizable) theory!

Examples:

- Maximally supersymmetric gauge theory in D=6,8,10 dimensions SYM_D
- Scalar field theory in D=4,6,8,10 dimensions ϕ^4_D
- Gauge theory in D=4,6,8 dimensions YM_D
- Supersymmetric Wess-Zumino model with quartic superpotential in D=4 Φ^4_4

These are the toy models for (super) gravity - our aim

The Scalar theory example

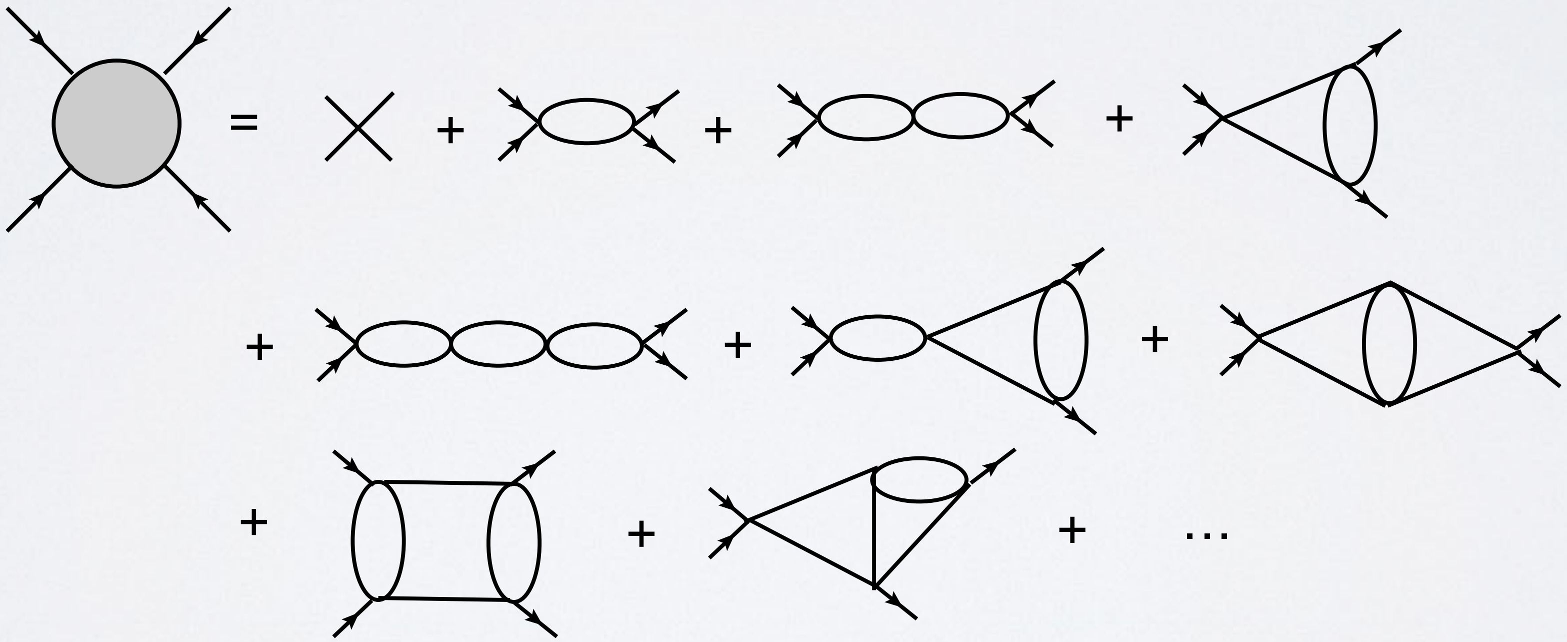
ϕ^4_D $\quad D = 4, 6, 8, 10 \quad [\lambda] = 2 - D/2$

Kazakov,19

2->2 scattering amplitude on shell $\quad m = 0 \quad s + t + u = 0$

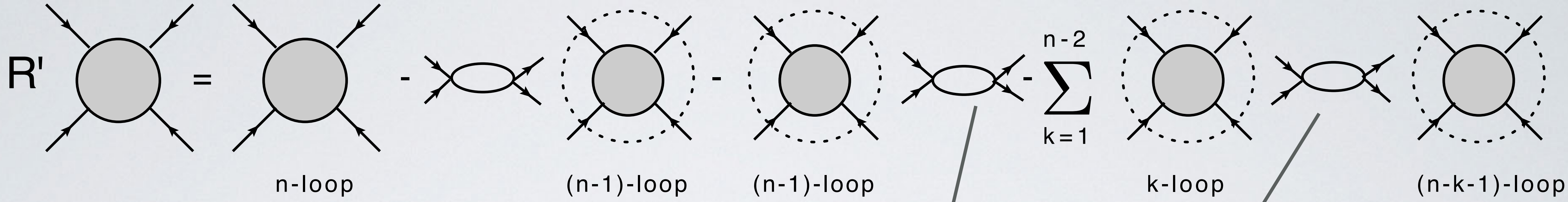
$$\Gamma_4(s,t,u) = \lambda(1 + \Gamma_s(s,t,u) + \Gamma_t(s,t,u) + \Gamma_u(s,t,u))$$

PT:
$$\Gamma_s = \sum_{n=1}^{\infty}(-z)^n S_n, \quad \Gamma_t = \sum_{n=1}^{\infty}(-z)^n T_n, \quad \Gamma_u = \sum_{n=1}^{\infty}(-z)^n U_n, \quad z \equiv \frac{\lambda}{\epsilon}$$

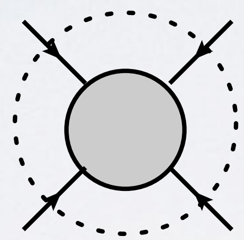


PT expansion (only s-channel is shown)

Recurrence Relations for the Leading Poles



- ··· terms with higher loop remaining diagrams

KR' 

$$= \frac{A_k^{(k)'}}{\epsilon^k} = (-1)^{k+1}\frac{A_k^{(k)}}{\epsilon^k}$$

$$\begin{aligned} nS_n(s,t,u) \;=\; & \frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\left(S_{n-1}(s,t',u')+T_{n-1}(s,t',u')+U_{n-1}(s,t',u')\right) \\ + \; & \frac{1}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\sum_{k=1}^{n-2}\sum_{p=0}^{(D/2-2)k}\sum_{l=0}^{p}\frac{1}{p!(p+D/2-2)!}\times \\ \times \; & \frac{d^p}{dt'^l du'^{p-l}}(S_k+T_k+U_k)\frac{d^p}{dt'^l du'^{p-l}}(S_{n-k-1}+T_{n-k-1}+U_{n-k-1})s^p[x(1-x)]^p t^l u^{p-l} \end{aligned}$$

$$t' = -xs, u' = -(1-x)s$$

Differential Equation

Summing up the recurrence relation $\sum_{n=2}^{\infty}(-z)^n$ one gets the diff equation

$$-\frac{d\Gamma_s(s,t,u)}{dz} = \frac{1}{2}\frac{\Gamma(D/2-1)}{\Gamma(D-2)}s^{D/2-2} \qquad \Gamma_s(z=0)=0$$

$$+\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\left[\Gamma_s(s,t',u')+\Gamma_t(s,t',u')+\Gamma_u(s,t',u')\right]\Big|_{\substack{t'=-xs,\\ u'=-(1-x)s}}$$

$$+\frac{1}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\sum_{p=0}^{\infty}\sum_{l=0}^{p}\frac{1}{p!(p+D/2-2)!}\times$$

$$\times\left(\frac{d^p}{dt'^l du'^{p-l}}(\Gamma_s+\Gamma_t+\Gamma_u)\Big|_{\substack{t'=-xs,\\ u'=-(1-x)s}}\right)^2 s^p[x(1-x)]^p t^l u^{p-l}$$

$$\frac{d\Gamma_s(s,t,u)}{d\log\mu^2} = -\frac{\lambda}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\sum_{p=0}^{\infty}\sum_{l=0}^{p}\frac{1}{p!(p+D/2-2)!}\times$$

$$\times\left(\frac{d^p\bar{\Gamma}_4(s,t',u')}{dt'^l du'^{p-l}}\Big|_{\substack{t'=-xs,\\ u'=-(1-x)s}}\right)^2 s^p[x(1-x)]^p t^l u^{p-l}$$

$$\Gamma_s(\log\mu^2=0)=0$$

SYM_D

Perturbation Expansion for the 4-point Amplitudes for any D

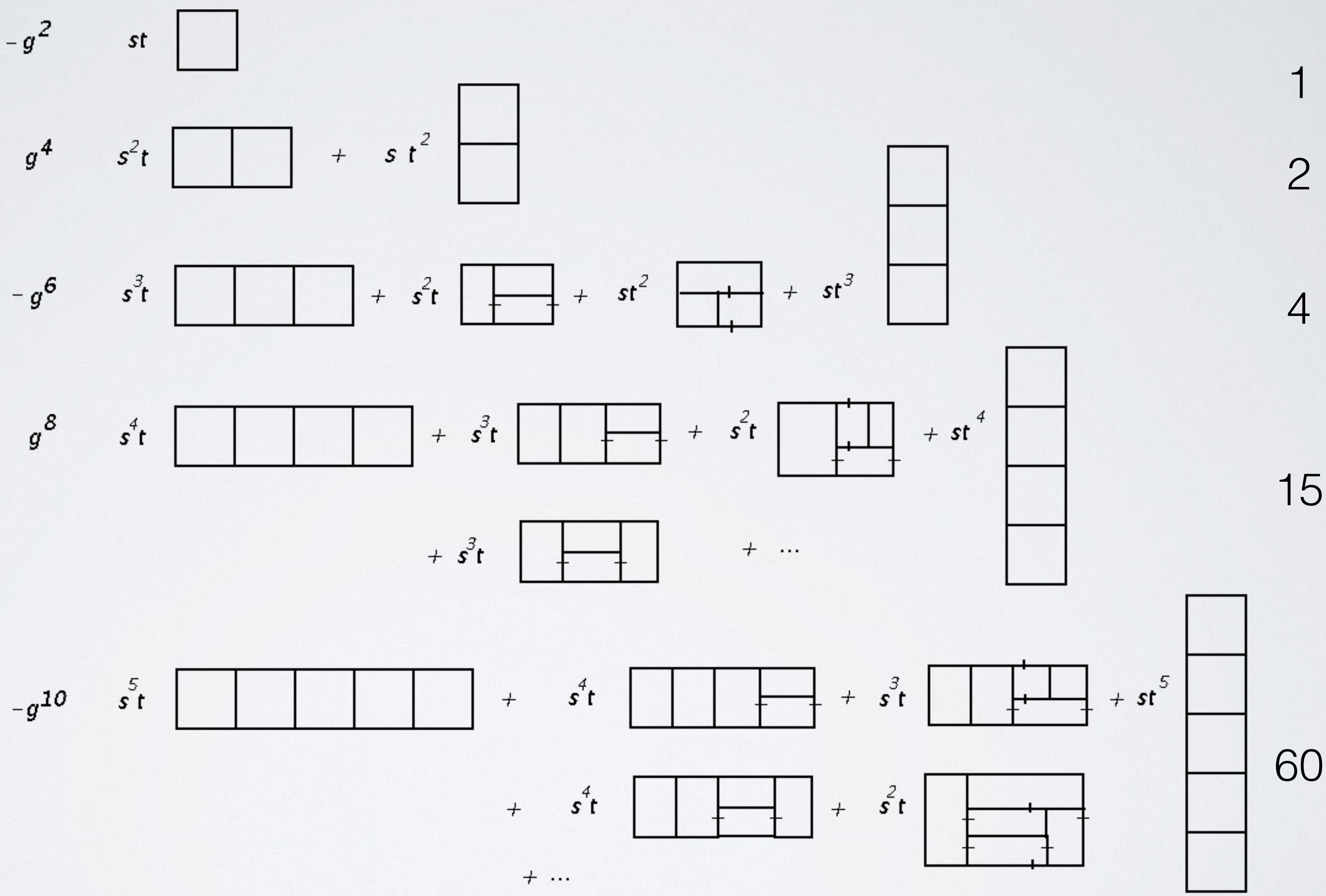
T. Dennen Yu-yin Huang 10 ,
S.Caron-Huot D.O'Connell 10

$$A_4/A_4^{tree}$$

No bubbles
No Triangles

First UV div at
L=[6/(D-4)] loops

IR finite



Universal expansion for any D in maximal SYM due to Dual conformal invariance

Bork,Kazakov,Kompaneets,Vlasenko, 13

Borlakov,,Kazakov,Tolkachev,Vlasenko, 15

SYM_D

D=6 N=2

S-channel $S_n(s,t)$ **T-channel** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact all-loop recurrence relation $S_3 = -s/3,\ T_3 = -t/3$

$$nS_n(s,t) = -2s \int_0^1 dx \int_0^x dy\ (S_{n-1}(s,t') + T_{n-1}(s,t'))$$

$n \geq 4$

$t' = t(x-y) - sy$

D=8 N=1

S-channel $S_n(s,t)$ **T-channel** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact all-loop recurrence relation $S_1 = \frac{1}{12},\ T_1 = \frac{1}{12}$

$$\begin{aligned} nS_n(s,t) &= -2s^2 \int_0^1 dx \int_0^x dy\ y(1-x)\ (S_{n-1}(s,t') + T_{n-1}(s,t'))|_{t'=tx+yu} \\ + \quad & s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!} \frac{d^p}{dt'^p}(S_k(s,t') + T_k(s,t')) \times \\ & \times \frac{d^p}{dt'^p}(S_{n-1-k}(s,t') + T_{n-1-k}(s,t'))|_{t'=-sx}\ (tsx(1-x))^p \end{aligned}$$

RG Equation

SYM_D

D=6 N=2

$$\Sigma(s,t,z) = z^{-2}\sum_{n=3}^{\infty}(-z)^n S_n(s,t)$$

$$\frac{d}{dz}\Sigma(s,t,z) = s - \frac{2}{z}\Sigma(s,t,z) + 2s\int_0^1 dx \int_0^x dy\ \left(\Sigma(s,t',z) + \Sigma(t',s,z)\right)|_{t'=xt+yu}$$

Linear equation

D=8 N=1

$$\Sigma(s,t,z) = \sum_{n=1}^{\infty}(-z)^n S_n(s,t)$$

$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{1}{12} + 2s^2\int_0^1 dx \int_0^x dy\ y(1-x)\ \left(\Sigma(s,t',z) + \Sigma(t',s,z)\right)|_{t'=tx+yu}$$
$$-s^4\int_0^1 dx\ x^2(1-x)^2\sum_{p=0}^{\infty}\frac{1}{p!(p+2)!}\left(\frac{d^p}{dt'^p}\left(\Sigma(s,t',z)+\Sigma(t',s,z)\right)|_{t'=-sx}\right)^2\ (tsx(1-x))^p.$$

Non-linear equation

- YM_D Both cubic and quartic vertices

  Equation is more complicated but has the same main features

- Wess-Zumino modern in D=4

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\ \bar{\Phi}\Phi + \int d^2\bar{\theta}\ \frac{g}{4!}\bar{\Phi}^4 + \int d^2\theta\ \frac{g}{4!}\Phi^4,$$

$$C =< \Phi\Phi\Phi\Phi >, \quad \bar{C} =< \bar{\Phi}\bar{\Phi}\bar{\Phi}\bar{\Phi} >, \quad M =< \bar{\Phi}\bar{\Phi}\Phi\Phi > . \qquad C = CS + CT + CU, \quad M + MS + MT + MU$$

RG Equations

$$\frac{dCS}{dz} = sg^2 MS \otimes (CS + CT + CU)\,,$$

$$\frac{dMS}{dz} = \frac{1}{2}[sg^2(MS \otimes MS + MT \otimes MT + MU \otimes MU)$$
$$+\bar{C}S \otimes CS + \bar{C}T \otimes CT + \bar{C}U \otimes CU],$$

$$A(s,t,u) \otimes B(s,t,u) = \int_0^1 dx \sum_{p=0}^{\infty} \sum_{l=0}^{p} \frac{1}{p!p!} \frac{d^p}{dt'^l du'^{p-l}}\ A(s,t',u') \frac{d^p}{dt'^l du'^{p-l}}\ B(s,t',u')|_{\substack{t' = -xs, \\ u' = -(1-x)s}} \quad s^p [x(1-x)]^p t$$

Solution of RG Equations - Genaral Case

$$\frac{d}{dz}A(z) = b_0\{-1 - 2\int_\Delta A(z) - \int_\bigcirc A^2(z)\}$$

In the r.h.s. one has a second degree polynomial:

- Two real roots - solution is an exponent (decreasing or increasing depending on a theory and kinematics) SYM_6

- Degenerate real root - solution with a pole at low (Asymptotic Freedom) or high (Zero Charge) energies depending on a kinematics ϕ^4_D

- Two complex roots - solution with infinite number of periodic poles in both directions SYM_8

Solution of RG Equation

$D=4$

$$s \sim t \sim u \sim E^2$$

$$\frac{d\bar{\Gamma}_4}{d\log\mu^2} = -\lambda\frac{3}{2}\bar{\Gamma}_4^2, \quad \bar{\Gamma}_4(\log\mu^2=0)=1 \quad \rightarrow \quad \bar{\Gamma}_4 = \frac{1}{1+\frac{3}{2}\lambda\log(\mu^2/E^2)}$$

General Solution for any D

$$\bar{\Gamma}_4(s,t,u) = \mathcal{P}\frac{1}{1+\frac{1}{2}\frac{\Gamma(D/2-1)}{\Gamma(D-2)}\lambda(s^{D/2-2}+t^{D/2-2}+u^{D/2-2})\log(\mu^2/E^2)}$$

$\mathcal{P}$ is the symbol of ordering in a sense of recurrence relation

$$\Gamma_4(s,t,u) = \mathcal{P}\frac{\lambda}{1+\lambda A_1^{(1)}\log(\mu^2/E^2)} = \mathcal{P}\sum_{n=0}^{\infty}(-\lambda)^n\log^n(\mu^2/E^2)(A_1^{(1)})^n$$

$$\mathcal{P}(A_1^{(1)})^n = \int_0^1 dx \sum_{k=0}^{n-1} \overrightarrow{\mathcal{P}(A_1^{(1)})^k}\, A_1^{(1)}\, \overleftarrow{\mathcal{P}(A_1^{(1)})^{n-1-k}},$$

High Energy Behaviour of the scattering amplitude in ϕ^4_D theory

$$\Gamma_4(s,t,u) = \mathcal{P}\frac{\lambda}{1+\frac{1}{2}\frac{\Gamma(D/2-1)}{\Gamma(D-2)}\lambda(s^{D/2-2}+t^{D/2-2}+u^{D/2-2})\log(\mu^2/E^2)}$$

$$s \sim t \sim u \sim E^2$$

$D=4$ $\quad 3/2>0$ $\quad$ As a result one has a Landau pole as $E\to\infty$

$D=6$ $\quad s+t+u=0$ $\quad$ All the leading divergences (logs) cancel in all loops

One can explicitly check that S_2 given above vanishes

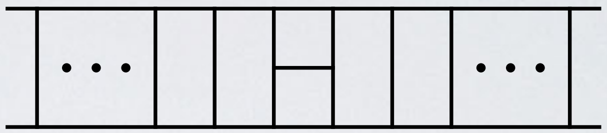
$D=8$ $\quad s^2+t^2+u^2>0$ $\quad$ has a Landau pole as $E\to\infty$

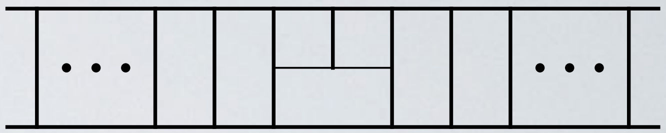
$D=10$ $\quad s^3+t^3+u^3=3stu>0$ $\quad s>0, t,u<0$ $\quad$ has a Landau pole as $E\to\infty$

<u>Conclusion:</u> $\quad \phi^4_D$ has a Landau pole as $E\to\infty$

Solution of RG equation D=6 N=2

Horizontal ladder + tennis court





$$\Sigma_L(s,z) = \frac{2}{s^2 z^2}(e^{sz} - 1 - sz - \frac{s^2 z^2}{2})$$

$$\Sigma_{L2} = \frac{1}{2s^2 z^2}\left[27(e^{z/3} - 1 - \frac{z}{3} - \frac{1}{2}\frac{z^2}{9} - \frac{1}{6}\frac{z^3}{27})(1 + 2\frac{t}{s}) - (e^z - 1 - sz - \frac{1}{2}z^2 - \frac{1}{6}z^3)\right]$$

In general case - numerical solution similar to the ladder approximation

$$\Sigma_s + \Sigma_t \sim e^{(s+t)z}$$

$$z \to \infty \qquad \begin{aligned} s+t &= -u > 0, \quad \Sigma \to \infty \\ s+u &= -t > 0, \quad \Sigma \to \infty \\ t+u &= -s < 0, \quad \Sigma \to const \end{aligned}$$

Solution of RG equation D=8 N=1

Borlakov,Kazakov,Tolkachev,Vlasenko, 16

Horizontal ladder

Diff equation
$$\frac{d}{dz}\Sigma_A = -\frac{1}{3!} + \frac{2}{4!}\Sigma_A - \frac{2}{5!}\Sigma_A^2 \qquad z = g^2 s^2/\epsilon$$

$$\Sigma_A(z) = -\sqrt{5/3}\frac{4\tan(z/(8\sqrt{15}))}{1-\tan(z/(8\sqrt{15}))\sqrt{5/3}} = \sqrt{10}\frac{\sin(z/(8\sqrt{15}))}{\sin(z/(8\sqrt{15})-z_0)}$$

$$\Sigma(z) = -(z/6 + z^2/144 + z^3/2880 + 7z^4/414720 + \dots) \qquad z_0 = \arcsin(\sqrt{3/8})$$

infinite number of poles

In general case - numerical solution similar to the ladder approximation possessing infinite number of poles in both directions

Resume

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- The UV divergences in non-renormalizable theories are local and can be removed by local counter terms like in renormalizable ones

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- **The UV divergences in non-renormalizable theories are local and can be removed by local counter terms like in renormalizable ones**

- **The main difference is that the renormalization constant Z depends on kinematics and acts like an operator rather than simple multiplication**

Bogoliubov Laboratory of Theoretical Physics

Resume

- The UV divergences in non-renormalizable theories are local and can be removed by local counter terms like in renormalizable ones
- The main difference is that the renormalization constant Z depends on kinematics and acts like an operator rather than simple multiplication
- Based on locality of the counter terms due to the Bogoliubov-Parasiuk theorem one can construct the recurrence relations that define all loop divergences starting from one loop

Resume

- **The UV divergences in non-renormalizable theories are local and can be removed by local counter terms like in renormalizable ones**

- **The main difference is that the renormalization constant Z depends on kinematics and acts like an operator rather than simple multiplication**

- **Based on locality of the counter terms due to the Bogoliubov-Parasiuk theorem one can construct the recurrence relations that define all loop divergences starting from one loop**

- **The recurrence relations can be converted into the generalized RG equations just like in renormalizable theories**

Resume

- The UV divergences in non-renormalizable theories are local and can be removed by local counter terms like in renormalizable ones

- The main difference is that the renormalization constant Z depends on kinematics and acts like an operator rather than simple multiplication

- Based on locality of the counter terms due to the Bogoliubov-Parasiuk theorem one can construct the recurrence relations that define all loop divergences starting from one loop

- The recurrence relations can be converted into the generalized RG equations just like in renormalizable theories

- The RG equations allow one to sum up the leading (subleading, etc) divergences in all loops and define the high-energy behaviour