Mass hierarchies from residual modular symmetries

in collaboration with S.T. Petcov, P.P. Novichkov

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Corfu *Workshop on the SM and Beyond* 2 September 2021

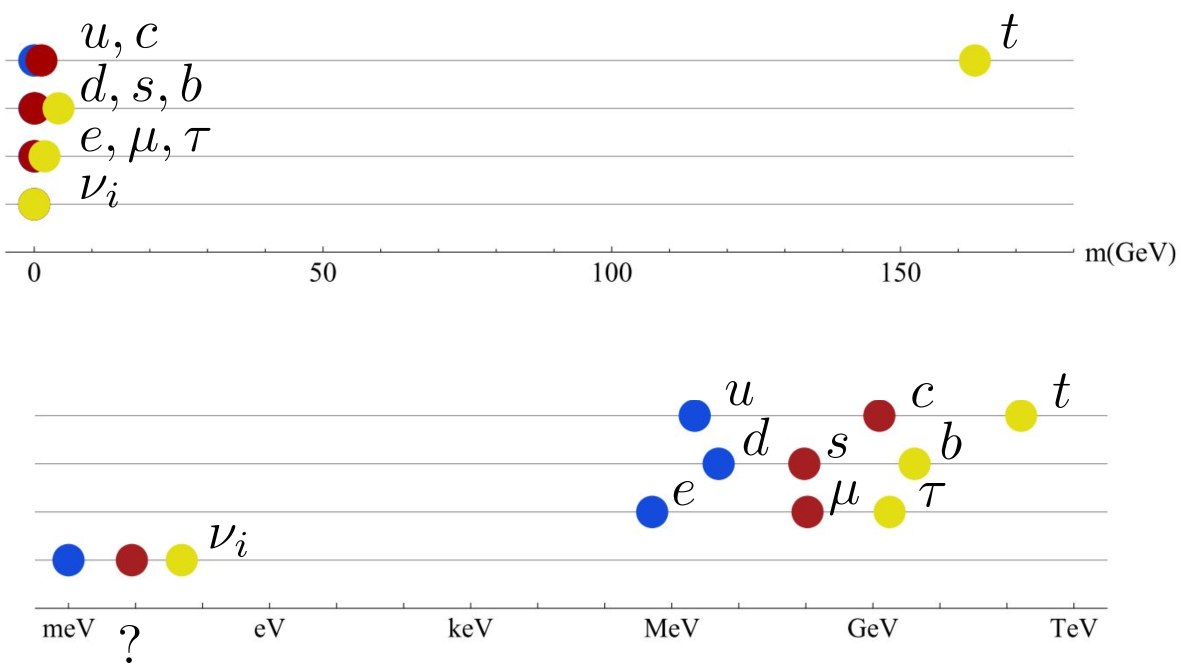
Modular flavour symmetries

Why?

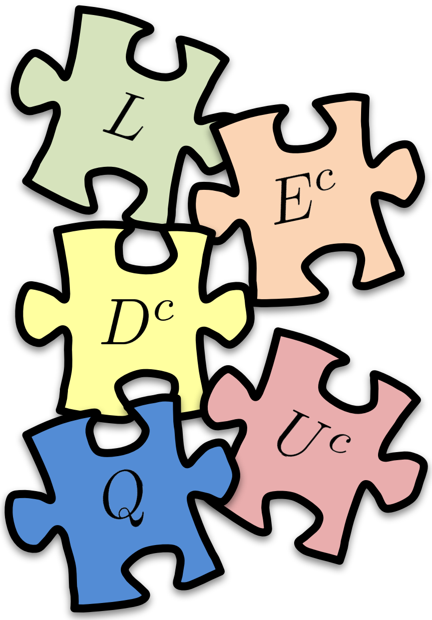
How?

What?

The flavour puzzle

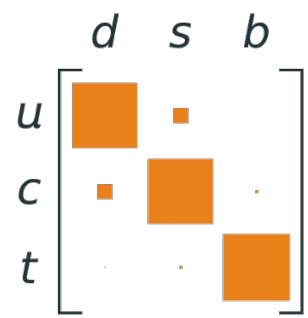


adapted from R. Toorop's PhD thesis

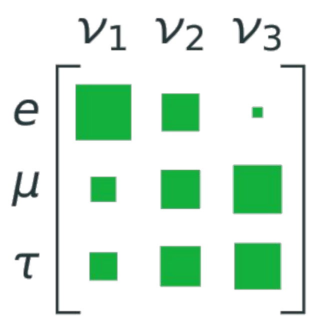


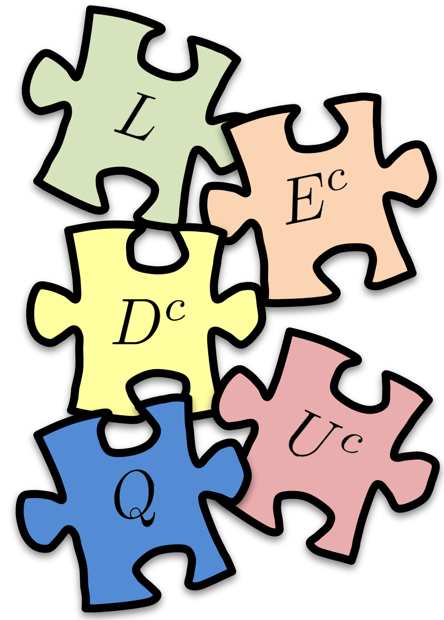
The flavour puzzle

$V_{\text{CKM}} \sim$



$U_{\text{PMNS}} \sim$





adapted from P. Novichkov's slides at PASCOS 2021

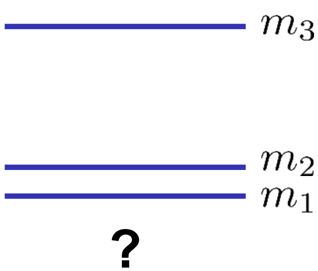
3ν flavour paradigm

Masses: ordering

$$\frac{\Delta m^2_{\odot}}{|\Delta m^2_A|} \sim \frac{1}{30}$$

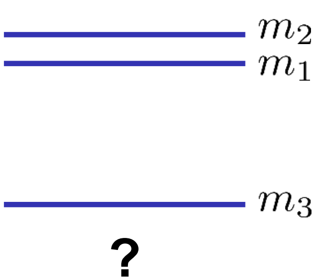
Normal ordering (NO) vs. Inverted ordering (IO)

$m_1 < m_2 < m_3$



vs.

$m_3 < m_1 < m_2$



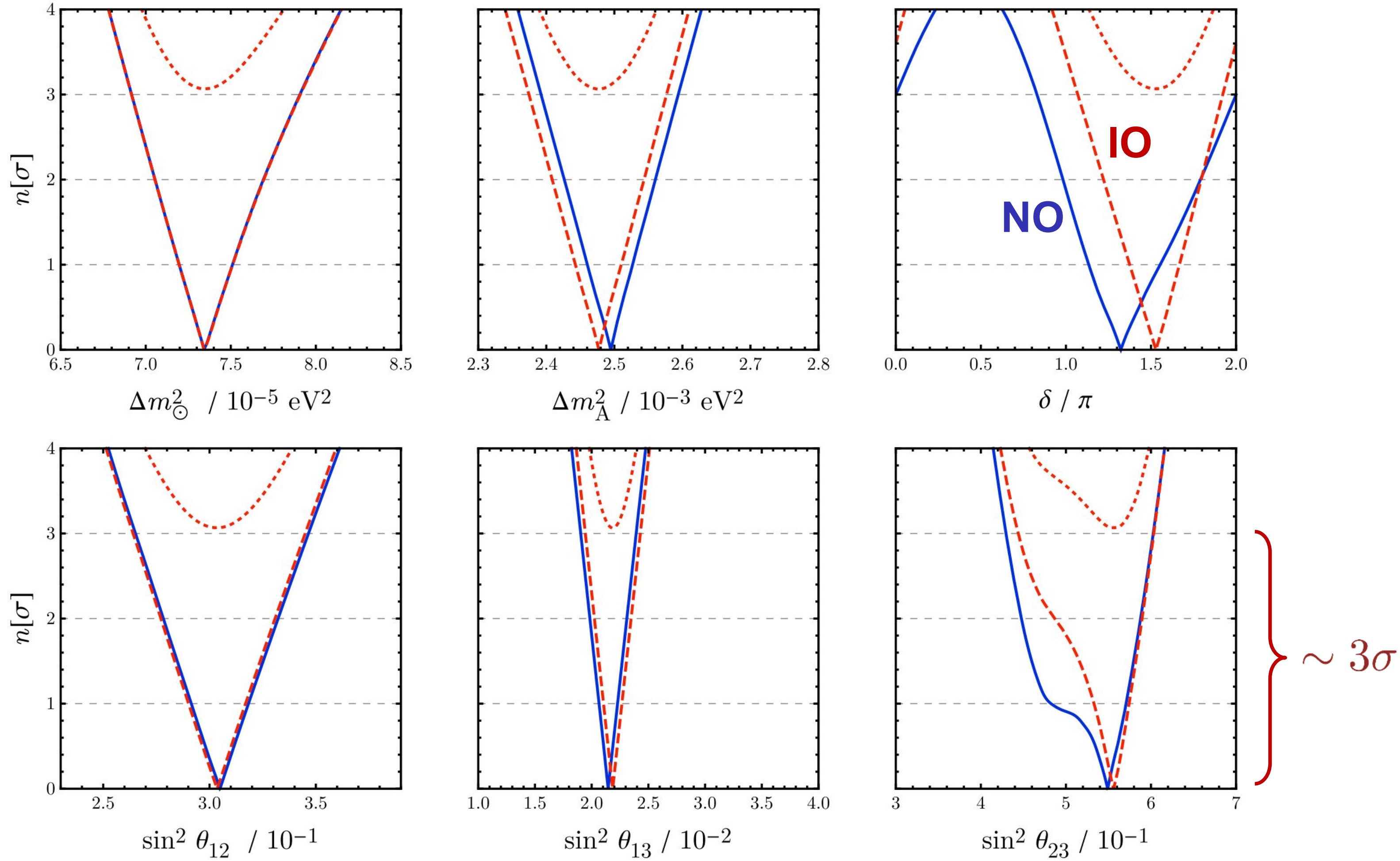
Mixing: parameterisation

$$c_{ij} \equiv \cos\theta_{ij}, \; s_{ij} \equiv \sin\theta_{ij}$$

$$U_{\mathrm{PMNS}} = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & e^{i\alpha_{21}/2} & \\ & & e^{i\alpha_{31}/2} \end{pmatrix}$$

3ν flavour paradigm (cont.)

from Capozzi et al. 1804.09678,
see also 1811.05487, **2003.08511**

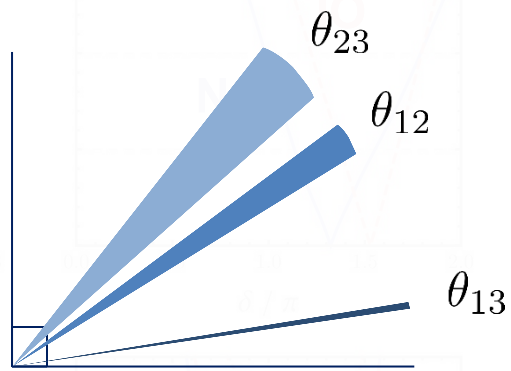


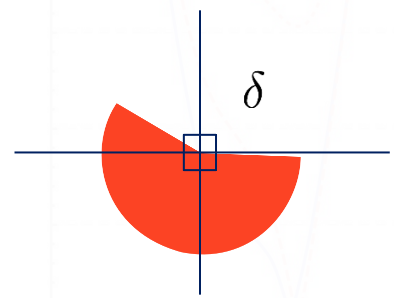
3ν flavour paradigm (cont.)

from Capozzi et al. 1804.09678,
see also 1811.05487, **2003.08511**

For a spectrum with NO:

Parameter	Best-fit value
$\Delta m^2_\odot$	7.34×10^{-5} eV^2
$\lvert\Delta m^2_{\mathrm{A}}\rvert$	2.49×10^{-3} eV^2
$\sin^2\theta_{12}$	0.304
$\sin^2\theta_{13}$	0.0214
$\sin^2\theta_{23}$	0.551
δ	$1.32\,\pi$

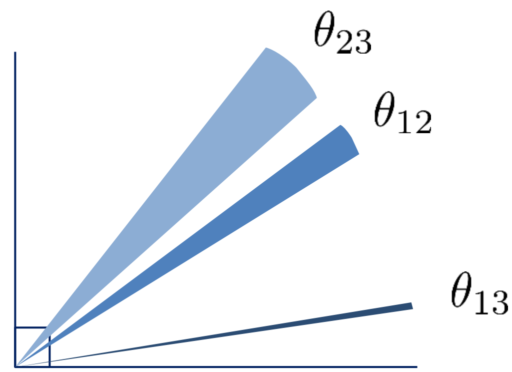


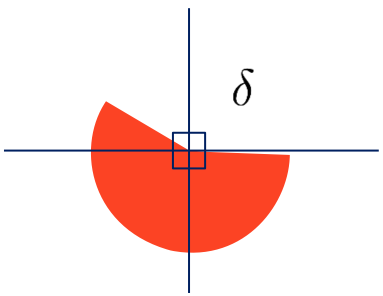


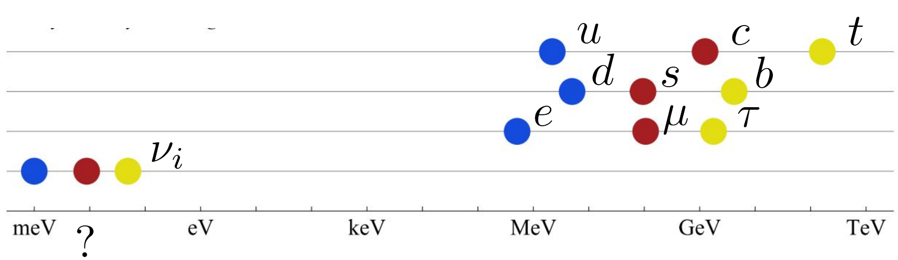
$\sin^2\theta_{12}$ / 10^{-1} $\sin^2\theta_{13}$ / 10^{-2} $\sin^2\theta_{23}$ / 10^{-1}

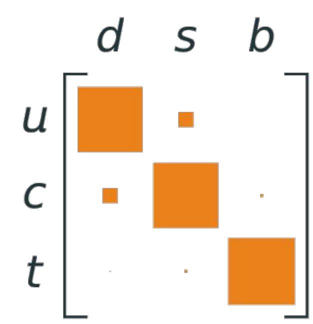
Is there an organizing principle behind this?

$$\frac{\Delta m^2_\odot}{|\Delta m^2_A|} \sim \frac{1}{30}$$

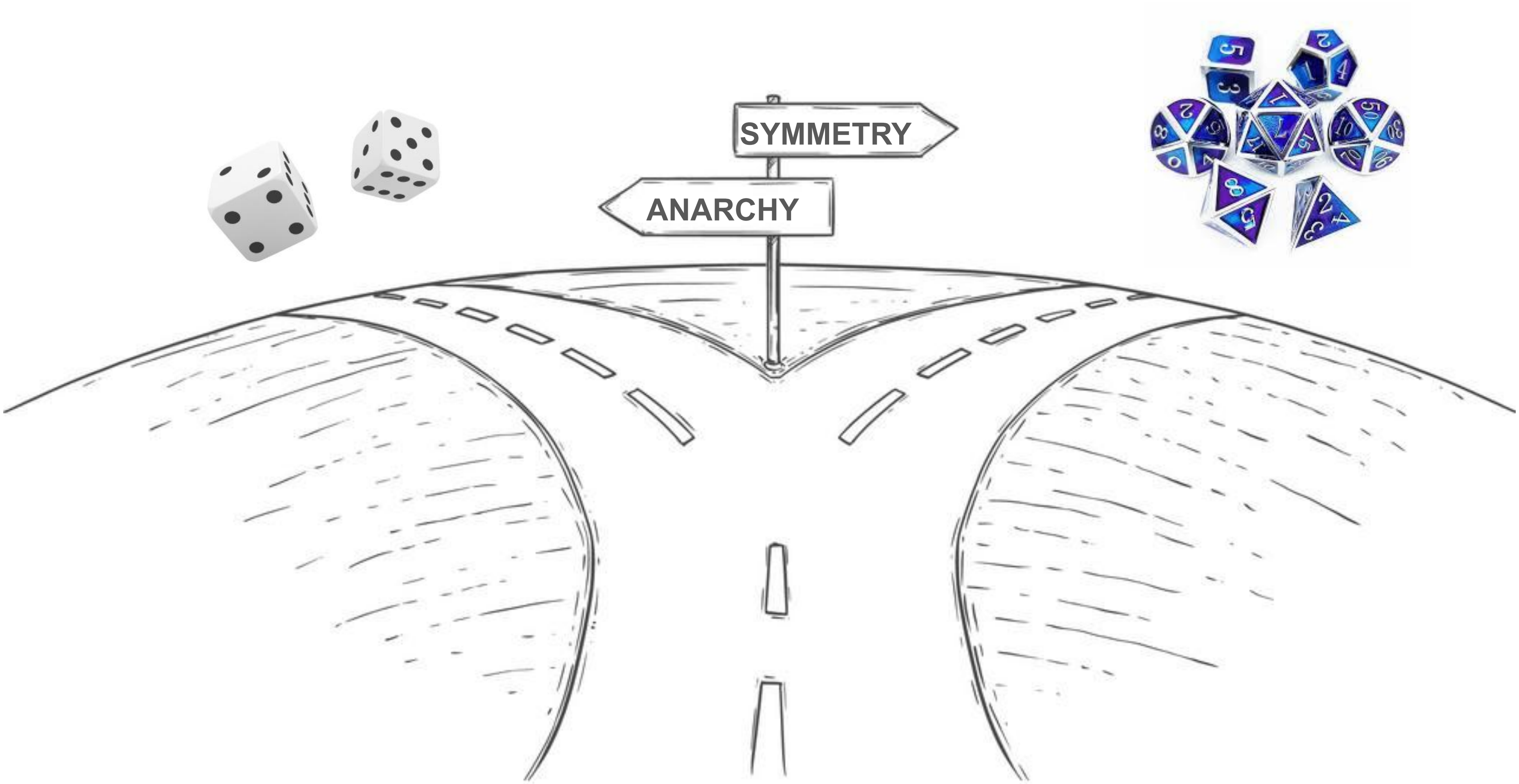




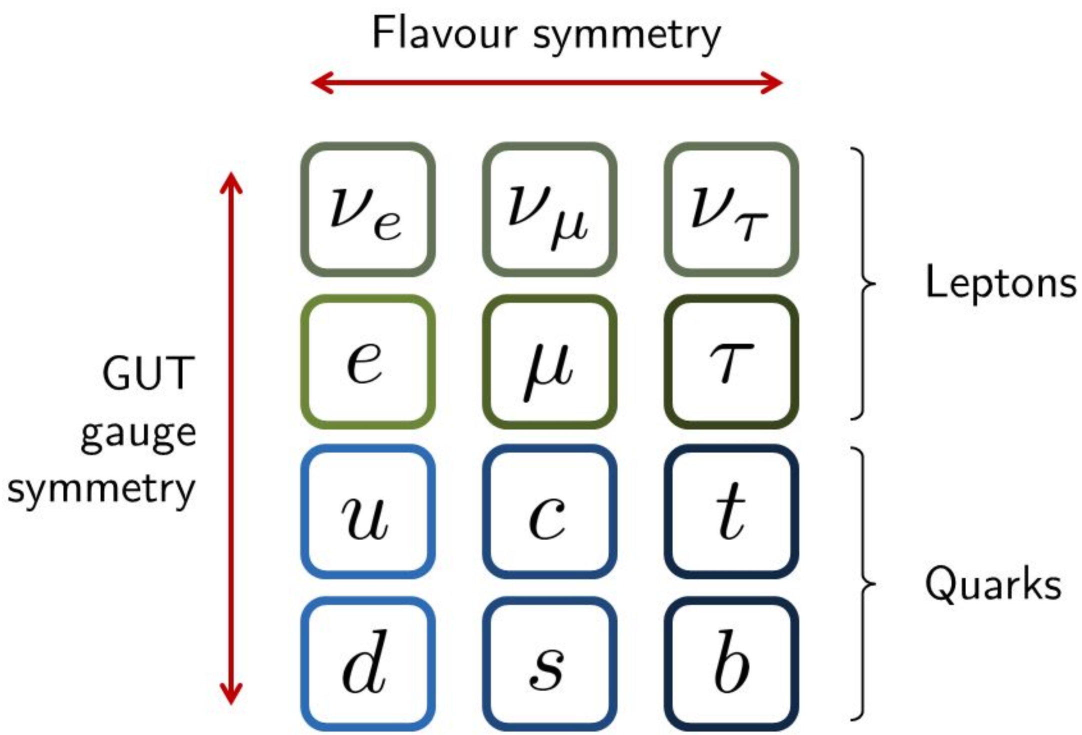




Is there an organizing principle behind this?



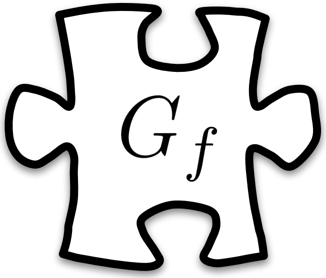
Flavour symmetries



For reviews, see: Altarelli and Feruglio (2010), Ishimori et al. (2010), King and Luhn (2013), Petcov (2017), Feruglio and Romanino (2019)

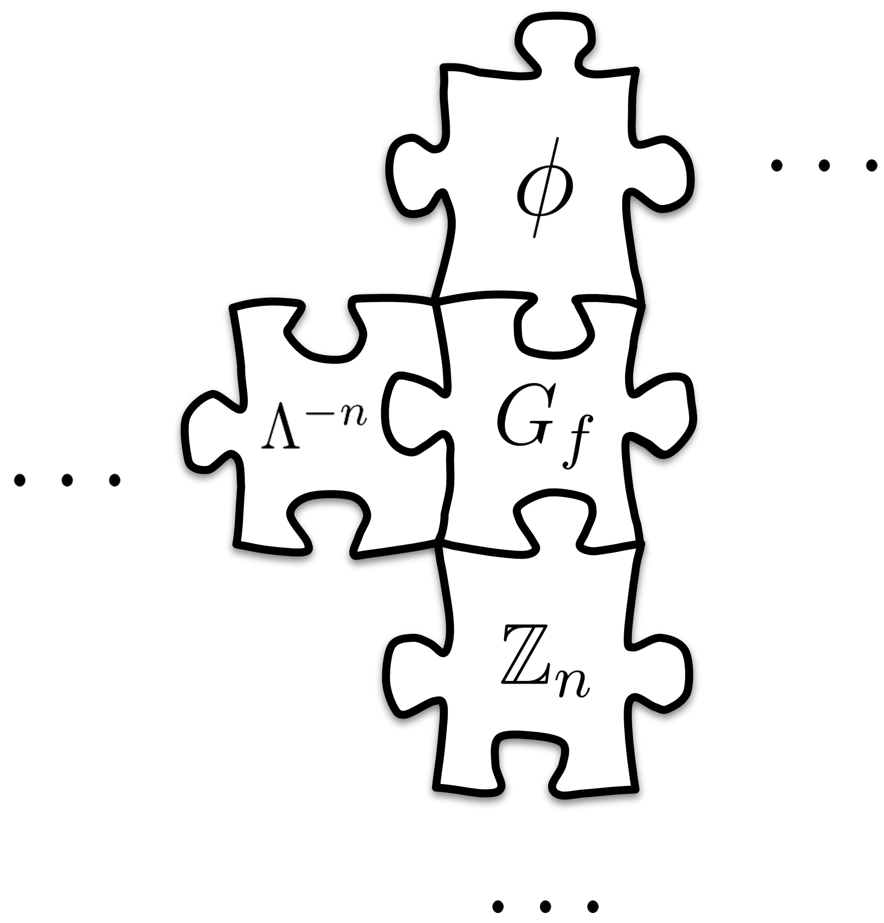
Problems with the usual approach

Non-Abelian discrete flavour symmetries

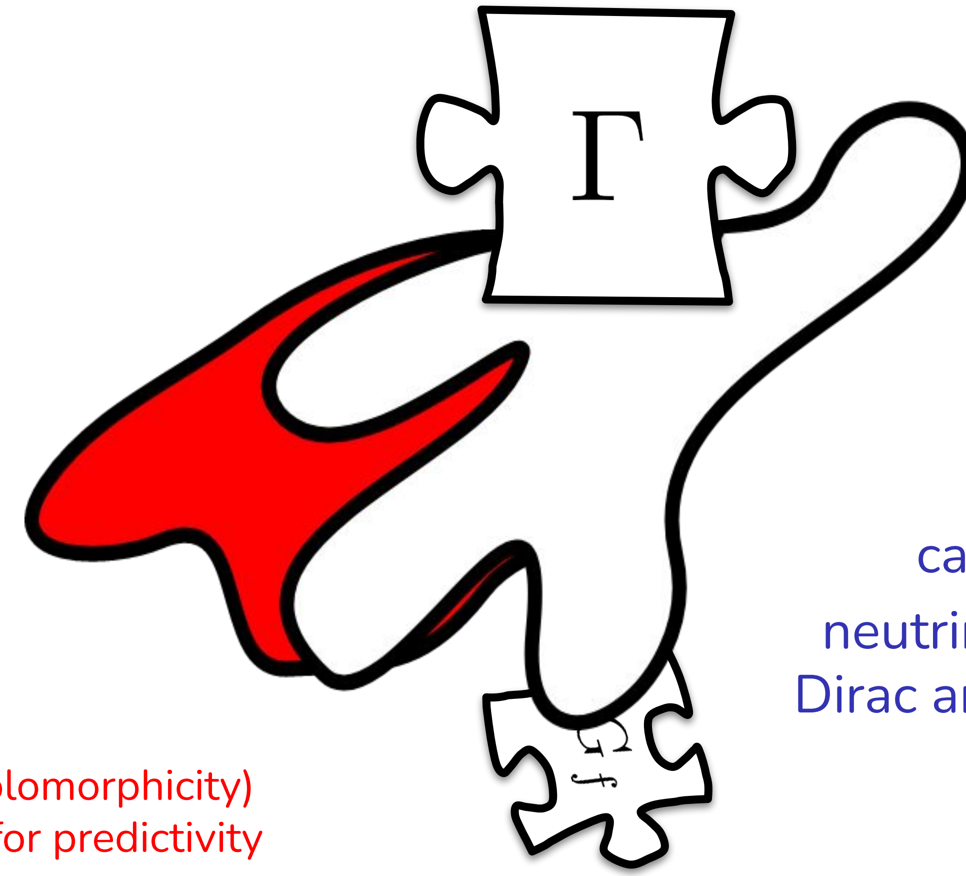


model-independent approaches relying on residual symmetries
constrain mixing and the Dirac phase

Problems with the usual approach



Modular symmetry to the rescue!



Feruglio, 1706.08749

can constrain all:

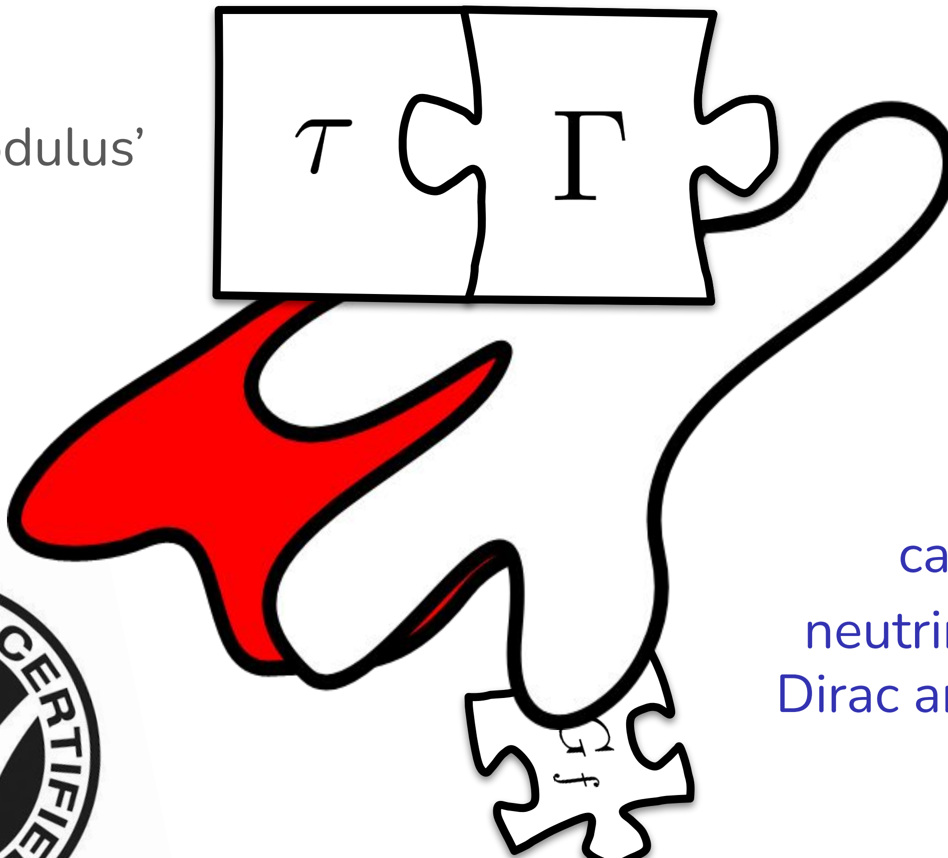
neutrino masses, mixing, Dirac and Majorana phases

SUSY (holomorphicity) **required** for predictivity

see also 2010.07952

Modular symmetry to the rescue!

‘modulus’



Feruglio,
1706.08749

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Modular symmetry to the rescue!

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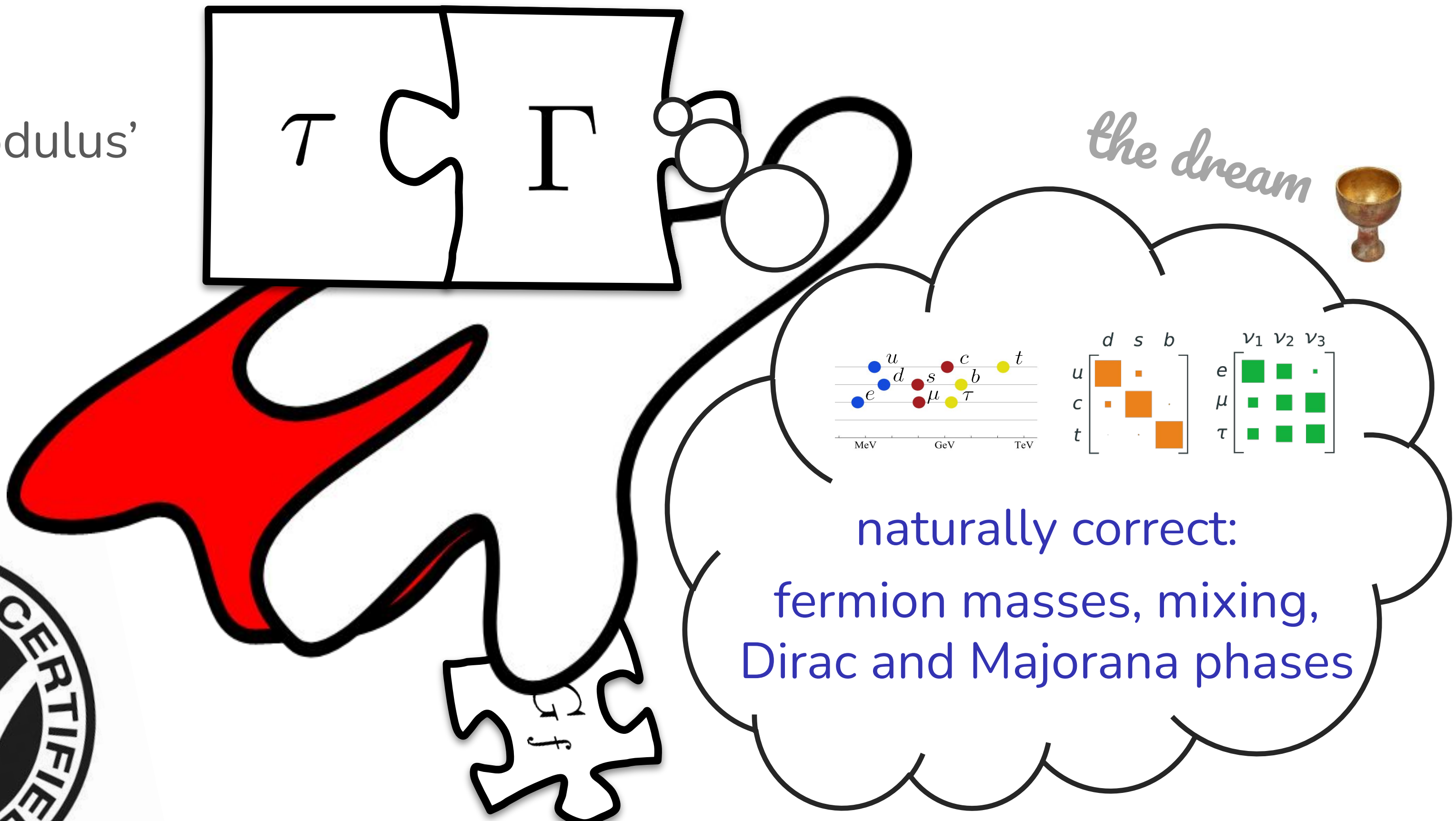
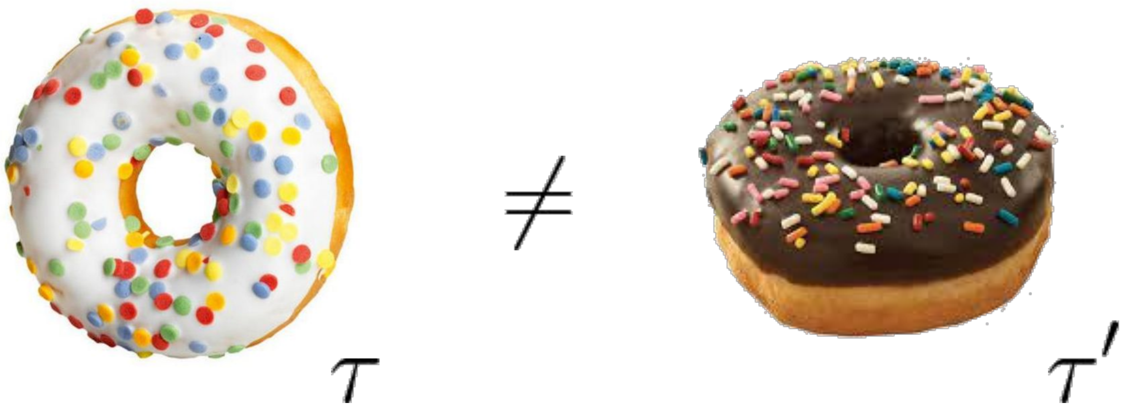


How?

The modulus



τ may describe a torus compactification

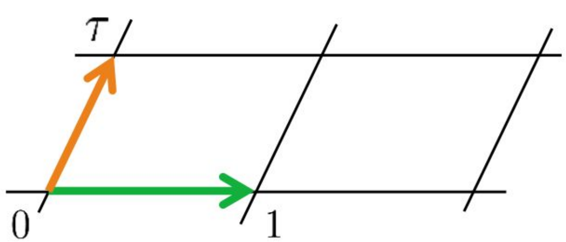
In the **bottom-up** modular approach τ is a dimensionless **spurion**

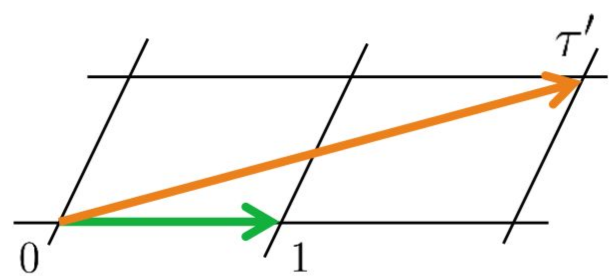
The modulus

$=$

τ

$$\tau' = \frac{a\tau+b}{c\tau+d}$$





$\tau \in \mathcal{H}$

$ad - bc = 1 \qquad a, b, c, d \in \mathbb{Z}$

The modulus

 $=$ 

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

$$\det \gamma = 1$$

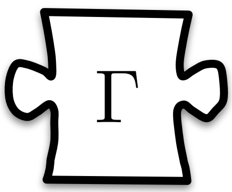
$$a, b, c, d \in \mathbb{Z}$$

the ***modular group***

$$\Gamma \equiv SL(2, \mathbb{Z}) = \{\gamma\}$$

The modular group

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$



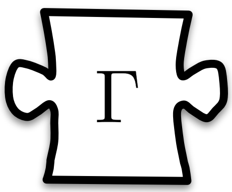
$$\equiv SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z}, \; \det\gamma = 1 \right\}$$

Presentation in terms of generators S, T, R:

$$S^2 = R, \quad (ST)^3 = R^2 = \mathbb{1}, \quad RT = TR$$

The modular group

$$\tau \to \frac{a\tau + b}{c\tau + d}$$



$$\Gamma \equiv SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z},\ \det\gamma = 1 \right\}$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} :$$

$$\tau \to -1/\tau$$

inverSion

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} :$$

$$\tau \to \tau + 1$$

Translation

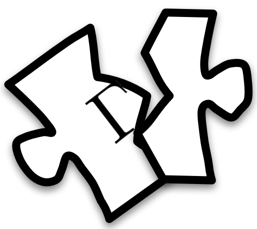
$$R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} :$$

$$\tau \to \tau$$

Redundant

The modular group

$$\langle \tau \rangle \not\to \frac{a\tau + b}{c\tau + d}$$



$$\equiv SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z}, \ \det\gamma = 1 \right\}$$

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Translation

$$R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} :$$

$$\tau \to \tau$$

Redundant

but can affect fields...

The field transformations

$$\psi \;\rightarrow\; (c\tau + d)^{-k}\, \rho(\gamma)\, \psi$$

The field transformations

NEW!

$$\psi \;\rightarrow\; \boxed{(c\tau + d)^{-k}}\; \rho(\gamma)\, \psi$$

Weight $k \in \mathbb{Z}$

The field transformations

NEW!

$$\psi \to \boxed{(c\tau + d)^{-k}} \; \boxed{\rho(\gamma)} \, \psi$$

Weight $k \in \mathbb{Z}$

"Almost trivial" representation of the modular group

$$\rho\Big(\Gamma(N)\Big) = \mathbb{1}$$

$$\rho\Big(T\,\Gamma(N)\Big) = \rho(T)$$

$$\rho\Big(S\,\Gamma(N)\Big) = \rho(S)$$

$$\dots$$

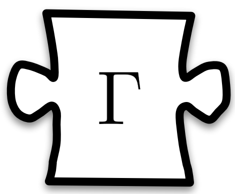
Feruglio, 1706.08749

The field transformations

$$\psi \to \boxed{(c\tau + d)^{-k}}\ \boxed{\rho(\gamma)}\ \psi$$

Weight $k \in \mathbb{Z}$

"Almost trivial" representation of the modular group

$\Gamma(N) \subset$ 

Principal congruence subgroup of level *N*

$$\Gamma(N) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

$$\rho\big(\Gamma(N)\big) = \mathbb{1}$$

$$\rho\big(T\,\Gamma(N)\big) = \rho(T)$$

$$\rho\big(S\,\Gamma(N)\big) = \rho(S)$$

$$\ldots$$

Feruglio, 1706.08749

$\rho(\gamma)$ **is effectively a representation of** $\Gamma'_N \equiv \Gamma/\Gamma(N)$

The finite modular groups

$\Gamma'_N \equiv \Gamma/\Gamma(N)$ *behave like flavour groups*

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2, \mathbb{Z}_4)$	$A'_5 \equiv SL(2, \mathbb{Z}_5)$

← drop the **R** generator

The finite modular groups

$\Gamma'_N \equiv \Gamma/\Gamma(N)$ *behave like flavour groups*

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Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$

← drop the **R** generator

$\Gamma_2 \simeq S_3$

Kobayashi et al., 1803.10391

$\Gamma_3 \simeq A_4$

Feruglio, 1706.08749

$\Gamma_4 \simeq S_4$

JP, Petcov, 1806.11040

$\Gamma_5 \simeq A_5$

Novichkov et al., 1812.02158

summary in Appendices of Novichkov, JP, Petcov, Titov, 1905.11970

$\Gamma'_3 \simeq A'_4$

Liu, Ding, 1907.01488

$\Gamma'_4 \simeq S'_4$

Novichkov, JP, Petcov, 2006.03058

$\Gamma'_5 \simeq A'_5$

Wang, Yu, Zhou, 2010.10159

For top-down, see e.g.:

Kobayashi et al., 1804.06644; Kobayashi, Tamba, 1811.11384; de Anda et al., 1812.05620; Baur et al., 1901.03251, 1908.00805; Kariyazono et al., 1904.07546; Nilles et al., 2001.01736, 2004.05200, 2006.03059; Kobayashi, Otsuka, 2001.07972, 2004.04518; Abe et al., 2003.03512; Ohki et al., 2003.04174; Kikuchi et al., 2005.12642

& many more... (>100)

Need modular forms

$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g(\psi_1 \ldots \psi_n)_{\mathbf{1}}$$

$$\psi \to (c\tau + d)^{-k} \rho(\gamma) \psi$$

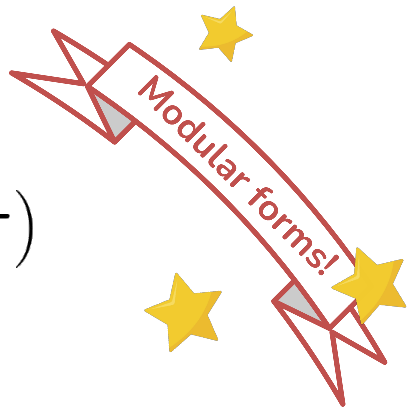
Need modular forms

$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g(Y(\tau)\,\psi_1 \ldots \psi_n)_{\mathbf{1}}$$

$$\psi \;\to\; (c\tau + d)^{-k}\;\rho(\gamma)\,\psi$$

$$Y(\tau) \;\to\; (c\tau + d)^{k_Y}\;\rho_Y(\gamma)\,Y(\tau)$$



Need modular forms

$$\psi \sim (\mathbf{r}, k)$$

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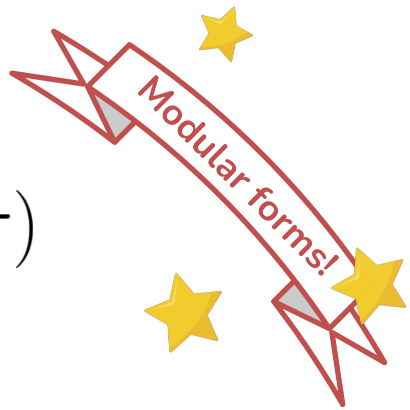
$$\psi \to (c\tau + d)^{-k}\ \rho(\gamma)\, \psi$$

$$Y(\tau) \to (c\tau + d)^{k_Y}\ \rho_Y(\gamma)\, Y(\tau)$$



$$\begin{cases} k_Y = k_1 + \ldots + k_n \\ \rho_Y \otimes \rho_1 \otimes \ldots \otimes \rho_n \supset \mathbf{1} \end{cases}$$

The modular forms

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$
$\dim \mathcal{M}_k(\Gamma(N))$	$k/2+1$	$k+1$	$2k+1$	$5k+1$

Not so many available!

*A **finite set** of functions for each* k_Y

Lowest-weight modular forms for each group:

$\Gamma_N^{(\prime)}$ $Y_{\mathbf{r}}^{(k)}$	$\Gamma_2 \simeq S_3$ $Y_{\mathbf{2}}^{(2)}$
$\Gamma'_3 \simeq A'_4$ $Y_{\hat{\mathbf{2}}}^{(1)}$	$\Gamma_3 \simeq A_4$ $Y_{\mathbf{3}}^{(2)}$
$\Gamma'_4 \simeq S'_4$ $Y_{\hat{\mathbf{3}}}^{(1)}$	$\Gamma_4 \simeq S_4$ $Y_{\mathbf{2}}^{(2)}, Y_{\mathbf{3'}}^{(2)}$
$\Gamma'_5 \simeq A'_5$ $Y_{\hat{\mathbf{6}}}^{(1)}$	$\Gamma_5 \simeq A_5$ $Y_{\mathbf{3}}^{(2)}, Y_{\mathbf{3'}}^{(2)}, Y_{\mathbf{5}}^{(2)}$

Example

Let's build a modular-invariant term!

$$W \supset N N$$

Example

Let's build a modular-invariant term!

$$W \supset NN$$

$$\Gamma_3 \simeq A_4$$

$$N \sim (\mathbf{3}, 1)$$

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Let's build a modular-invariant term!

$$\Gamma_3 \simeq A_4$$

$$N \sim (\mathbf{3}, 1)$$

$$W \supset \Lambda \left(N \otimes N \otimes Y_{\mathbf{3}}^{(2)} \right)_{\mathbf{1}}$$

Example

Let's build a modular-invariant term!

$$W \supset N\,N$$

$$\Gamma_3 \simeq A_4$$

$$N \sim (\mathbf{3}, 1)$$

$$W \supset \Lambda \left(N \otimes N \otimes Y_{\mathbf{3}}^{(2)} \right)_{\mathbf{1}}$$

$$Y_{\mathbf{3}}^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}$$

$$M_N = \Lambda \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

so now we can build models...

Example: an S_4 lepton model

Novichkov, JP, Petcov, Titov, 1811.04933

Ingredients: **Choose group, field content**

$$\psi \sim (\mathbf{r}, k)$$

Example: an S4 lepton model

Novichkov, JP, Petcov, Titov, 1811.04933

Ingredients: Choose group, field content $\psi \sim (\mathbf{r}, k)$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}} H_d$$
$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}} ,$$

Procedure: Fit couplings and *τ* $\min \chi^2(\tau,\, g'/g,\, g^2/\Lambda,\, \alpha, \beta, \gamma)$

Example: an S4 lepton model

Novichkov, JP, Petcov, Titov, 1811.04933

Ingredients: **Choose group, field content** $\psi \sim (\mathbf{r}, k)$

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$g' \in \mathbb{C}$ only physical phase

Procedure: **Fit couplings and *τ*** $\min \chi^2(\tau,\, g'/g,\, g^2/\Lambda,\, \alpha, \beta, \gamma)$

Example: an S4 lepton model

Novichkov, JP, Petcov, Titov, 1811.04933

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Procedure: Fit couplings and τ $\min \chi^2(\tau,\, g'/g,\, g^2/\Lambda,\, \alpha, \beta, \gamma)$

$$\text{gCP} \;\Rightarrow\; g' \in \mathbb{R}$$

Novichkov, JP, Petcov, Titov, 1905.11970

τ can be the only source of CPV

Example: an S4 lepton model (results)

Novichkov, JP, Petcov, Titov, 1811.04933

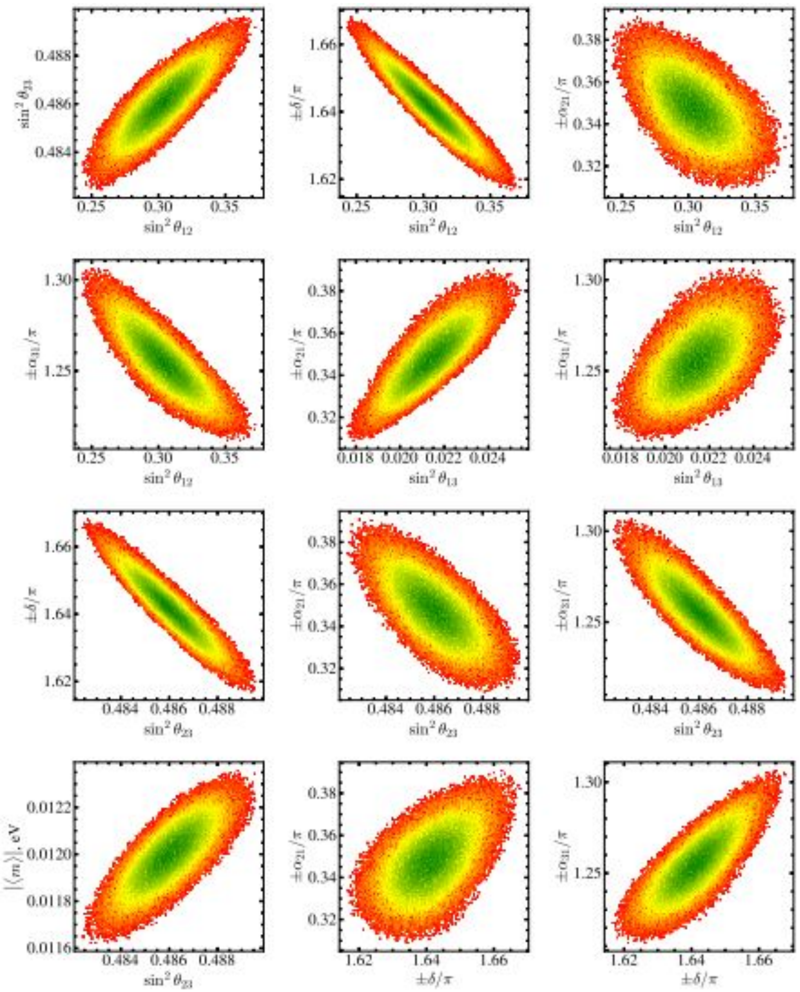
$$\sin^2\theta_{23} \sim 0.49$$
$$\delta \sim 1.6\pi$$
$$\alpha_{21} \sim 0.3\pi$$
$$\alpha_{31} \sim 1.3\pi$$

$$|\langle m\rangle|_{\beta\beta} \sim 12 \text{ meV}$$
$$\textstyle\sum_i m_i \sim 0.08 \text{ eV}$$

7 (4) parameters

vs.

12 (9) observables



What's new?

Mass hierarchies from modular symmetry?

- Usually fermion mass hierarchies are put in **by hand**: hierarchies (or cancellations) between superpotential parameters

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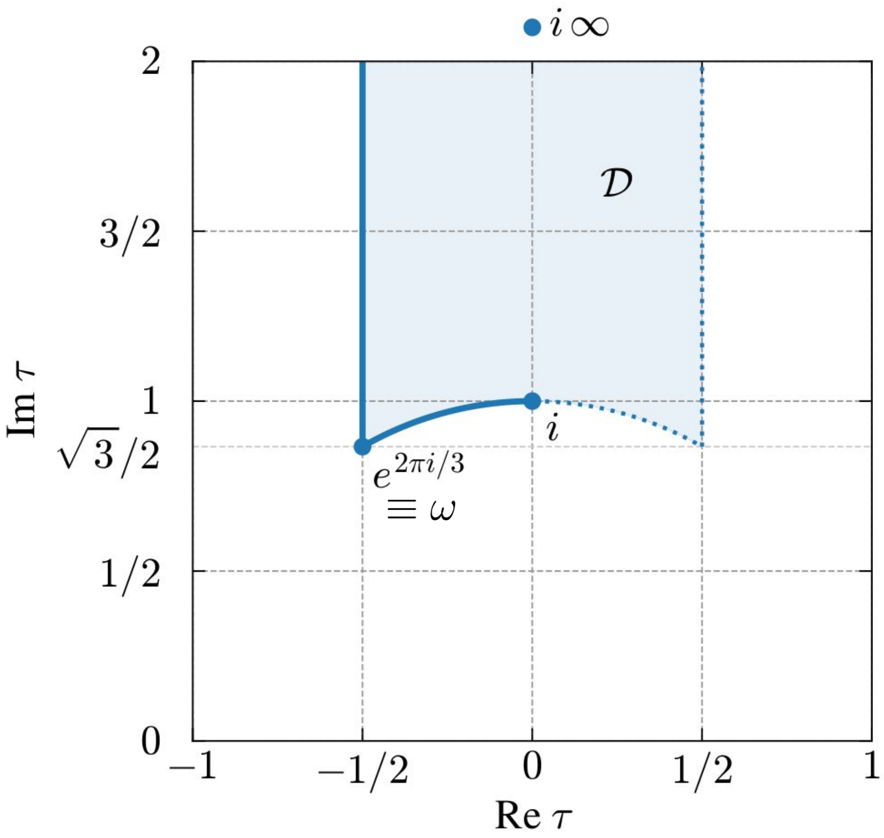
- **Existing approaches** - new (weighted) scalars which enter the mass matrices a la Froggatt-Nielsen. Weights are analogous to FN charges

  Criado, Feruglio, King, 1908.11867
  King, King, 2002.00969

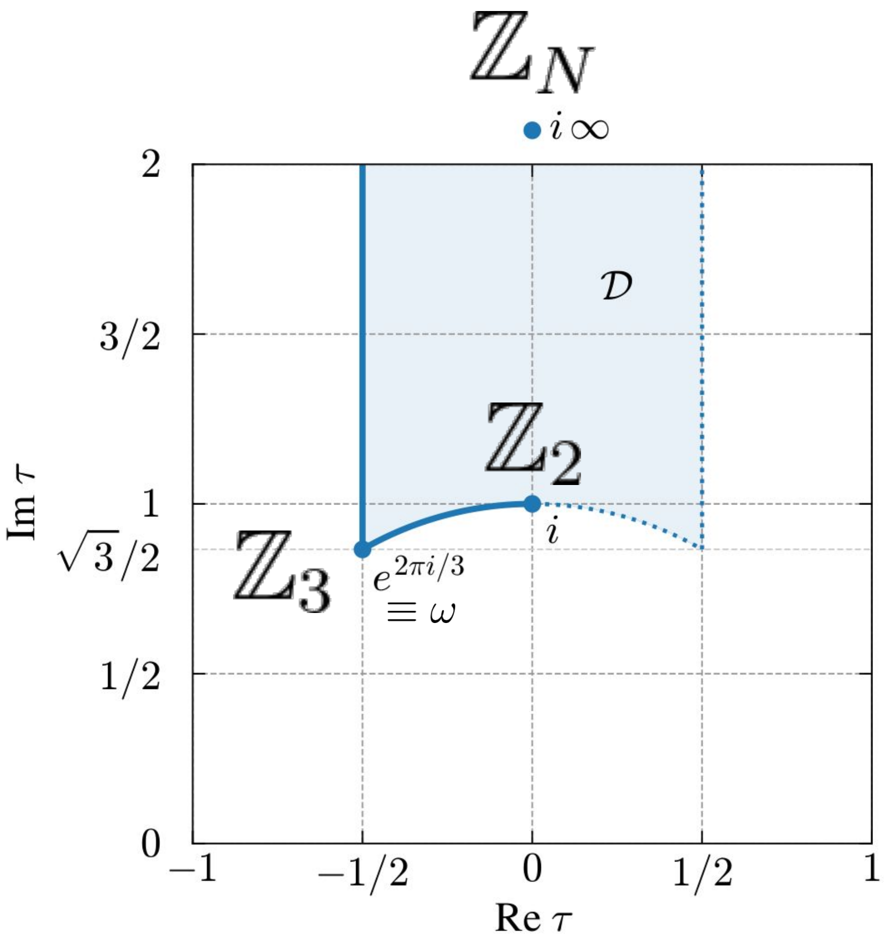
- **Our approach** - No new scalars, mechanism uses **only τ**, common weights across generations (unlike FN charges)

Residual modular symmetries



- The **fundamental domain** is enough
- **Any τ** breaks the modular symmetry

Residual modular symmetries



- The **fundamental domain** is enough
- **Any *τ*** breaks the modular symmetry
- At special values of *τ*, some **residual symmetry** remains

Key idea:

some couplings vanish as we approach a symmetric point

Corrections to vanishing couplings

$$\tau = \tau_{\text{sym}}$$

$$M \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Key idea:

some couplings vanish as we approach a symmetric point

Corrections to vanishing couplings

$$\tau = \tau_{\text{sym}} \qquad\qquad \epsilon = |\tau - \tau_{\text{sym}}| > 0$$

$$M \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad M \sim \begin{pmatrix} 1 & \epsilon^{\cdots} & \epsilon^{\cdots} \\ \epsilon^{\cdots} & \epsilon^{\cdots} & \epsilon^{\cdots} \\ \epsilon^{\cdots} & \epsilon^{\cdots} & \epsilon^{\cdots} \end{pmatrix}$$

In the vicinity of the sym. point, the couplings are $\mathcal{O}(\epsilon^l)$

Key idea:

some couplings vanish as we approach a symmetric point

Decompositions under residual groups

(determine $\mathcal{O}(\epsilon^l)$)

τ_{sym}	Residual sym.	Possible powers ϵ^l
i	$\mathbb{Z}_2$	$l = 0, 1$
ω	$\mathbb{Z}_3$	$l = 0, 1, 2$
$i\infty$	$\mathbb{Z}_N$	$l = 0, 1, \ldots, N$

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Feruglio, Gherardi, Romanino, Titov, 2101.08718 (for A_4, m_e=0)

$$\psi^c \, M \, \psi$$

$$\psi \rightsquigarrow \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots}$$

$$\psi^c \rightsquigarrow \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots}$$

In general, depend on weights

Determined for all $N \leqslant 5$

Example: hierarchical mass matrix (A5)

$$\psi \sim (\mathbf{3}, k)$$
$$\psi^c \sim (\mathbf{3}', k^c)$$

$$\Rightarrow$$

Under the residual group of

$$\tau_{\text{sym}} = i\infty$$

$$\psi \rightsquigarrow 1_0 \oplus \mathbf{1}_1 \oplus \mathbf{1}_4$$
$$\psi^c \rightsquigarrow 1_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3$$

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For $\psi^c M \psi$, we expect:

$$M \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix}$$

with $\epsilon = e^{-2\pi\,\mathrm{Im}\,\tau/5}$

$\Rightarrow$

fermion spectrum

$$\sim (1, \epsilon, \epsilon^4)$$ ✔

Indeed the case, provided enough modular forms contribute to *M*

(otherwise, $m_e = 0$)

Scan of possible mass patterns

Performed for 3 generations, for all $N \leqslant 5$

e.g. fermion spectra for multiplets of modular A5

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k + k^c \equiv 0$	 $k + k^c \equiv 1$	 $k + k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{3}$	$\mathbf{3}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{3}$	$\mathbf{3}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^4)$
$\mathbf{3}'$	$\mathbf{3}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{3}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^4)$
$\mathbf{3}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^3)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$

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$\mathbf{3}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^4)$
$\mathbf{3}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^3)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$

Promising hierarchical patterns

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$ $\tau \simeq i\infty$	
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	

Promising hierarchical patterns

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$
			$\tau \simeq i\infty$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$ with $\mathbf{1}_a \neq (\mathbf{1}_b)^*$
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes [\mathbf{2} \oplus \mathbf{1}$, or $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}']$, $\mathbf{3}' \otimes [\mathbf{2} \oplus \mathbf{1}'$, or $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}']$, $\hat{\mathbf{3}}' \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}']$, $\hat{\mathbf{3}} \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}']$
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes \mathbf{3}'$

Promising hierarchical patterns (try leptons)

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$ $\tau \simeq i\infty$	
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	$\hat{\mathbf{3}}' \otimes (\hat{\mathbf{2}} \oplus \hat{\mathbf{1}})$
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes \mathbf{3}'$

$L \sim (\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}, 2),\ E^c \sim (\hat{\mathbf{3}}', 2),\ N^c \sim (\mathbf{3}, 1)$

8 parameters

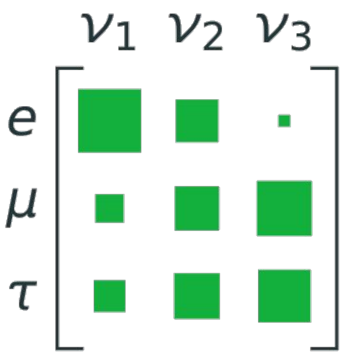
$L \sim (\mathbf{3}, 3),\ E^c \sim (\mathbf{3}', 1),\ N^c \sim (\hat{\mathbf{2}}, 2)$

8 parameters

Fits are fine-tuned :(

Wrong PMNS in the symmetric limit:
parameters are driven into cancellations

How to avoid fine-tuning (in the lepton sector)



$$\xrightarrow{\tau \to \tau_{\mathrm{sym}}} \begin{bmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix} \text{ or } \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}$$

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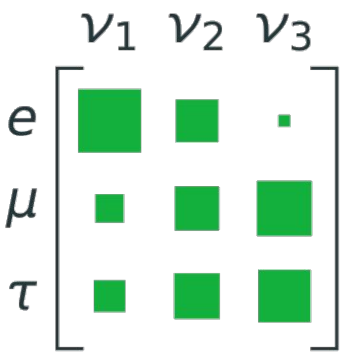

$$\xrightarrow{\tau \to \tau_{\mathrm{sym}}} \begin{bmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix} \text{ or } \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}$$

Reyimuaji, Romanino, 1801.10530

$$1. \begin{cases} L \sim 1 \oplus 1 \oplus 1 \\ E^c \sim 1 \oplus \mathbf{r} \not\supset 1 \end{cases} \qquad 2. \begin{cases} L \sim \mathbf{1} \oplus \mathbf{1} \oplus \bar{\mathbf{1}} \\ E^c \sim \bar{\mathbf{1}} \oplus \mathbf{r} \not\supset \mathbf{1}, \bar{\mathbf{1}} \end{cases}$$

$$3. \; m_e = m_\mu = m_\tau = 0$$

$$4. \; m_{\nu_1} = m_{\nu_2} = m_{\nu_3} = 0$$

for mixing near symmetric points, see also Okada, Tanimoto, 2009.14242

Promising hierarchical patterns (leptons)

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}_{E^c} \otimes \mathbf{r}_L$	Case
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$	1 or 4
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$	2
			$\tau \simeq i\infty$	$[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}'] \otimes [\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}']$, $[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}''] \otimes [\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'']$	2
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a, \text{ or } \mathbf{2} \oplus \mathbf{1}^{(\prime)}, \text{ or } \hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$	1 or 4
5	A'_5	–	–	–	–

1. $\begin{cases} L \sim 1 \oplus 1 \oplus 1 \\ E^c \sim 1 \oplus \mathbf{r} \not\supset 1 \end{cases}$

2. $\begin{cases} L \sim \mathbf{1} \oplus \mathbf{1} \oplus \bar{\mathbf{1}} \\ E^c \sim \bar{\mathbf{1}} \oplus \mathbf{r} \not\supset \mathbf{1}, \bar{\mathbf{1}} \end{cases}$

3. $m_e = m_\mu = m_\tau = 0$

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N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}_{E^c}\otimes\mathbf{r}_L$	Case
2	S_3	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	$[\mathbf{2}\oplus$ [illegible]	1 or 4
3	A'_4	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	[illegible] $\mathbf{1}']\otimes[\mathbf{1}$ [illegible]	2
			$\tau\simeq i\infty$	[illegible]	2
4	S'_4	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	$[\mathbf{3}_a$, or [illegible] $]\otimes[\mathbf{1}_b\oplus\mathbf{1}_b\oplus\mathbf{1}'_b]$	1 or 4
5	A'_5	–	–	–	–

1. $\begin{cases} L \sim 1\oplus 1\oplus 1 \\ E^c \sim 1\oplus \mathbf{r} \not\supset 1 \end{cases}$

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3. $m_e = m_\mu = m_\tau = 0$

4. $m_{\nu_1} = m_{\nu_2} = m_{\nu_3} = 0$

Example: lepton model close to ω

Only model from a scan requiring minimal # params., $m_e > 0$, and Dirac phase within 2σ range (otherwise unconstrained):

$$L \sim (\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}', 2),\ E^c \sim (\hat{\mathbf{3}}, 4),\ N^c \sim (\mathbf{3}', 1)$$

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$$M_e \propto \begin{pmatrix} 1 & \alpha - 2\beta & 2\sqrt{3}i\gamma \\ \sqrt{3}\epsilon & \sqrt{3}(\alpha + 2\beta)\epsilon & 2i\gamma\epsilon \\ \frac{5}{2}\epsilon^2 & \left(\frac{5}{2}\alpha - \beta\right)\epsilon^2 & -\frac{5}{\sqrt{3}}i\gamma\epsilon^2 \end{pmatrix}$$

$$\epsilon \sim \left|\tau - e^{2\pi i/3}\right|$$

$$M_\nu \propto \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a \\ 1 & a & 2i\sqrt{\frac{2}{3}}b \end{pmatrix}$$

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Only model from a scan requiring minimal # params., $m_e > 0$, and Dirac phase within 2σ range (otherwise unconstrained):

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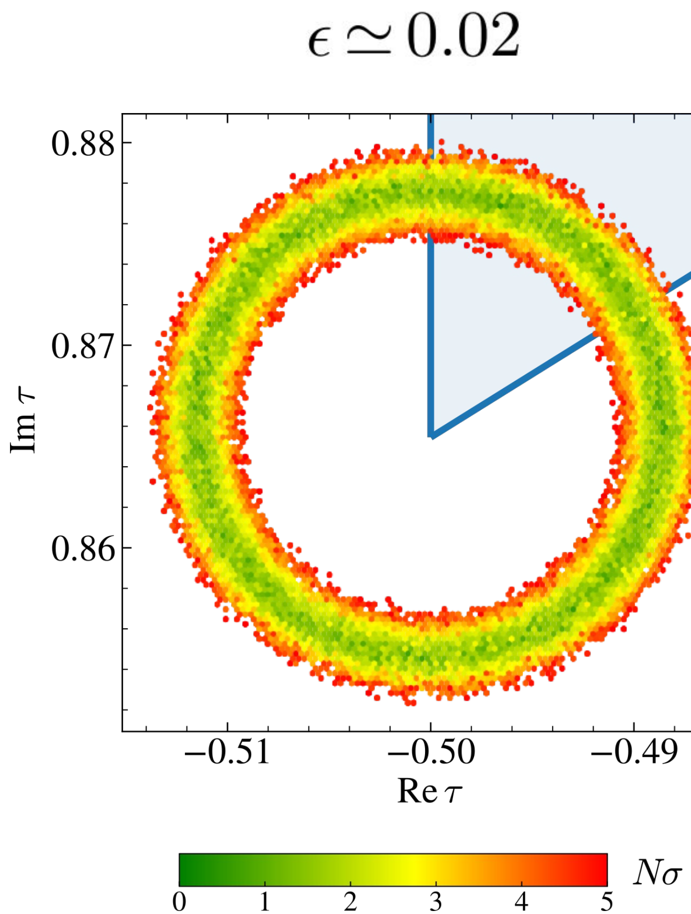
$$M_e \propto \begin{pmatrix} 1 & \alpha - 2\beta & 2\sqrt{3} i\gamma \\ \sqrt{3}\epsilon & \sqrt{3}(\alpha + 2\beta)\epsilon & 2i\gamma\epsilon \\ \frac{5}{2}\epsilon^2 & \left(\frac{5}{2}\alpha - \beta\right)\epsilon^2 & -\frac{5}{\sqrt{3}} i\gamma\epsilon^2 \end{pmatrix} \qquad \epsilon \sim \left|\tau - e^{2\pi i/3}\right|$$

$$M_\nu \propto \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a \\ 1 & a & 2i\sqrt{\frac{2}{3}} b \end{pmatrix}$$

$\epsilon \simeq 0.02$	$\alpha = 2.45 \pm 0.44$
$a = 1.5 \pm 0.15$	$\beta = 2.14 \pm 0.32$
$b = 2.22 \pm 0.17$	$\gamma = 0.91 \pm 0.05$

Example: lepton model close to ω



$$m_e = \mathcal{O}(\epsilon^2)$$
$$m_\mu = \mathcal{O}(\epsilon)$$
$$m_\tau = \mathcal{O}(1)$$

$$\text{NO}, \quad m_{\nu_1} = 0$$
$$\delta = \pi + \mathcal{O}(10^{-6})$$
$$m_{\beta\beta} = (1.44 \pm 0.33)\ \text{meV}$$

Naturally allows for **hierarchies**, **large mixing**, and some **predictivity**

Summary

Summary

- **Modular symmetry** may strongly constrain masses and mixing.
- Fields carrying a non-trivial modular weight transform with a **scale factor** in addition to the usual unitary rotation.
- To build invariants one needs modular forms, which are functions of a **single complex parameter** τ.
- Fermion **mass hierarchies** can naturally arise if τ is in the vicinity of a point of residual symmetry.

$$\tau_{\rm sym} = i,\ i\infty,\ \omega$$

Σας ευχαριστώ!

Backup slides

Modular-invariant SUSY actions

Ferrara et al, '89

$$W(\psi;\tau) = \sum_n \sum_{\{i_1,\dots,i_n\}} \sum_s g_{i_1 \dots i_n, s} \left(Y_{i_1 \dots i_n, s}(\tau)\, \psi_{i_1} \dots \psi_{i_n}\right)_{\mathbf{1},s}$$

$$\mathcal{S} = \int d^4x\, d^2\theta\, d^2\overline{\theta}\; K(\psi,\overline{\psi};\tau,\overline{\tau}) + \int d^4x\, d^2\theta\; W(\psi;\tau) + \text{h.c.}$$

τ is a dimensionless spurion: once its value is fixed, it **parameterises all** modular sym. breaking

One may argue that *Y*'s play the role of flavons, but structures are **completely fixed** given the modulus VEV

Modular-invariant SUSY actions

$$W(\psi;\tau) = \sum_{n} \sum_{\{i_1,\ldots,i_n\}} \sum_{s} g_{i_1\,\ldots\,i_n,s} \left(Y_{i_1\,\ldots\,i_n,s}(\tau)\, \psi_{i_1} \ldots \psi_{i_n}\right)_{\mathbf{1},s}$$

$$\begin{cases} \tau \to \gamma\tau = \dfrac{a\tau+b}{c\tau+d} \\ \psi_i \to (c\tau+d)^{-k_i} \rho_i(\gamma)\,\psi_i \\ Y(\tau) \to Y(\gamma\tau) = (c\tau+d)^{k_Y} \rho_Y(\gamma)\, Y(\tau) \end{cases} \quad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

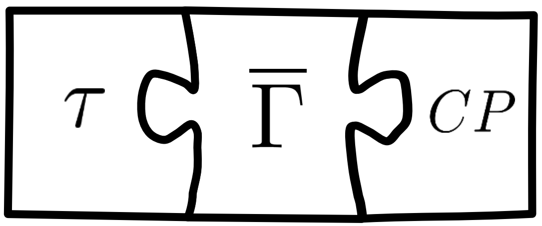
weights

k_Y positive & even, for PSL(2,Z)

$Y(\tau)$ are **modular forms** obeying
$$\begin{cases} k_Y = k_{i_1} + \ldots + k_{i_n} \\ \rho_Y \otimes \rho_{i_1} \otimes \ldots \otimes \rho_{i_1} \supset \mathbf{1} \end{cases}$$

Live in linear spaces of finite dimension

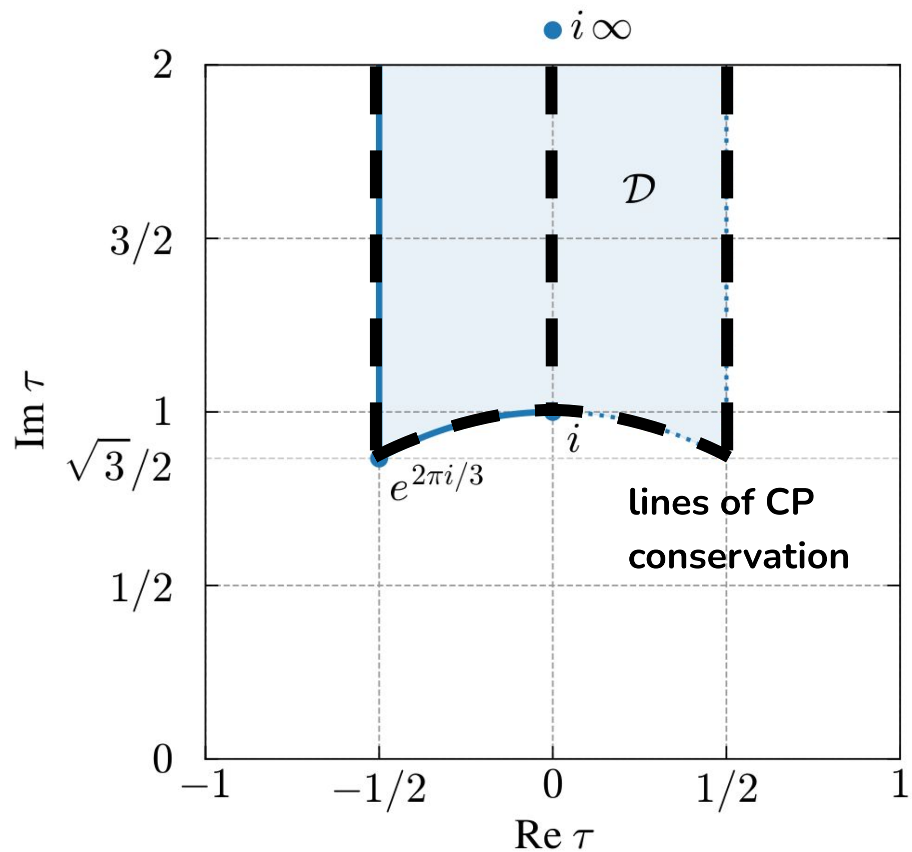
Combining modular and CP symmetries



$$\tau \xrightarrow{\text{CP}} -\tau^*$$

$$\psi(x) \xrightarrow{\text{CP}} X_{\mathbf{r}}^{\text{CP}}\,\overline{\psi}(x_{\text{P}})$$

$$Y(\tau) \xrightarrow{\text{CP}} X_{\mathbf{r}}^{\text{CP}}\,Y^*(\tau)$$



SUSY breaking effects?

- **RGEs & threshold corrections** need to be considered, depend on tan β and unknown SUSY spectrum
- **SUSY-breaking** corrections can be made negligible via separation of scales (power counting argument)
- Under reasonable conditions, predictions may be unaffected

Feruglio and Criado, 1807.01125

Constraints on the Kähler potential?

- **Kähler** not constrained by the symmetry.
- Under a modular transformation, invariant up to:

$$K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) \;\rightarrow\; K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) + f(\chi_i; \tau) + f(\overline{\chi}_i; \overline{\tau})$$

- Minimal choice:

$$K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) = -h\,\Lambda_0^2 \log(-i(\tau - \overline{\tau})) + \sum_i \frac{|\chi_i|^2}{(-i(\tau - \overline{\tau}))^{k_i}}$$

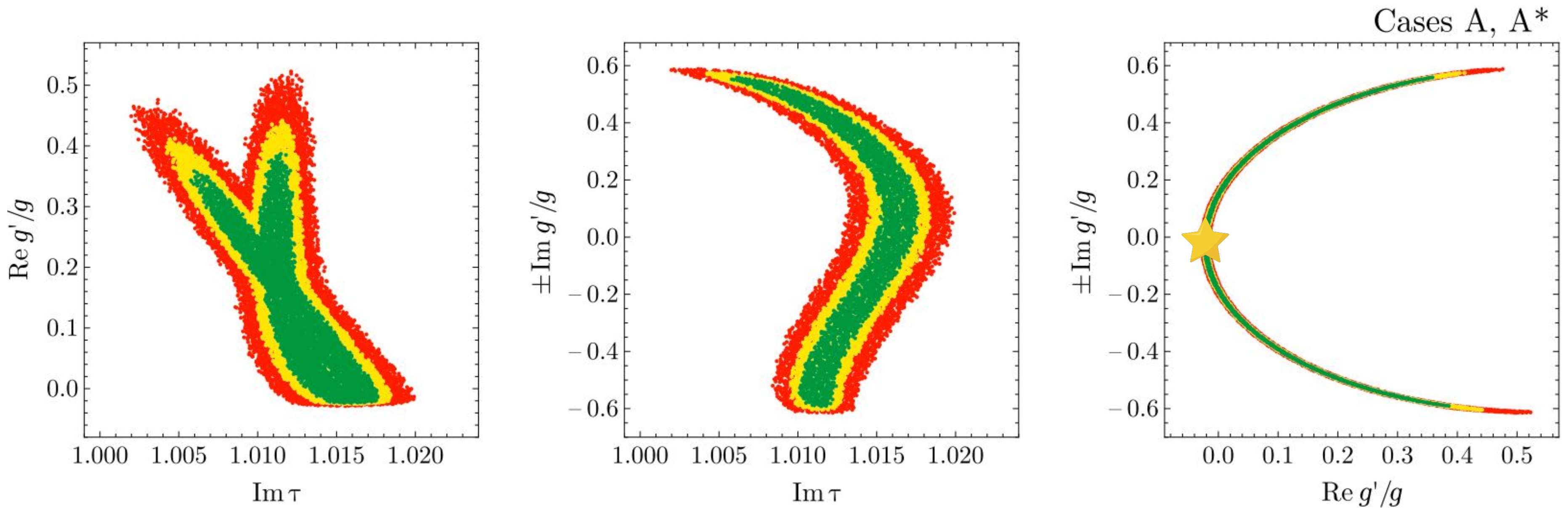
should be justified from the top-down

Chen, Ramos-Sánchez and Ratz, 1909.06910

- Further constraints may arise from combining modular group + traditional finite flavour symmetry

Nilles, Ramos-Sanchez, Vaudrevange, 2004.05200

Correlations between parameters in the first S4 example model



see Novichkov, JP, Petcov, Titov, 1811.04933

Decompositions under residual groups: A5’

$\mathbf{r}$	$\mathbb{Z}_4^S\,(\tau = i)$	$\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R\,(\tau = \omega)$	$\mathbb{Z}_5^T \times \mathbb{Z}_2^R\,(\tau = i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^\pm$	$1_0^\pm$
$\hat{\mathbf{2}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^\mp \oplus \mathbf{1}_{k+2}^\mp$	$\mathbf{1}_2^\mp \oplus \mathbf{1}_3^\mp$
$\hat{\mathbf{2}}'$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^\mp \oplus \mathbf{1}_{k+2}^\mp$	$\mathbf{1}_1^\mp \oplus \mathbf{1}_4^\mp$
$\mathbf{3}$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^\pm \oplus \mathbf{1}_{k+1}^\pm \oplus \mathbf{1}_{k+2}^\pm$	$1_0^\pm \oplus \mathbf{1}_1^\pm \oplus \mathbf{1}_4^\pm$
$\mathbf{3}'$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^\pm \oplus \mathbf{1}_{k+1}^\pm \oplus \mathbf{1}_{k+2}^\pm$	$1_0^\pm \oplus \mathbf{1}_2^\pm \oplus \mathbf{1}_3^\pm$
$\mathbf{4}$	$\mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^\pm \oplus \mathbf{1}_k^\pm \oplus \mathbf{1}_{k+1}^\pm \oplus \mathbf{1}_{k+2}^\pm$	$\mathbf{1}_1^\pm \oplus \mathbf{1}_2^\pm \oplus \mathbf{1}_3^\pm \oplus \mathbf{1}_4^\pm$
$\hat{\mathbf{4}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^\mp \oplus \mathbf{1}_k^\mp \oplus \mathbf{1}_{k+1}^\mp \oplus \mathbf{1}_{k+2}^\mp$	$\mathbf{1}_1^\mp \oplus \mathbf{1}_2^\mp \oplus \mathbf{1}_3^\mp \oplus \mathbf{1}_4^\mp$
$\mathbf{5}$	$\mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^\pm \oplus \mathbf{1}_{k+1}^\pm \oplus \mathbf{1}_{k+1}^\pm \oplus \mathbf{1}_{k+2}^\pm \oplus \mathbf{1}_{k+2}^\pm$	$1_0^\pm \oplus \mathbf{1}_1^\pm \oplus \mathbf{1}_2^\pm \oplus \mathbf{1}_3^\pm \oplus \mathbf{1}_4^\pm$
$\hat{\mathbf{6}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^\mp \oplus \mathbf{1}_k^\mp \oplus \mathbf{1}_{k+1}^\mp \oplus \mathbf{1}_{k+1}^\mp \oplus \mathbf{1}_{k+2}^\mp \oplus \mathbf{1}_{k+2}^\mp$	$1_0^\mp \oplus 1_0^\mp \oplus \mathbf{1}_1^\mp \oplus \mathbf{1}_2^\mp \oplus \mathbf{1}_3^\mp \oplus \mathbf{1}_4^\mp$

More stabilizers?

- Despite working with representations of the quotients, our theories are **fully modular invariant**
- To have canonical kinetic terms,

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \Rightarrow \quad g_i \to (c\tau + d)^{-k_{Y_i}} g_i$$

- e.g. in a particular model,

$$(\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots) \to$$
$$\left(\frac{a\tau + b}{c\tau + d},\, (c\tau + d)^{-2}\,\beta/\alpha,\, (c\tau + d)^{-2}\,\gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots\right)$$

these different parameter sets lead to the same observables

see section 4 of Novichkov, JP, Petcov, Titov, 1811.04933