Cluster Algebras for Feynman Integrals

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DESY

CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

CORFU2021 Workshop on the Standard Model and Beyond
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PRL 126 091603 (2021) with Dima Chicherin, Johannes Henn

Scattering amplitudes

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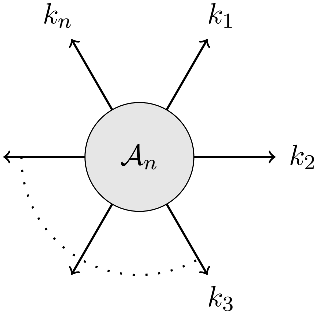

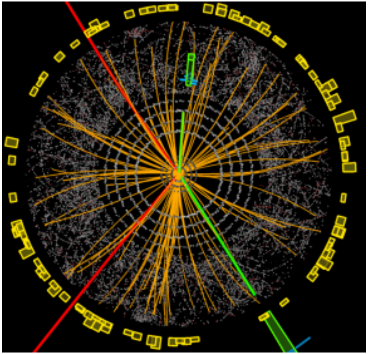

The arena where perturbative quantum field theory confronts experiment.

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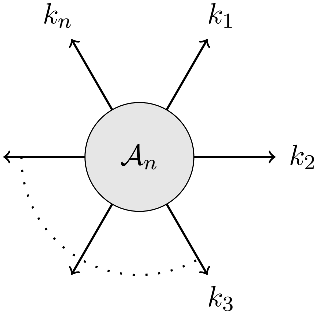

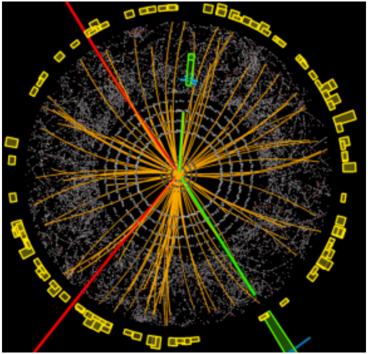

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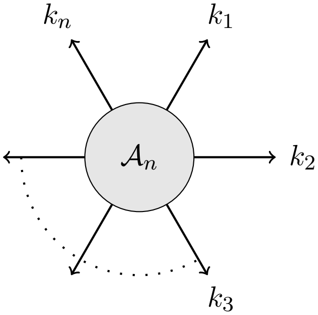

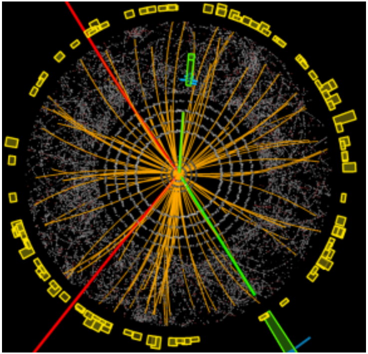

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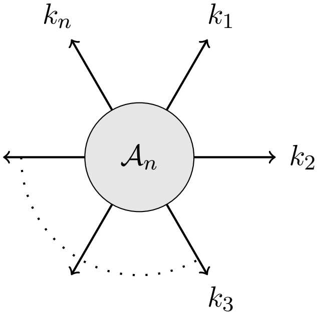

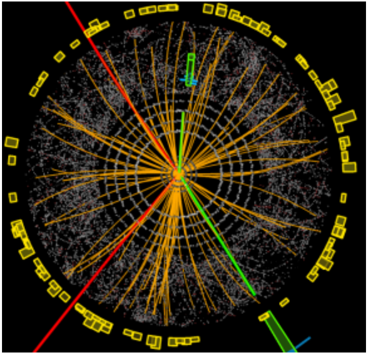

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Especially relevant in light of High-Luminosity LHC 2027-2037. [Zerlauth Talk]

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All of them vital for recent state of the art calculation of 2-loop 5-point 1-mass planar master integrals, relevant for W-boson production + 2 jets, [Abreu,Ita,Moriello,Page,Tschernow,Zeng] [Canko,Papadopoulos,Syrrakos]

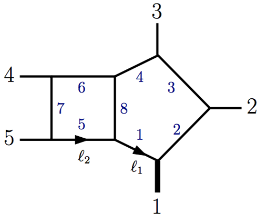

Image Credit: 2005.04195

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Tremendously successful in describing singularities of n-particle planar amplitudes $\mathcal{A}_n$ in $\mathcal{N} = 4$ SYM.
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Discovered cluster algebras encode singularities of a wealth of physically relevant examples, including QCD corrections to amplitudes for pp→ Higgs+jet!

Outline

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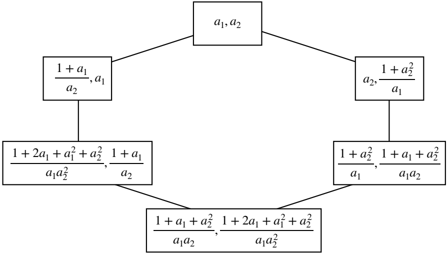

Exchange graph: Clusters=vertices, mutations=edges

Geometric Interpretation of Cluster Algebras

Finite cluster algebras classified by Dynkin diagrams. For A_n:

- Cluster = triangulation of $(n+3)$-gon by noncrossing diagonals
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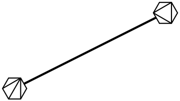

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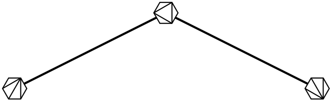

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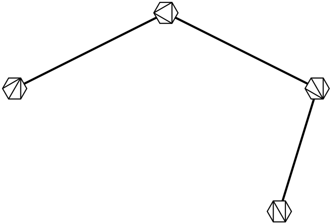

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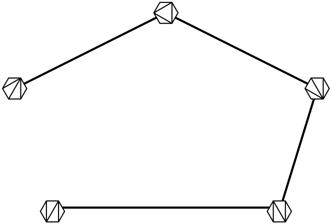

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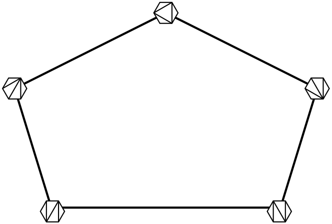

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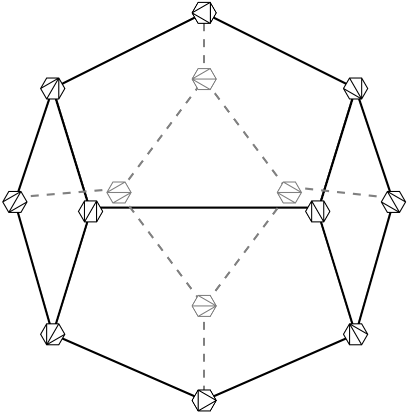
Example: $A_3 =$ hexagon exchange graph



Adapted from 1810.08149

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Collection of ϕ_{α_i} : **symbol alphabet** Φ

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The right variables & their applications

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Potential amplitude singularities when cluster coordinates $a_m = 0, \infty$

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- *variables a_m of a Grassmannian $Gr(4, n)$ cluster algebra*
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Potential amplitude singularities when cluster coordinates $a_m = 0, \infty$

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Essential information for computing $\mathcal{A}_n$ via amplitude bootstrap
[PoS CORFU2019 Review:Caron-Huot,Dixon,Drummond,Dulat,Foster,Gurdogan,Hippel,McLeod,GP]

Cluster Algebras and $\mathcal{N} = 4$ Amplitudes

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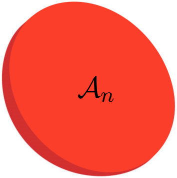

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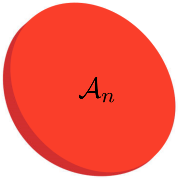

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- $n = 6$: 7 loops (MHV)
- $n = 7$: 4 loops

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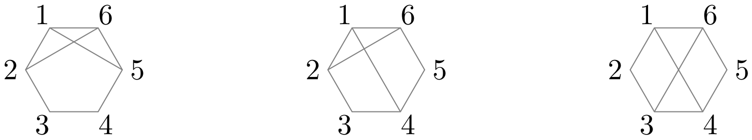

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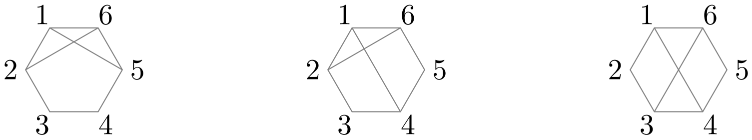

For physical $n = 6, 7$ functions, equivalent to *extended Steinmann relations*. Massively reduces size of function space. [Caron-Huot,Dixon,DulatMcLeod,Hippel,GP]

The Genetic Material of $\mathcal{N} = 4$ SYM Amplitudes

DNA

Amplitude symbol

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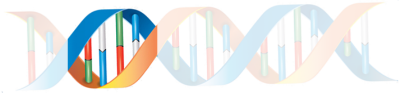

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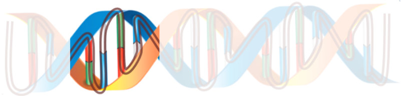

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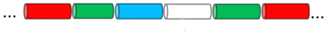

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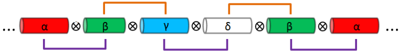

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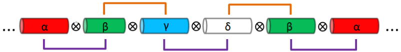

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Could cluster algebras provide the genetic material of generic quantum field theories?

Searching for Cluster Algebras in QFT

Look at scalar Feynman integrals with massless internal propagators in dimensional regularization.

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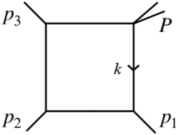

$$= \int \frac{d^D k}{k^2 (k+p_1)^2 (k+p_1+p_2)^2 (k-P)^2}$$

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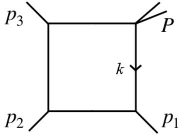

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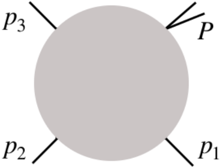
For any given process, scalar integrals related by integration-by-parts identities. Basis in the vector space they span=master integrals. [Chetyrkin,Tkachov]

Main Example: Four-point functions with one leg offshell/massive



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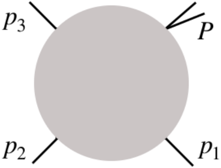

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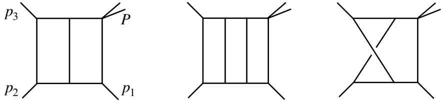

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- Alphabet of all known master integrals: [Gehrmann,Remiddi] [Di Vita, Mastrolia, Schubert, Yundin]

$$\Phi_{\mathsf{2dHPL}} = \{z_1, z_2, z_3, 1 - z_1, 1 - z_2, 1 - z_3\}\,,$$

"2-dimensional HPLs" [Gehrmann,Remiddi]

Identifying Candidate Cluster Algebras

	2dHPL		
# independent variables	2		

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$\boxed{\text{2dHPLs} = C_2 \text{ polylogarithms!}}$

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Physical Significance: C_2 Cluster Algebra Underlies Higgs Amplitudes!

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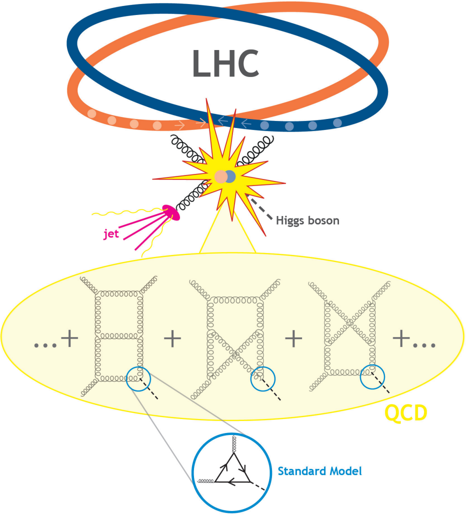
  [Gehrmann, Jaquier, Glover, Koukoutsakis] [Duhr]

  in heavy top mass limit



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to all orders in ϵ. Important structural information for manipulating $\mathbf{f}$, e.g. analytic continuation.

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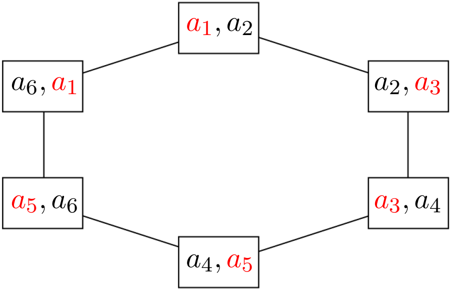

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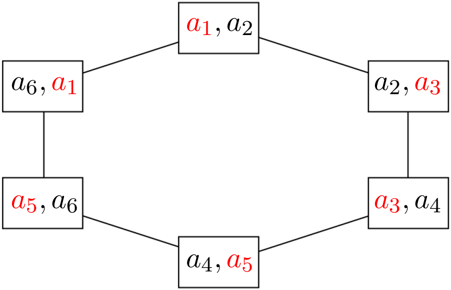

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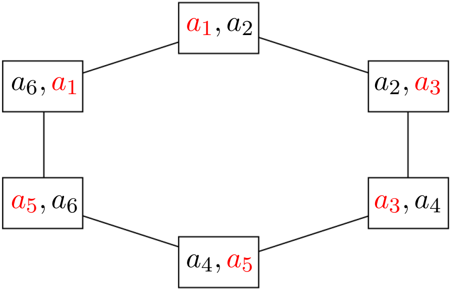

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weight	1	2	3	4	5	6	7	8
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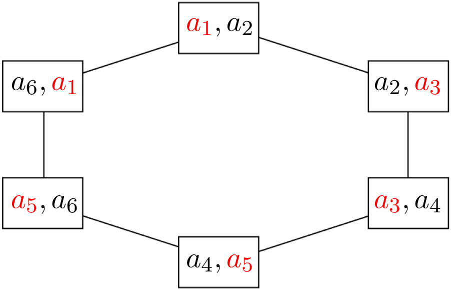

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Application: In parallel work, used to bootstrap $\mathcal{N} = 4$ SYM analogue of Higgs amplitude through 5 loops [Dixon,Mcleod,Wilhelm]

More examples of cluster algebras from one-loop Feynman integrals

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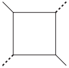
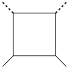
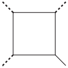
	3	9	$\subset A_3$
	3	10	$\subset C_3$
	4	16	$\subset C_4$

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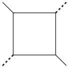
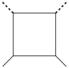
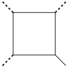
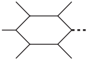
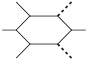
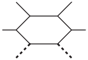
	3	9	$\subset A_3$
	3	10	$\subset C_3$
	4	16	$\subset C_4$

Finite 6D integrals	# variables	# letters	cluster algebra
	4	16	D_4
	5	24	$\subset D_5$
	5	27	$\lim \mathrm{Tr}(4,8)$

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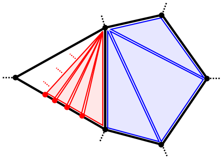

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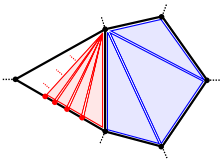

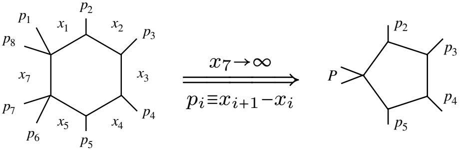

From appropriately symmetrized limit, obtain all letters appearing in finite (hard) part of two-loop five-gluon amplitude in QCD!

Conclusions

The beautiful mathematics of cluster algebras underlie the analytic structure of several physically relevant Feynman integrals & processes!

- Higgs+jet amplitudes to all orders in ϵ
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Could cluster algebras and generalizations provide an organizing principle ("genetic material") that simplifies future collider physics calculations?

Cluster Algebras and $\mathcal{N} = 4$ Amplitudes: The right variables

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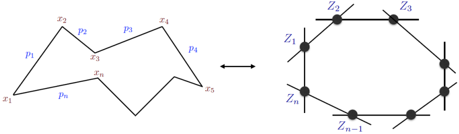

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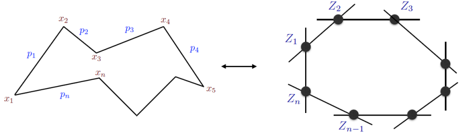

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Cluster $\mathcal{A}$-coordinates a_m: Certain homogeneous polynomials of $\langle ijkl\rangle$

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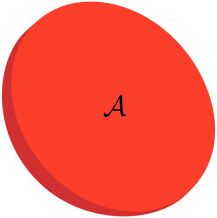

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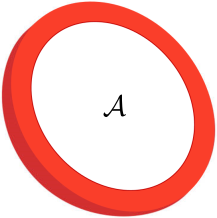

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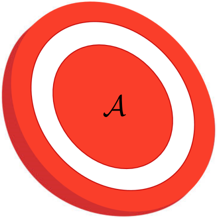

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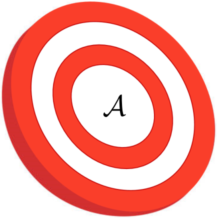

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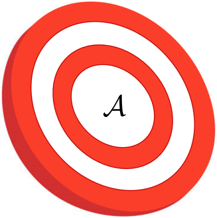

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See also $S(A_n^{(2)}) \to A_n^{(2)}$, $S(\mathcal{A}_7) \to \mathcal{A}_7$ work [Golden(,Paulos),Spradlin(,Volovich)] [Dixon,Liu] [Golden,McLeod]

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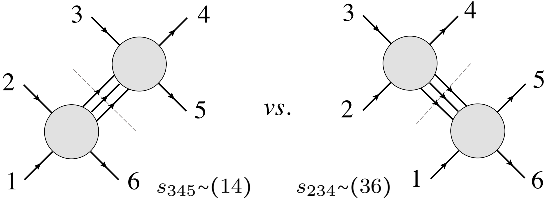

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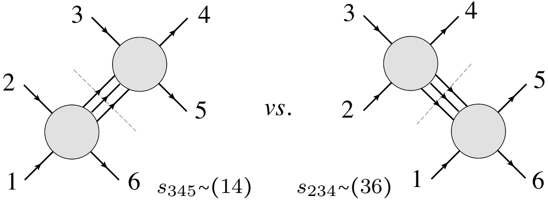

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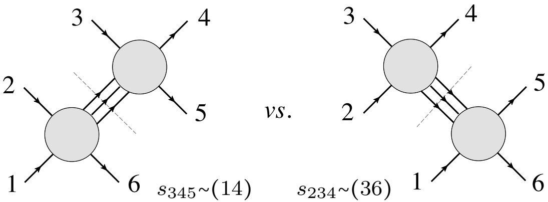

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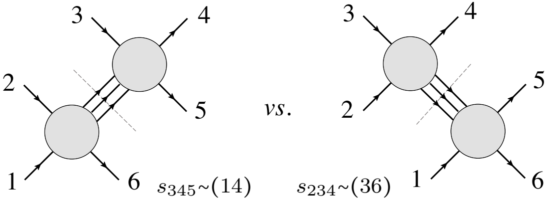

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- Recently confirmed in planar 5-pt 1-mass master integrals.
  [Abreu,Ita,Moriello,Page,Tschernow,Zeng]

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Comparing the two matrices,

$$\boxed{\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}}$$

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$$\text{Mutation on } a_k: \quad a_k' = a_k^{-1}\left(\prod_{i=1}^{d+m} a_i^{[b_{ik}]_+} + \prod_{i=1}^{d+m} a_i^{[-b_{ik}]_+}\right),\ k \le n\,,$$

all other a_j unchanged. b_{ij} elements of B, which itself mutates as

$$b_{ij}' = \begin{cases} -b_{ij} & \text{for } i = k \text{ or } j = k \\ b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}\left[b_{kj}\right]_+ & \text{otherwise} \end{cases},$$

with $[x]_+ = \max(0, x)$.

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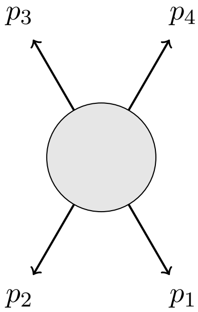

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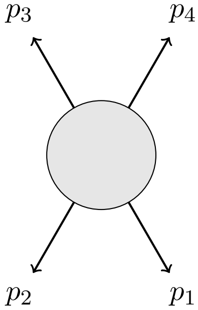

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Alphabet of known integrals: [Henn,Smirnov^2] [Panzer] [Henn,Mistlberger,Smirnov,Wasser]

$$\Phi = \{z, 1+z\}$$

(Nonpositive) Harmonic Polylogarithms [Remiddi,Vermaseren]

Instead, consider next-to simplest case as our main example.

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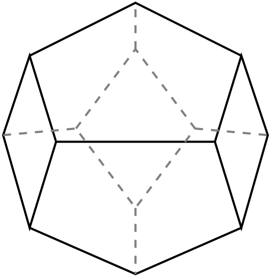

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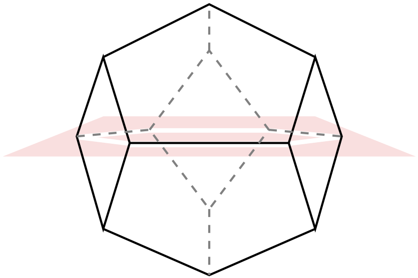
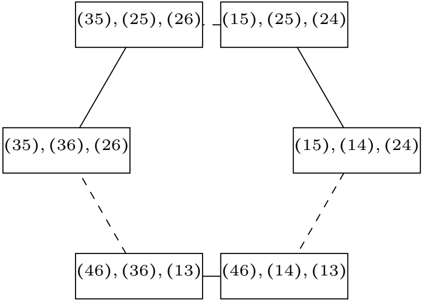
- Nevertheless, C_2 is parity (=up-down reflection) invariant surface of A_3!



Adjacency Interpretation: Embedding C_2 in A_3 II

Zooming in on A_3 equator:



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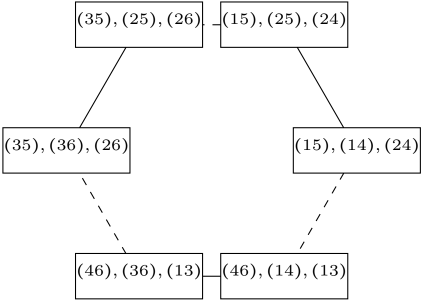

- Only P-invariant combinations of A_3 coordinates (ij) needed.

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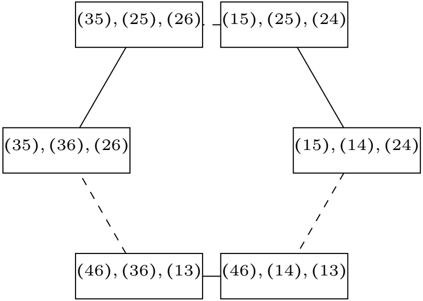

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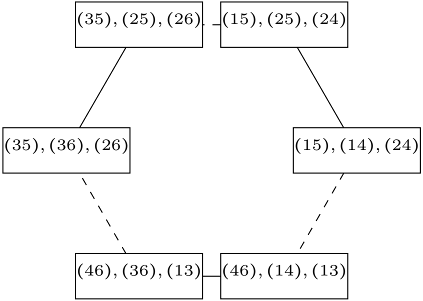

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C_2 adjacency = extended Steinmann relations for $\mathcal{A}_6$!

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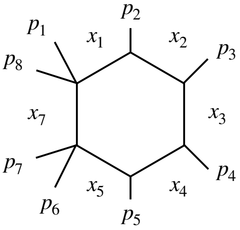

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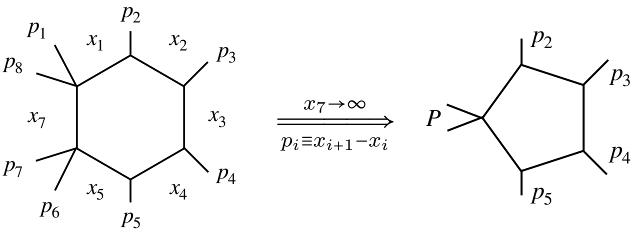

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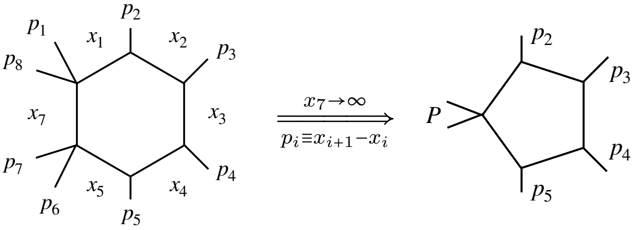

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Specifically, 27 letters of $D = 6$ 2-mass hard hexagon $=$ 1-loop 1-mass pentagon alphabet except $\Delta_3 = \lambda(P^2, s_{23}, s_{45}), \Delta_5 = \det(2p_i \cdot p_j)\big|_{i,j\leq 4}$ $+$ 8 2-loop letters ($\sqrt{\Delta_5}$ rationalized by mom.twistors).

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