# Taming distributional celestial amplitudes

## Corfu, 2021

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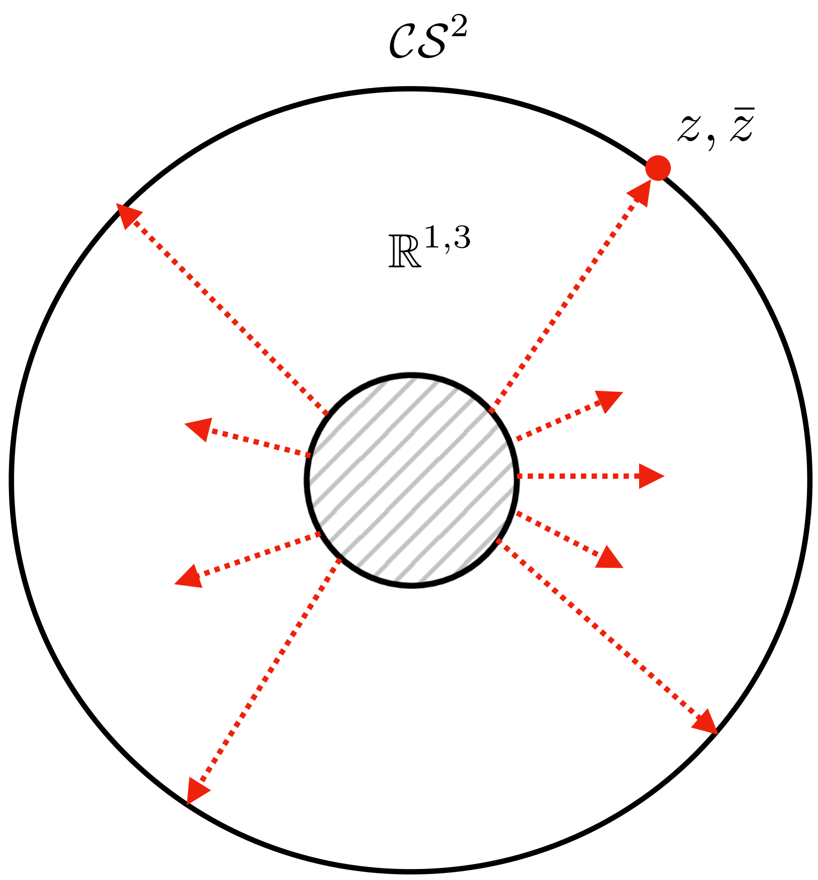
Based on [Sharma '21]

# Locality in the sky?

[Pasterski, Shao '17]




Momentum Eigenstates

Boost Eigenstates

$$|\omega, z, \bar{z}, \ell, \epsilon\rangle \longrightarrow |z, \bar{z}, h, \bar{h}, \epsilon\rangle = \int_0^\infty \frac{\mathrm{d}\omega}{\omega}\, \omega^\Delta\, |\omega, z, \bar{z}, \ell, \epsilon\rangle$$

$$h = \frac{\Delta + \ell}{2}\,,\quad \bar{h} = \frac{\Delta - \ell}{2}$$

$$\epsilon = \pm 1$$

$$\updownarrow$$

$$\mathcal{O}^{\epsilon}_{h,\bar{h}}(z, \bar{z})$$

Local?

**Shadow transform**

$$\mathbf{S}[\mathcal{O}^{\epsilon}_{h,\bar{h}}](z, \bar{z}) = \int_{\mathbb{C}} \frac{\mathrm{d}^2 w\; \mathcal{O}^{\epsilon}_{h,\bar{h}}(w, \bar{w})}{(w - z)^{2-2h}\,(\bar{w} - \bar{z})^{2-2\bar{h}}}$$

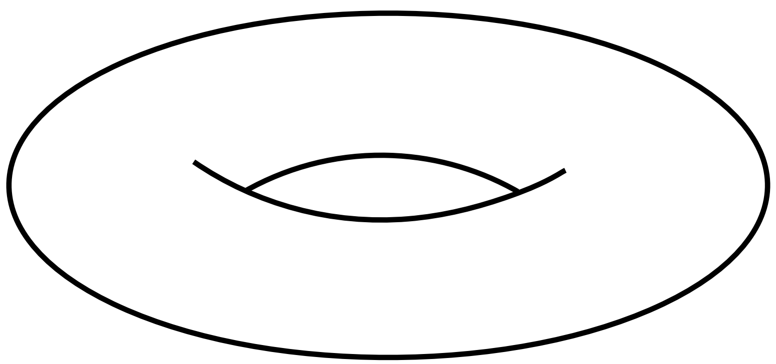
Bulk dual state

$$|z, \bar{z}, 1-h, 1-\bar{h}, \epsilon\rangle_{\mathbf{S}}$$

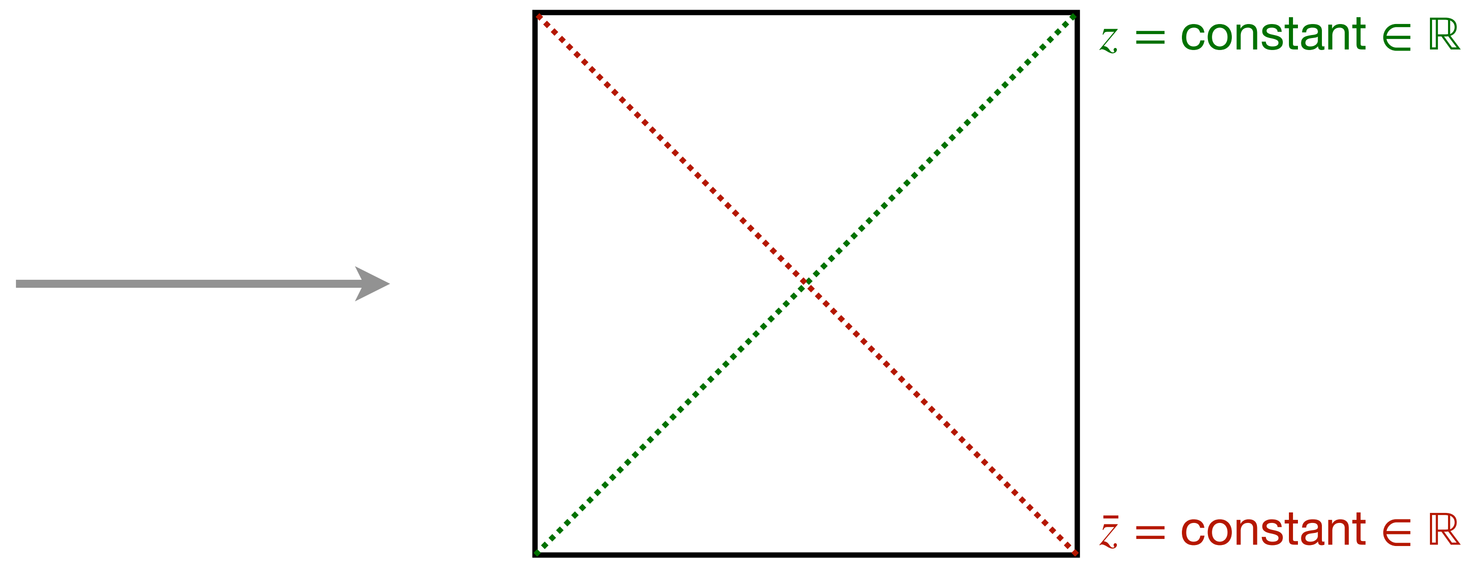
# Light transforms

[Atanasov, Ball, Melton, Raclariu, Strominger '21]

**Celestial torus**
**of $\mathbb{R}^{2,2}$**



$$ds^2 = dz\, d\bar{z}$$




Bulk dual state

**Lights transforms**

$$\mathbf{L}[\mathcal{O}^{\epsilon}_{h,\bar{h}}](z,\bar{z}) = \int_{\mathbb{R}} \frac{dw\ \mathcal{O}^{\epsilon}_{h,\bar{h}}(w,\bar{z})}{(w-z)^{2-2h}} \qquad |z,\bar{z},1-h,\bar{h},\epsilon\rangle_{\mathbf{L}}$$

$$\bar{\mathbf{L}}[\mathcal{O}^{\epsilon}_{h,\bar{h}}](z,\bar{z}) = \int_{\mathbb{R}} \frac{d\bar{w}\ \mathcal{O}^{\epsilon}_{h,\bar{h}}(z,\bar{w})}{(\bar{w}-\bar{z})^{2-2\bar{h}}} \qquad |z,\bar{z},h,1-\bar{h},\epsilon\rangle_{\bar{\mathbf{L}}}$$

[Atanasov, Melton, Raclariu, Strominger '21] [Kravchuk, Simmons-Duffin '18]

# Previously in celestial news

- Shadow transform $\longrightarrow$ adjoint operation in 2d CCFT.

  [Crawley, Miller, Narayanan, Strominger '21]

- Shadow transforms used for gluon conformal block expansions.

  [Fan, Fotopoulos, Stieberger, Taylor, Zhu '21]

- Light transform exchanges observed in scalar conformal block expansions.

  [Atanasov, Melton, Raclariu, Strominger '21]

- Light transform unravels symmetries of self-dual CCFT.

  [Himwich, Pate, Singh '21] [Strominger '21] [Guevara, Himwich, Pate, Strominger '21]

# Today...

- Light transformed celestial amplitudes
- Light transformed celestial OPE
- Connections with twistors

# Ambidextrous choices

[Pasterski, Shao, Strominger '17]

$$\mathcal{A}_n(z_i,\bar{z}_i,\Delta_i,\ell_i,\epsilon_i) = \int_{\mathbb{R}_+^n} \prod_{j=1}^{n} \frac{\mathrm{d}\omega_j}{\omega_j}\, \omega_j^{\Delta_j}\, A_n(\epsilon_i\omega_i q_i,\ell_i)\, \delta^4\Big(\sum_{j=1}^{n} \epsilon_i\omega_i q_i\Big)$$

Celestial amplitude

Momentum space amplitude

$$q_{i\,\alpha\dot\alpha} = \begin{pmatrix} z_i\bar{z}_i & z_i \\ \bar{z}_i & 1 \end{pmatrix}_{\alpha\dot\alpha}$$

$$\epsilon_i = \pm 1$$

$$\mathcal{L}_n(z_i,\bar{z}_i,\Delta_i,\ell_i,\epsilon_i) = \int_{\mathbb{R}^n} \prod_{\bar{\mathsf{a}}} \frac{\mathrm{d}\bar{w}_{\bar{\mathsf{a}}}}{(\bar{w}_{\bar{\mathsf{a}}}-\bar{z}_{\bar{\mathsf{a}}})^{2-2\bar{h}_{\bar{\mathsf{a}}}}} \prod_{\mathsf{a}} \frac{\mathrm{d}w_{\mathsf{a}}}{(w_{\mathsf{a}}-z_{\mathsf{a}})^{2-2h_{\mathsf{a}}}}$$
$$\times\, \mathcal{A}_n(z_{\bar{\mathsf{a}}},\bar{w}_{\bar{\mathsf{a}}},\Delta_{\bar{\mathsf{a}}},-,\epsilon_{\bar{\mathsf{a}}}\,;\,w_{\mathsf{a}},\bar{z}_{\mathsf{a}},\Delta_{\mathsf{a}},+,\epsilon_{\mathsf{a}})$$

$\mathsf{a} \longrightarrow$ Positive helicity gluon/graviton

$\bar{\mathsf{a}} \longrightarrow$ Negative helicity gluon/graviton

# Gluon examples

$$\mathcal{A}_n(i^{\epsilon_i}_{\Delta_i,\ell_i}) \equiv \mathcal{A}_n(z_i, \bar{z}_i, \Delta_i, \ell_i, \epsilon_i) \propto \delta(\beta), \qquad \beta = \mathrm{i}\sum_{j=1}^{n}(\Delta_j - 1)$$

**2 points**

$$\mathcal{A}_2(1^+_{\Delta_1,+1}, 2^-_{\Delta_2,-1}) = \delta(\beta)\,\delta(z_{12})\,\delta(\bar{z}_{12})$$

$$z_{ij} = z_i - z_j$$

$$\Longrightarrow \mathcal{L}_2(1^+_{\Delta_1,+1}, 2^-_{\Delta_2,-1}) = \delta(\beta)\int_{\mathbb{R}^2} \frac{\mathrm{d}w_1\,\mathrm{d}\bar{w}_2\;\delta(w_1 - z_2)\,\delta(\bar{z}_1 - \bar{w}_2)}{(w_1 - z_1)^{1-\Delta_1}\,(\bar{w}_2 - \bar{z}_2)^{1-\Delta_2}}$$

$$= \frac{\delta(\beta)}{z_{21}^{1-\Delta_1}\;\bar{z}_{12}^{1-\Delta_2}}\,.$$

# Gluon examples

**3 points**

$$\mathcal{A}_3(1^-_{\Delta_1,-1},2^-_{\Delta_2,-1},3^+_{\Delta_3,+1}) = \delta(\beta)\,\Theta\!\left(\frac{z_{13}}{z_{12}}\right)\Theta\!\left(\frac{z_{32}}{z_{12}}\right)\frac{\delta(\bar z_{13})\,\delta(\bar z_{23})}{z_{12}^{-\Delta_3}\,z_{32}^{2-\Delta_1}\,z_{13}^{2-\Delta_2}}$$

$$\begin{aligned}\mathcal{L}_3(1^-_{\Delta_1,-1},2^-_{\Delta_2,-1},3^+_{\Delta_3,+1}) &= \delta(\beta)\int_{\mathbb{R}^3}\frac{\mathrm{d}\bar w_1}{(\bar w_1-\bar z_1)^{1-\Delta_1}}\,\frac{\mathrm{d}\bar w_2}{(\bar w_2-\bar z_2)^{1-\Delta_2}}\,\frac{\mathrm{d}w_3}{(w_3-z_3)^{1-\Delta_3}}\\ &\quad\times\Theta\!\left(\frac{z_1-w_3}{z_1-z_2}\right)\Theta\!\left(\frac{w_3-z_2}{z_1-z_2}\right)\frac{\delta(\bar w_1-\bar z_3)\,\delta(\bar w_2-\bar z_3)}{z_{12}^{-\Delta_3}\,(w_3-z_2)^{2-\Delta_1}\,(z_1-w_3)^{2-\Delta_2}}\\ &= \frac{\delta(\beta)}{z_{12}^{-\Delta_3}\,\bar z_{31}^{1-\Delta_1}\,\bar z_{32}^{1-\Delta_2}}\int_{z_2}^{z_1}\frac{\mathrm{sgn}(z_{12})\,\mathrm{d}w_3}{(w_3-z_3)^{1-\Delta_3}\,(w_3-z_2)^{2-\Delta_1}\,(z_1-w_3)^{2-\Delta_2}}\end{aligned}$$

$$\Longrightarrow\quad \mathcal{L}_3(1^-_{\Delta_1,-1},2^-_{\Delta_2,-1},3^+_{\Delta_3,+1}) = \frac{\delta(\beta)\,B(\Delta_1-1,\Delta_2-1)\,\mathrm{sgn}(z_{12})}{z_{13}^{\Delta_1-1}\,z_{23}^{\Delta_2-1}\,\bar z_{31}^{1-\Delta_1}\,\bar z_{32}^{1-\Delta_2}}\,.$$

**Parity conjugate**

$$\mathcal{L}_3(1^+_{\Delta_1,+1},2^+_{\Delta_2,+1},3^-_{\Delta_3,-1}) = \frac{\delta(\beta)\,B(\Delta_1-1,\Delta_2-1)\,\mathrm{sgn}(\bar z_{12})}{z_{31}^{1-\Delta_1}\,z_{32}^{1-\Delta_2}\,\bar z_{13}^{\Delta_1-1}\,\bar z_{23}^{\Delta_2-1}}\,.$$

# Observations

$$\mathcal{L}_2(1^+_{\Delta_1,+1}, 2^-_{\Delta_2,-1}) = \frac{\delta(\beta)}{z_{21}^{1-\Delta_1}\,\bar{z}_{12}^{1-\Delta_2}}\,.$$

$$\mathcal{L}_3(1^+_{\Delta_1,+1}, 2^+_{\Delta_2,+1}, 3^-_{\Delta_3,-1}) = \frac{\delta(\beta)\,B(\Delta_1-1,\Delta_2-1)\,\mathrm{sgn}(\bar{z}_{12})}{z_{31}^{1-\Delta_1}\,z_{32}^{1-\Delta_2}\,\bar{z}_{13}^{\Delta_1-1}\,\bar{z}_{23}^{\Delta_2-1}}\,.$$

- Ambidextrous prescription absorbs *all* momentum conserving delta functions.
- Brings amplitude to standard conformally covariant power law.
- Can read off light transform celestial OPE.

$$\begin{aligned}&\mathbf{L}[O^a_{\Delta_1,+}](z_1,\bar{z}_1)\,\mathbf{L}[O^b_{\Delta_2,+}](z_2,\bar{z}_2)\\ &\qquad\qquad\sim \mathrm{sgn}(\bar{z}_{12})\,B(\Delta_1-1,\Delta_2-1)\,f^{abc}\,\mathbf{L}[O^c_{\Delta_1+\Delta_2-1,+}](z_2,\bar{z}_2)+\cdots\end{aligned}$$

# Graviton examples

**2 points**

$$\mathcal{A}_2(1^+_{\Delta_1,+2}, 2^-_{\Delta_2,-2}) = \delta(\beta)\,\delta(z_{12})\,\delta(\bar{z}_{12})$$

**3 points**

$$\mathcal{A}_3(1^+_{\Delta_1,+2}, 2^+_{\Delta_2,+2}, 3^-_{\Delta_3,-2}) = \delta(\beta+\mathrm{i})\,\Theta\left(\frac{\bar{z}_{13}}{\bar{z}_{12}}\right)\Theta\left(\frac{\bar{z}_{32}}{\bar{z}_{12}}\right)\frac{\delta(z_{13})\,\delta(z_{23})}{\bar{z}_{12}^{-2-\Delta_3}\,\bar{z}_{32}^{2-\Delta_1}\,\bar{z}_{13}^{2-\Delta_2}}$$

$$\downarrow$$

$$\mathcal{L}_2(1^+_{\Delta_1,+2}, 2^-_{\Delta_2,-2}) = \frac{\delta(\beta)}{z_{21}^{-\Delta_1}\,\bar{z}_{12}^{-\Delta_2}}\,.$$

$$\mathcal{L}_3(1^+_{\Delta_1,+2}, 2^+_{\Delta_2,+2}, 3^-_{\Delta_3,-2}) = \frac{\delta(\beta+\mathrm{i})\,B(\Delta_1-1,\Delta_2-1)\,|\bar{z}_{12}|}{z_{31}^{-\Delta_1}\,z_{32}^{-\Delta_2}\,\bar{z}_{13}^{\Delta_1-1}\,\bar{z}_{23}^{\Delta_2-1}}\,.$$

**Celestial OPE**

$$\mathbf{L}[G_{\Delta_1,+}](z_1,\bar{z}_1)\,\mathbf{L}[G_{\Delta_2,+}](z_2,\bar{z}_2) \sim |\bar{z}_{12}|\;B(\Delta_1-1,\Delta_2-1)\,\mathbf{L}[G_{\Delta_1+\Delta_2,+}](z_2,\bar{z}_2)+\cdots$$

# Comparing celestial OPE

[Pate, Raclariu, Strominger, Yuan '19] [Fan, Fotopoulos, Taylor '19]

**For boost eigenstates**

$$O^a_{\Delta_1,+}(z_1,\bar z_1)\,O^b_{\Delta_2,+}(z_2,\bar z_2) \sim \frac{1}{z_{12}}\,B(\Delta_1-1,\Delta_2-1)\,f^{abc}\,O^c_{\Delta_1+\Delta_2-1,+}(z_2,\bar z_2)+\cdots$$

$$G_{\Delta_1,+}(z_1,\bar z_1)\,G_{\Delta_2,+}(z_2,\bar z_2) \sim \frac{\bar z_{12}}{z_{12}}\,B(\Delta_1-1,\Delta_2-1)\,G_{\Delta_1+\Delta_2,+}(z_2,\bar z_2)+\cdots$$

**For light transformed states**

$$\mathbf{L}[O^a_{\Delta_1,+}](z_1,\bar z_1)\,\mathbf{L}[O^b_{\Delta_2,+}](z_2,\bar z_2) \sim \mathrm{sgn}(\bar z_{12})\,B(\Delta_1-1,\Delta_2-1)\,f^{abc}\,\mathbf{L}[O^c_{\Delta_1+\Delta_2-1,+}](z_2,\bar z_2)+\cdots$$

$$\mathbf{L}[G_{\Delta_1,+}](z_1,\bar z_1)\,\mathbf{L}[G_{\Delta_2,+}](z_2,\bar z_2) \sim |\bar z_{12}|\,B(\Delta_1-1,\Delta_2-1)\,\mathbf{L}[G_{\Delta_1+\Delta_2,+}](z_2,\bar z_2)+\cdots$$

Non-singular OPE? 🤔

# Twistor eigenstates

**Spinor-helicity**

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}\,, \qquad \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}} \in \mathbb{R}^2$$

Real and independent in $\mathbb{R}^{2,2}$

**Witten's 1/2 Fourier transform**

$$|Z,\ell\rangle_{\mathbb{T}} \equiv |\lambda,\mu,\ell\rangle_{\mathbb{T}} := \int_{\mathbb{R}^2} \mathrm{d}^2\bar{\lambda}\ \mathrm{e}^{\mathrm{i}[\mu\,\bar{\lambda}]}\,|\lambda,\bar{\lambda},\ell\rangle$$

Twistor eigenstate

Momentum eigenstate

[Witten '03]

[Mason, Skinner '09]

[Arkani-Hamed, Cachazo, Cheung, Kaplan '09]

$$\mu^{\dot{\alpha}} \xleftrightarrow{\text{Fourier conjugate}} \bar{\lambda}_{\dot{\alpha}}$$

**Little group**

$$|r\,Z,\ell\rangle_{\mathbb{T}} = r^{-2\ell-2}\,|Z,\ell\rangle_{\mathbb{T}}\,, \qquad r \in \mathbb{R}^*$$

**Twistors**

$$Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha) \in \mathbb{RP}^3$$

# Primary twistor eigenstates

$$(\mu^{\dot\alpha},\lambda_\alpha)\sim r\,(\mu^{\dot\alpha},\lambda_\alpha) \xrightarrow{\text{Scale fixing}} \mu^{\dot\alpha}=\begin{pmatrix}\bar z\\ 1\end{pmatrix}^{\dot\alpha}\,, \quad \lambda_\alpha=\epsilon\,\omega\begin{pmatrix}z\\ 1\end{pmatrix}_\alpha$$

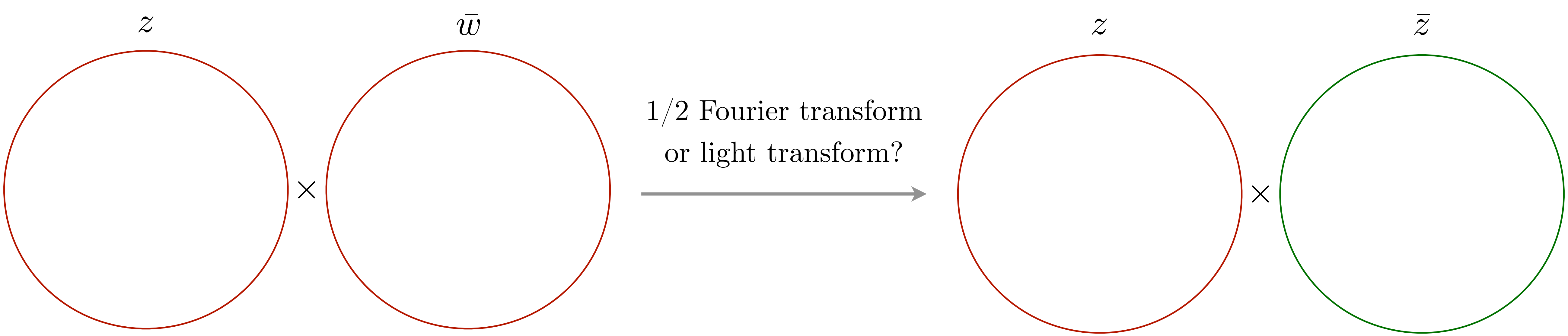
**Conformal primary twistor eigenstate**

$$|z,\bar z,h,1-\bar h,\epsilon\rangle_{\mathbb{T}} := \int_0^\infty \frac{\mathrm{d}\omega}{\omega}\,\omega^{2h}\,|\lambda,\mu,\ell\rangle_{\mathbb{T}} \overset{?}{=} |z,\bar z,h,1-\bar h,\epsilon\rangle_{\bar{\mathbf{L}}}$$

Point on twistor space of $\mathrm{AdS}_3/\mathbb{Z}$ slices

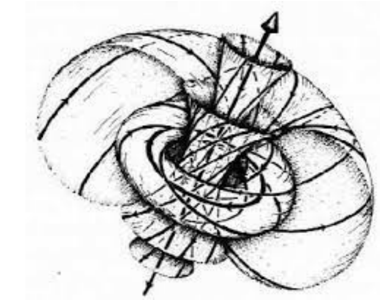
Transforms as conformal primary with these weights

"1/2 Mellin transform"




(Considered as an auxiliary copy of the celestial torus.)

# All roads lead to twistors...



$$
\begin{aligned}
&|z, \bar{z}, h, 1-\bar{h}, \epsilon\rangle_{\mathbb{T}} \\
&\qquad = \epsilon^{-2\ell}\,\Gamma(2-2\bar{h})\left(\mathrm{e}^{\pi\mathrm{i}(1-\bar{h})}\,|z, \bar{z}, h, 1-\bar{h}, \epsilon\rangle_{\bar{\mathbf{L}}} + \mathrm{e}^{\pi\mathrm{i}\bar{h}}\,|z, \bar{z}, h, 1-\bar{h}, -\epsilon\rangle_{\bar{\mathbf{L}}}\right)
\end{aligned}
$$

$$
\begin{tikzcd}
\textbf{Momentum eigenstates} \arrow[r, "1/2\text{ Fourier transform}"] \arrow[d, "\text{Mellin transform}"'] & \textbf{Twistor eigenstates} \arrow[d, "1/2\text{ Mellin transform}"] \\
\textbf{Boost eigenstates} \arrow[r, "\text{Light transforms}"] & \textbf{Primary twistor eigenstates}
\end{tikzcd}
$$

→ Original motivation for ambidextrous prescription.

[Arkani-Hamed, Cachazo, Cheung, Kaplan '09]

# Questions and speculations

- Light transformed boost eigenstates dual to local operators?
- In usual CFTs, correlators involving only light transformed local operators vanish. Reconcile with CCFT results? [Kravchuk, Simmons-Duffin '18]
- Maybe original boost eigenstates dual to *non-local* operators?
- Light transforms "refine" the transform to twistor eigenstates.
- Is celestial holography the origin of the success of twistor strings?
- Twistorial origins of $\mathrm{w}_{1+\infty}$ symmetries in $(2,2)$ signature in a light transform basis? [Strominger '21] (See Mason's talk for story in Lorentzian signature.)