# Classical soft graviton theorem

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# PLAN

1. Summary of the results

2. Classical derivation

3. Relation to soft theorem

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|---|---|
| Alok Laddha, A.S. | arXiv:1806.01872 |
| Biswajit Sahoo, A.S. | arXiv:1808.03288 |
| Arnab Priya Saha, Biswajit Sahoo, A.S. | arXiv:1912.06413 |
| Biswajit Sahoo | arXiv:2008.04376 |
| Biswajit Sahoo, A.S. | arXiv:2105.08739 |

# Results

Consider a classical scattering in space

A set of objects with asymptotic four momenta $p'_1, \cdots p'_m$ come together, interact via complicated forces, and disperse as a set of other objects with asymptotic four momenta $p_1, \cdots p_n$.

$$p_i^2 \equiv -(p_i^0)^2 + \vec{p}_i^2 = -m_i^2, \quad p_i'^2 = -m_i'^2, \quad i = 1, 2, \cdots$$

We shall choose the origin of space-time to be in the region where the scattering event takes place

Detector D placed at $\mathcal{I}^+$ – a far way point $\vec{x}$ – detects $h_{\mu\nu} \equiv (g_{\mu\nu} - \eta_{\mu\nu})/2$ around time $t_0$:

$$t_0 = R/c + \text{correction}, \quad R \equiv |\vec{x}|$$

The correction is due to the gravitational drag on the gravitational radiation.

Define retarded time at the detector:

$$\mathbf{u} \equiv \mathbf{t} - \mathbf{t_0}$$

Our focus will be on the late and early time tail of the radiation – the value of $h_{\mu\nu}$ at D at large positive u and large negative u.

Define $e_{\mu\nu}$ via:

$$e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,h_{\rho\sigma} \quad \Leftrightarrow \quad h_{\mu\nu} = e_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,e_{\rho\sigma}$$

Up to gauge transformations and corrections of order $R^{-2}$,

$$\boxed{\begin{aligned} e_{\mu\nu} &= A_{\mu\nu} + \frac{1}{u}\,B_{\mu\nu} + \frac{\ln|u|}{u^2}\,F_{\mu\nu} + \mathcal{O}(u^{-2}), \quad \text{for large positive u} \\ e_{\mu\nu} &= \frac{1}{u}\,C_{\mu\nu} + \frac{\ln|u|}{u^2}\,G_{\mu\nu} + \mathcal{O}(u^{-2}), \quad \text{for large negative u} \end{aligned}}$$

$A_{\mu\nu}, B_{\mu\nu}, F_{\mu\nu}, C_{\mu\nu}, G_{\mu\nu}$ are given solely by the momenta of the ingoing and outgoing objects without requiring any knowledge of the details of the scattering process.

$$A^{\mu\nu} = \frac{2\,G}{R\,c^3}\left[-\sum_{i=1}^{n} p_i^{\mu}\, p_i^{\nu}\,\frac{1}{n.p_i} + \sum_{i=1}^{m} p_i'^{\mu}\, p_i'^{\nu}\,\frac{1}{n.p_i'}\right], \quad R \equiv |\vec{x}|, \quad n \equiv (1, \vec{x}/R)$$

$$B^{\mu\nu} = -\frac{4\,G^2}{R\,c^7}\left[\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{p_i.p_j}{\{(p_i.p_j)^2 - m_i^2 m_j^2 c^4\}^{3/2}}\left\{\frac{3}{2}m_i^2 m_j^2 c^4 - (p_i.p_j)^2\right\}\right.$$

$$\times\,\frac{p_i^{\mu}}{n.p_i}\,(n.p_j\, p_i^{\nu} - n.p_i\, p_j^{\nu})$$

$$\left. - \left\{\sum_{j=1}^{n} p_j.n \sum_{i=1}^{n}\frac{1}{p_i.n}\,p_i^{\mu} p_i^{\nu} - \sum_{j=1}^{m} p_j'.n \sum_{i=1}^{m}\frac{1}{p_i'.n}\,p_i'^{\mu} p_i'^{\nu}\right\}\right]$$

$$C^{\mu\nu} = \frac{4\,G^2}{R\,c^7}\left[\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{p_i'.p_j'}{\{(p_i'.p_j')^2 - m_i'^2 m_j'^2 c^4\}^{3/2}}\left\{\frac{3}{2}m_i'^2 m_j'^2 c^4 - (p_i'.p_j')^2\right\}\right.$$

$$\left.\times\,\frac{p_i'^{\mu}}{n.p_i'}\,(n.p_j'\, p_i'^{\nu} - n.p_i'\, p_j'^{\nu})\right].$$

$$F^{\mu\nu} = 2\,\frac{G^3}{R\,c^{11}}\Bigg[4\left\{\sum_{j=1}^{n} p_j.n \sum_{\ell=1}^{n} p_\ell.n \sum_{i=1}^{n} \frac{p_i^\mu p_i^\nu}{p_i.n} - \sum_{j=1}^{m} p'_j.n \sum_{\ell=1}^{m} p'_\ell.n \sum_{i=1}^{m} \frac{p_i'^\mu p_i'^\nu}{p'_i.n}\right\}$$

$$+4\sum_{\ell=1}^{n} p_\ell.n \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{p_i.n}\,\frac{p_i.p_j}{\{(p_i.p_j)^2 - p_i^2 p_j^2\}^{3/2}}\{2(p_i.p_j)^2 - 3p_i^2 p_j^2\}\{n.p_j\, p_i^\mu\, p_i^\nu - n.p_i\, p_i^{(\mu}\, p_j^{\nu)}\}$$

$$+2\sum_{\ell=1}^{m} p'_\ell.n \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{1}{p'_i.n}\,\frac{p'_i.p'_j}{\{(p'_i.p'_j)^2 - p_i'^2 p_j'^2\}^{3/2}}\{2(p'_i.p'_j)^2 - 3p_i'^2 p_j'^2\}\{n.p'_j\, p_i'^\mu\, p_i'^\nu - n.p'_i\, p_i'^{(\mu}\, p_j'^{\nu)}\}$$

$$+\sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{\ell=1\\ \ell\neq i}}^{n} \frac{1}{p_i.n}\,\frac{p_i.p_j}{\{(p_i.p_j)^2 - p_i^2 p_j^2\}^{3/2}}\{2(p_i.p_j)^2 - 3p_i^2 p_j^2\}\,\frac{p_i.p_\ell}{\{(p_i.p_\ell)^2 - p_i^2 p_\ell^2\}^{3/2}}$$

$$\{2(p_i.p_\ell)^2 - 3p_i^2 p_\ell^2\}\{n.p_j\, p_i^\mu - n.p_i\, p_j^\mu\}\,\{n.p_\ell\, p_i^\nu - n.p_i\, p_\ell^\nu\}\Bigg]\,,$$

$$G^{\mu\nu} = -2\,\frac{G^3}{R\,c^{11}}\Bigg[2\sum_{\ell=1}^{n} p'_\ell.n \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{1}{p'_i.n}\,\frac{p'_i.p'_j}{\{(p'_i.p'_j)^2 - p_i'^2 p_j'^2\}^{3/2}}\{2(p'_i.p'_j)^2 - 3p_i'^2 p_j'^2\}$$

$$\{n.p'_j\, p_i'^\mu\, p_i'^\nu - n.p'_i\, p_i'^{(\mu}\, p_j'^{\nu)}\}$$

$$-\sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} \sum_{\substack{\ell=1\\ \ell\neq i}}^{m} \frac{1}{p'_i.n}\,\frac{p'_i.p'_j}{\{(p'_i.p'_j)^2 - p_i'^2 p_j'^2\}^{3/2}}\{2(p'_i.p'_j)^2 - 3p_i'^2 p_j'^2\}\,\frac{p'_i.p'_\ell}{\{(p'_i.p'_\ell)^2 - p_i'^2 p_\ell'^2\}^{3/2}}$$

$$\{2(p'_i.p'_\ell)^2 - 3p_i'^2 p_\ell'^2\}\{n.p'_j\, p_i'^\mu - n.p'_i\, p_j'^\mu\}\,\{n.p'_\ell\, p_i'^\nu - n.p'_i\, p_\ell'^\nu\}\Bigg]\,.$$

$$\mathbf{e}_{\mu\nu} = \mathbf{A}_{\mu\nu} + \frac{1}{\mathbf{u}}\,\mathbf{B}_{\mu\nu} + \frac{\ln|\mathbf{u}|}{\mathbf{u}^2}\,\mathbf{F}_{\mu\nu} + \mathcal{O}(\mathbf{u}^{-2}), \quad \textbf{for large positive u}$$

$$\mathbf{e}_{\mu\nu} = \frac{1}{\mathbf{u}}\,\mathbf{C}_{\mu\nu} + \frac{\ln|\mathbf{u}|}{\mathbf{u}^2}\,\mathbf{G}_{\mu\nu} + \mathcal{O}(\mathbf{u}^{-2}), \quad \textbf{for large negative u}$$

**$A_{\mu\nu}$: memory term**

**– a permanent change in the state of the detector after the passage of gravitational waves**

**Zeldovich, Polnarev; Braginsky, Grishchuk; Braginsky, Thorne; $\cdots$**

**– related to the leading soft graviton theorem** **Strominger; $\cdots$**

**$B_{\mu\nu}, F_{\mu\nu}, C_{\mu\nu}, G_{\mu\nu}$: tail terms**

**– related to logarithmic terms in the subleading and subsusbleading soft graviton theorem**

**Laddha, A.S.; Sahoo, A.S.**
**Saha, Sahoo, A.S.; Sahoo**

1. The result is a statement in classical GR, even though it was originally suggested by quantum soft graviton theorem.

2. $A_{\mu\nu}, B_{\mu\nu}, F_{\mu\nu}, C_{\mu\nu}, G_{\mu\nu}$ can be expressed in terms of the momenta of incoming and outgoing objects without knowing what forces operated and how the objects moved during the scattering.

3. For charged particles there are known corrections to this formula due to long range electromagnetic interactions

4. In the expressions for $A_{\mu\nu}$, $B_{\mu\nu}$ and $F_{\mu\nu}$, the sum over final state particles i,j includes integration over outgoing flux of radiation, regarded as a flux of massless particles.

For $A_{\mu\nu}$ this gives the 'non-linear memory' term

Christodoulou; Thorne; Blanche, Damour; Bieri, Garfinkle; $\cdots$

Due to some miraculous cancellation, in $B_{\mu\nu}$ and $F_{\mu\nu}$ the contribution from massless final states can be expressed in terms of massive state momenta

– makes it easier to calculate these since we do not need to find the gravitational wave spectrum emitted during the scattering.

In $B_{\mu\nu}$, drop massless particles / radiation contribution in the sum over final states, and add

$$-\frac{4\,G^2}{R\,c^7}\left[P_F^\mu P_F^\nu - P_I^\mu P_I^\nu\right]$$

$P_I$: total incoming momentum

$P_F$: total outgoing momentum carried by massive particles

In $F_{\mu\nu}$, drop massless particles / radiation contribution in the sum over final states, and add

$$-\frac{8\,G^3}{R\,c^{11}}\left[n.P_F\,P_F^\mu\,P_F^\nu - n.P_I\,P_I^\mu P_I^\nu\right]$$

Note: These are not new formulæ but follow from manipulating the results shown earlier

**Furthermore, if the final state has at most one massive object and arbitrary number of massless particles, then $B_{\mu\nu}$ and $F_{\mu\nu}$ become totally independent of the final state momenta.**

**This has somewhat unusual consequences.**

**Suppose we have a pair of massless / ultra-relativistic particles passing each other at large impact parameter.**

**A far away detector detects some gravitational wave profile due to late time acceleration of these particles under each other's gravitational field.**

**If we now reduce the impact parameter the particles will scatter and emit gravitational waves**

**Except for the memory term, the late time gravitational wave profile detected in the faraway detector will remain unchanged since it is insensitive to final momenta**

If we reduce the impact parameter even further, the particles will coalesce to form a black hole together with massless radiation.

The late time gravitational wave profile at the faraway detector still remains the same as if nothing has happened

– there is only one massive object in the final state!

The same argument tells us that the 1/u and $\ln|u|/u^2$ tails vanish for binary black hole merger

One object (bound system)
$\to$ one object + gravitational radiation

– equivalent to one object $\to$ one object for $B_{\mu\nu}$ or $F_{\mu\nu}$ computation

– no $B_{\mu\nu}$ or $F_{\mu\nu}$

5. So far in our formulæ there is no spin dependence.

– expected to arise at order $u^{-2}$

Ghosh, Sahoo, arXiv:2106.10741

After taking Fourier transform these results also give the power spectrum $P(\omega)$ of soft gravitational radiation.

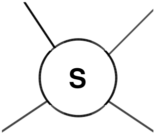
Example: Power spectrum of soft gravitational radiation for the scattering of a pair of massless particles in cm frame at large impact parameter / small deflection angle

$$\frac{E^2}{\pi^2}\,\frac{\theta_s^2}{2}\left\{1+2\ln 2+\ln\theta_s^{-2}+\mathcal{O}(\theta_s^3)\right\}$$
$$+\quad \frac{E^4}{2\pi^4}\omega^2(\ln\omega)^2\left[1-\frac{\theta_s^2}{2}\left\{1+2\ln 2+\ln\theta_s^{-2}\right\}+\mathcal{O}(\theta_s^3)\right]+\mathcal{O}(\omega^2\ln\omega)\,.$$

E: energy of each particle, $\theta_s$ deflection angle

– confirms a conjecture by Ciafaloni, Colferai and Veneziano on the sign of the $\omega^2(\ln\omega)^2$ term, but the coefficient differs by a factor of 2 from the conjectured value.

# Classical derivation




1. We divide the space-time into the scattering region S where complicated interactions take place and the asymptotic region F where the particles interact via long range gravitational force.

2. We iteratively solve the coupled equation of matter and gravity in the asymptotic region F

– matter equations are evolved forward for incoming particles and backward for outgoing particles since initial and final momenta are known (includes hard radiation emitted from S)

– gravitational equations in F are always evolved forward using retarded Green's function

We write

$$\mathbf{g}_{\mu\nu} = \eta_{\mu\nu} + 2\,\mathbf{h}_{\mu\nu}, \quad \mathbf{e}_{\mu\nu} = \mathbf{h}_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,\mathbf{h}_{\rho\sigma}, \quad \mathbf{e}^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}\mathbf{e}_{\alpha\beta}$$

and rewrite Einstein's equation in de Donder gauge $\partial^\mu \mathbf{e}_{\mu\nu} = 0$, as

$$\Box\,\mathbf{e}^{\mu\nu} = -8\,\pi\,\mathbf{G}\,\mathbf{T}^{\mu\nu}(\mathbf{x}), \qquad \Box \equiv \eta^{\rho\sigma}\partial_\rho\partial_\sigma \qquad \mathbf{T}^{\mu\nu} \equiv \mathbf{T}^{\mathbf{X}\mu\nu} + \mathbf{T}^{\mathbf{h}\mu\nu}$$

$\mathbf{T}^{\mathbf{X}\mu\nu}$: matter stress tensor

$\mathbf{T}^{\mathbf{h}\mu\nu}$ captures all terms quadratic and higher order in $\mathbf{h}_{\rho\sigma}$ on the left hand side of Einstein's equation.

From now on all indices will be raised and lowered by the flat metric $\eta$ and we shall set c=1

Note: We are not assuming weak gravity at this stage.

$\Box\, \mathbf{e}^{\mu\nu} = -8\,\pi\,\mathbf{G}\,\mathbf{T}^{\mu\nu}$ can be 'solved' as:

$$\mathbf{e}^{\mu\nu}(\mathbf{x}) = -8\,\pi\,\mathbf{G} \int \mathbf{d}^4\mathbf{y}\,\mathbf{G_r}(\mathbf{x},\mathbf{y})\,\mathbf{T}^{\mu\nu}(\mathbf{y})$$

$\mathbf{G_r}(\mathbf{x},\mathbf{y})$: retarded Green's function in flat space-time

Using explicit form of $\mathbf{G_r}$ one finds that for large $\mathbf{R} \equiv |\vec{\mathbf{x}}|$,

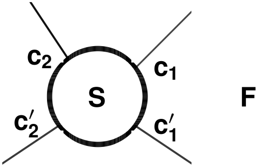
$$\tilde{\mathbf{e}}^{\mu\nu}(\omega,\vec{\mathbf{x}}) = \frac{2\,\mathbf{G}}{\mathbf{R}}\,\mathbf{e}^{\mathbf{i}\,\omega\,\mathbf{R}}\,\widehat{\mathbf{T}}^{\mu\nu}(\mathbf{k}), \quad \mathbf{k} = \omega(\mathbf{1},\hat{\mathbf{n}}), \quad \hat{\mathbf{n}} \equiv \vec{\mathbf{x}}/\mathbf{R}$$

$$\tilde{\mathbf{e}}^{\mu\nu}(\omega,\vec{\mathbf{x}}) = \int \mathbf{dt}\,\mathbf{e}^{\mathbf{i}\omega\mathbf{t}}\,\mathbf{e}^{\mu\nu}(\mathbf{t},\vec{\mathbf{x}}), \qquad \widehat{\mathbf{T}}^{\mu\nu}(\mathbf{k}) \equiv \int \mathbf{d}^4\mathbf{x}\,\mathbf{e}^{-\mathbf{ik.x}}\,\mathbf{T}^{\mu\nu}(\mathbf{x})$$

The memory, $\mathbf{u}^{-1}$ and $\mathbf{u}^{-2}\ln\mathbf{u}$ terms in $\mathbf{e}^{\mu\nu}$ arise from $1/\omega$, $\ln\omega$ and $\omega(\ln\omega)^2$ terms in $\widehat{\mathbf{T}}^{\mu\nu}$

$$\tilde{e}^{\mu\nu}(\omega, \vec{x}) = \frac{2\,G}{R}\, e^{i\,\omega\,R}\, \widehat{T}^{\mu\nu}(k), \quad \widehat{T}^{\mu\nu}(k) \equiv \int d^4x\, e^{-ik.x}\, T^{\mu\nu}(x)$$

We shall divide the integration region over x into two parts:




1. Scattering region S: A region of large size L around x=0.

2. Asymptotic region F: Complement of S

Since our goal is to compute terms in $\widehat{T}_{\mu\nu}$ that are non-analytic as $\omega \to 0$, we can ignore the contribution from the finite region S in $\int d^4x$.

In the asymptotic region, we can regard $T^{X\mu\nu}$ as due to the incoming and outgoing object trajectories, moving under each others' long range gravitational field.

$$
\begin{aligned}
T^{X\mu\nu}(x) \equiv & \sum_{i=1}^{n} m_i \int_0^{\infty} d\tau\, \delta^{(4)}(x - X_i(\tau))\, \frac{dX_i^{\mu}}{d\tau}\, \frac{dX_i^{\nu}}{d\tau} \\
& + \sum_{i=1}^{m} m_i' \int_{-\infty}^{0} d\tau\, \delta^{(4)}(x - X_i'(\tau))\, \frac{dX_i'^{\mu}}{d\tau}\, \frac{dX_i'^{\nu}}{d\tau} + \cdots ,
\end{aligned}
$$

$$
T^{\mu\nu}(x) = T^{X\mu\nu}(x) + T^{h\mu\nu}(x),
$$

$$
\Box\, e^{\mu\nu} = -8\,\pi\, G\, T^{\mu\nu}\,,
$$

$$
\frac{d^2 X_i^{\mu}}{d\tau^2} = -\Gamma^{\mu}_{\nu\rho}(X(\tau))\, \frac{dX_i^{\nu}}{d\tau}\, \frac{dX_i^{\rho}}{d\tau}, \qquad \frac{d^2 X_i'^{\mu}}{d\tau^2} = -\Gamma^{\mu}_{\nu\rho}(X'(\tau))\, \frac{dX_i'^{\nu}}{d\tau}\, \frac{dX_i'^{\rho}}{d\tau}\,,
$$

**Boundary conditions:**

$$
X_i^{\mu}(\tau = 0) = c_i^{\mu}, \qquad \lim_{\tau\to\infty} \frac{dX_i^{\mu}}{d\tau} = V_i^{\mu} = \frac{1}{m_i}\, p_i^{\mu},
$$

$$
X_i'^{\mu}(\tau = 0) = c_i'^{\mu}, \qquad \lim_{\tau\to-\infty} \frac{dX_i'^{\mu}}{d\tau} = V_i'^{\mu} = \frac{1}{m_i'}\, p_i'^{\mu}\,.
$$

**– difference from earlier approach**

**Goldberger, Ridgway; Kosower, Maybee, O'Connell; $\cdots$**

**We solve these equations iteratively, starting with the solution:**

$$\mathbf{e}_{\mu\nu} = \mathbf{0}, \quad \mathbf{X}_i^\mu(\tau) = \mathbf{c}_i^\mu + \mathbf{V}_i^\mu\,\tau = \mathbf{c}_i^\mu + \frac{1}{\mathbf{m}_i}\,\mathbf{p}_i^\mu\,\tau,$$

$$\mathbf{X}_i'^\mu(\tau) = \mathbf{c}_i'^\mu + \mathbf{V}_i'^\mu\,\tau = \mathbf{c}_i'^\mu + \frac{1}{\mathbf{m}_i'}\,\mathbf{p}_i'^\mu\,\tau\,.$$

**This generates a series expansion in G M $\omega$, possibly with corrections involving ln $\omega$ factors.**

**In order to get $\omega^{-1}$ and ln $\omega$ terms, it is enough to do one iteration.** Saha, Sahoo, A.S.

**For $\omega^2 \ln\omega$ term, we need one more iteration.** Sahoo

**Taking Fourier transform we recover the results quoted earlier.**

# Relation to soft theorem

Soft theorems give the result for an amplitude with M soft gravitons in terms of the amplitude without the soft gravitons

Weinberg; Cachazo, Strominger; $\cdots$

This is simplest in D $\geq$ 5 where the S-matrix is free from IR divergence.

General structure

$$A_{soft+hard} = S_M\, A_{hard}$$

$S_M$: a matrix differential operator involving orbital and spin angular momenta of external states

The classical limit is taken by taking the

Laddha, A.S.

– hard particles to have mass $>>$ $M_{pl}$, and

– by replacing the orbital and spin angular momentum operators by classical spin and orbital angular momenta of external states.

In the classical limit, the soft theorem simplifies to:

$$\mathbf{A}_{\text{soft+hard}} = \left\{ \prod_{\alpha=1}^{M} \mathbf{S}_\alpha \right\} \times \mathbf{A}_{\text{hard}}$$

with $S_\alpha$ depending only on the quantum number of the $\alpha$-th soft particle and those of the hard particles.

We can now find the conditional probability of emitting M soft gravitons of certain quantum number for given set of ‘classical’ hard incoming and outgoing particles

$$\frac{1}{M!} |S|^{2M} \times (\text{phase space factor})^M \times |A_{\text{hard}}|^2$$

By maximizing this with respect to M, we find the classical number of soft gravitons with given quantum numbers.

This can be used to determine the (Fourier transform of the) gravitational wave-form up to a phase.

In D $\geq$ 5, this formula has been tested in many examples.

Soft factor gives the gravitational wave-form without any additional phase.

However in D=4 there are two subtleties.

1. $S_\alpha$ depends on the angular momenta of the hard particles.

$$J_i^{\mu\nu} = x_i^\mu p_i^\nu - x_i^\nu p_i^\mu + S_i^{\mu\nu}, \quad S_i^{\mu\nu} : \text{spin}$$

Due to the long range gravitational force of the other particles, $x_i^\mu$ has logarithmic correction to its trajectory:

$$x_i^\mu = p_i^\mu \tau / m_i + c_i^\mu \ln \tau + \cdots$$

$J_i^{\mu\nu}$ diverges as $\tau \to \infty$.

We use the wave-length of the gravitational wave as an ad hoc infra-red cut-off on $\tau$.

2. Due to long range gravitational force on the soft gravitons, they have a Coulomb phase:

$$\exp[i\,\omega\,R] \quad \Rightarrow \quad \exp[i\,\omega\,\{R - C\ln R\}]$$

We replace $\ln R$ by $\ln(R\,\omega)$ with the intuition that the Coulomb drag on a wave of wavelength $\lambda$ acts over the distance $\lambda$ to R

– correctly captures the effect of gravitational backscattering

Peters, Blanchet, Goldberger, Ross, Rothstein, $\cdots$

With these two ansatz on cut-off, we get the results for the gravitational wave-form quoted earlier after taking a Fourier transform.

In D=4, the classical results from soft theorem should be regarded as conjectures rather than derivations.

Nevertheless, soft theorem leads to the correct conjectures!

More importantly, it teaches us what questions might have universal answers independent of the details of the scattering process, generalizing the memory effect.

e.g. we get universal results by asking for soft radiation for given initial and final hard particles, instead of just for given initial data.

# For the future:

Power law decay in u comes from non-analytic function in frequency space

– arise from IR divergent terms and should therefore be determined by soft physics.

1. Can we develop a systematic procedure for computing all higher order terms in the large u expansion?

2. Does the magical independence of the result on the final state massless particle data continue to hold?

3. Do all such terms vanish for the black hole merger problem?