CORFU 2021

Celestial OPE Blocks

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Motivation : CFT "Road Map"

1) $\langle O_i O_j O_k \rangle = \dfrac{C_{ijk}}{|z_{ij}|^{\Delta_k-\Delta_i-\Delta_j}|z_{ik}|^{\Delta_j-\Delta_i-\Delta_k}|z_{jk}|^{\Delta_i-\Delta_j-\Delta_k}}$

ii) $O_i O_j \sim \dfrac{C_{ijk} O_k}{|z_{ij}|^{\Delta_k-\Delta_i-\Delta_j}}$ for $\langle O_i O_j \rangle = \dfrac{\delta_{ij}}{|z_{ij}|^{\Delta_i}}$

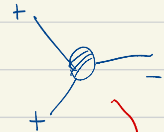
iii) $G(z,\bar z) = \sum C_{ijm} C_{mlk}\, G^{ij\to lk}_{\Delta_m}(z,\bar z)$

$\hookrightarrow \langle O_i O_j O_l O_k \rangle = \dfrac{G(z,\bar z)}{|z_{ij}|^{\#}|z_{lk}|^{\#}}$

$\hookrightarrow$ Crossing eqs, etc..

We only need 3-pt functions as building blocks!

CCFT (e.g. YM)

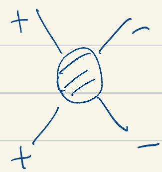


- Lorentzian
- Distributional

???

$O^+_{\Delta_1} O^+_{\Delta_2} \sim \dfrac{B(\Delta_1-1,\Delta_2-1)}{z_{12}}\, O^+_{\Delta_1+\Delta_2-1}$

???



$= \sum_{\Delta} (\ldots)\, G_\Delta(z,\bar z)$

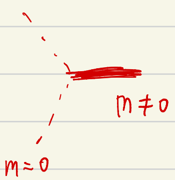
- Euclidean & Lorentzian
- Distributional

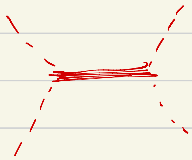
see talks by Tom & Ana

2 questions:

- What is the relation between the 3-pt functions & the OPE !!!
- What is the relation between euclidean & Lorentzian? How does the spectrum differs?

In this talk we will answer these two questions for the scalar CFT

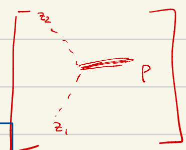




$$\rightsquigarrow \langle O_{\Delta_1} O_{\Delta_2} O^{(m)}_{\Delta_3} \rangle$$

$$\langle O_{\Delta_1} O_{\Delta_2} O_{\Delta_3} O_{\Delta_4} \rangle$$

[Lam & Shao; Nandan, Schreiber, Volovich, Zlotnikov; Zlotnikov & Law; Atanasov, Melton, Raclariu, Strominger; Chang, Huang, Huang, Li ...]

$$\langle O_{\Delta_1} O_{\Delta_2} O^{m}_{\Delta_3} \rangle = \int dw_1\, w_1^{\Delta_1 - 1} \int dw_2\, w_2^{\Delta_2 - 1} \int [d^3 p]\, \varphi^{(m)}_{\Delta_3}(z_3; p) \Big[$$  $$\Big]$$

Pasterski & Shao: $\boxed{\varphi^m_\Delta$: Complete basis of wavefunctions: $\Delta_3 \in 1 + i\mathbb{R}$ "principal series"$}$

Will come back to Yang-Mills at the end of the talk!

The main tool we introduce is the Celestial OPE block

$$O_{\Delta_1}(z_1, \bar{z}_1)\, O_{\Delta_2}(z_2, \bar{z}_2) = \int_{\Delta_3 \in 1 + i\mathbb{R}} d\Delta_3\, d^2 z_3\, (\Delta_3 - 1)^2\, \mathcal{K}(\Delta_i; z_i; \bar{z}_i)\, O^{m}_{\Delta_3}(z_3, \bar{z}_3)$$

$\hookrightarrow \langle O_{\Delta_1} \tilde{O}_{\Delta_2} \rangle = (\varphi_{\Delta_1}, \varphi_{2-\Delta_2})$

$$O_{\Delta_1}(z_1,\bar{z}_1)\, O_{\Delta_2}(z_2,\bar{z}_2) = \int_{\Delta_3 \in 1+i\mathbb{R}} d\Delta_3\, d^2 z_3\, (\Delta_3 - 1)^2\, \mathcal{K}(\Delta_i, z_i, \bar{z}_i)\, O^m_{\Delta_3}(z_3, \bar{z}_3)$$

$\hookrightarrow$ completeness of wavefunctions

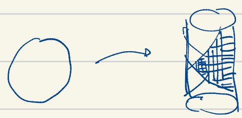
1) Is covariant under conformal symmetry (includes all descendants)

$SL(2,\mathbb{C})$ $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

2) "OPE coefficients" $\mathcal{K}(\Delta_i, z_i, \bar{z}_i)$ can be completely fixed from

- 3pt & 2pt functions
- Poincare symmetry

3) can be used for both Lorentzian or Euclidean CCFTs



$\int d^2 z$ (sphere) $\int dz\, d\bar{z}$ (Lorentzian cylinder) $\supset$ celestial torus.

4) OPE limit: extract primaries, shadows & light-rays in the OPE

5) Related to higher-points constructibility ; partial waves?

1) $$K'(\Delta_i, z_i', \bar{z}_i') = \left(\frac{a z_1 + b}{c z_1 + d}\right)^{\Delta_1} \left(\frac{a z_2 + b}{c z_2 + d}\right)^{\Delta_2} \left(\frac{a z_3 + b}{c z_3 + d}\right)^{2-\Delta_3} \times \text{c.c.} \times K(\Delta_i, z_i, \bar{z}_i)$$

2) Translation covariance reads

$$[P^N_{\Delta_1}, O_{\Delta_1}] O_{\Delta_2} + O_{\Delta_1} [P^N_{\Delta_2}, O_{\Delta_2}] = \int d\Delta_3 \, d^2 z_3 (\Delta_3 - 1)^2 K(\Delta_i, z_i) \left[P^{(m)\,N}_{\Delta_3} \, O^{(m)}_{\Delta_3} \right]$$

where $$P^N_\Delta(z) = q^N(z) \, e^{\partial_\Delta}$$

$$P^{m(N)}_\Delta(z) = \frac{m}{2}\left[\left(\partial\bar{\partial} q^N + \frac{\partial q^N \bar{\partial} + \bar{\partial} q^N \partial}{\Delta - 1} + \frac{q^N \partial\bar{\partial}}{(\Delta-1)^2}\right) e^{-\partial_\Delta} + \frac{\Delta \, q^N}{\Delta - 1} e^{\partial_\Delta}\right]$$

Integrating $\partial_\Delta, \partial_z, \bar{\partial}_{\bar{z}}$ by parts

$$\int d\Delta_3 \, d^2 z_3 (\Delta_3 - 1)^2 K(\Delta_i, z_i) \left[P^{(m)\,N}_{\Delta_3} \, O^{(m)}_{\Delta_3} \right] = \int d\Delta_3 \, d^2 z_3 (\Delta_3 - 1)^2 P^{(m)N}_{2-\Delta_3} [K(\Delta_i, z)] \, O^{(m)}_{\Delta_3}$$

So $$\left(P^N_{\Delta_1} + P^N_{\Delta_2} - P^N_{2-\Delta_3}\right) K(\Delta_i, z_i, \bar{z}_i) = 0$$

From I) & II) [Law & Zlotnikov]

$$K(\Delta_1, \Delta_2, 2-\Delta_3) = \lambda \frac{B\left(\frac{\Delta_1 - \Delta_2 + \Delta_3}{2}, \frac{\Delta_2 - \Delta_1 + \Delta_3}{2}\right)}{|z_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |z_{13}|^{\Delta_1 + \Delta_3 - \Delta_2} |z_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}}$$

We can see how the block contain the 3pt data by contracting:

$$\langle O_{\Delta_1} O_{\Delta_2} O^{(m)}_{\Delta_3'} \rangle = \int d\Omega_3 \, K(\Delta_i, z_i) \underbrace{\langle O^{(m)}_{\Delta_3} O^{(m)}_{\Delta_3'} \rangle}$$

$$-2\pi \frac{\delta(i(\Delta_3 + \Delta_3' - 2))}{(\Delta_3 - 1)^2} \delta^2(z_3 - z_3') + \frac{1}{\Delta_3 - 1} \frac{\delta(i(\Delta_3 - \Delta_3'))}{|z_3 - z_3'|^{2\Delta_3}}$$

$$\Rightarrow \langle O_{\Delta_1} O_{\Delta_2} O^{(m)}_{\Delta_3} \rangle = K(\Delta_1, \Delta_2, 2 - \Delta_3; z_1, z_2, z_3)$$

Importantly, both terms contribute since the block is "shadow symmetric", i.e. symmetric under $\Delta \leftrightarrow 2 - \Delta$

$$\tilde{O}^m_\Delta = \int \frac{d^2 z_p}{2\pi} \frac{O^m_\Delta(z_p)}{|z - z_p|^{2(2-\Delta)}} = \frac{O^m_{2-\Delta}}{\Delta - 1}$$ "no preferred choice"

So we can alternatively write the block as

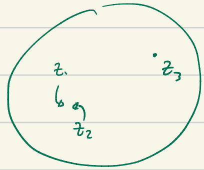
$$O_{\Delta_1} O_{\Delta_2} = \int_{\Delta_3 \in 1 + i\mathbb{R}^+} d\Delta_3 \, (\Delta_3 - 1)^2 \int d^2 z \, \tilde{K}(\Delta_i, z_i) \, O^{(m)}_{\Delta_3}$$

Interestingly, this form contains the OPE data of both $O^{(m)}_{\Delta_3}$ and its shadow $O^{(m)}_{2-\Delta_3}$ (locally independent)

OPE limit & Extracting local operators:

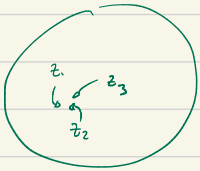
similar to partial waves → conformal blocks

1) $|z_{31}| \sim |z_{32}| \gg |z_{12}|$



$$\int d^2 z_3 \lim_{z_1 \to z_2} K(\Delta_i, z_i)\, O^m_{\Delta_3} = \underbrace{B\left(\frac{\Delta_1-\Delta_2+2-\Delta_3}{2}, \frac{\Delta_2-\Delta_1+2-\Delta_3}{2}\right)}_{|z_{12}|^{\#}} O^m_{2-\Delta_3}$$

ii) $|z_{31}| \sim |z_{12}|$



"blow up"

$$\int d^2 z_3\, K(\Delta_i, z_i)\, O^m_{\Delta_3} \longrightarrow \underbrace{B\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}, \frac{\Delta_2-\Delta_1+\Delta_3}{2}\right)}_{|z_{12}|^{\#}} O^m_{\Delta_3}$$

Analytic continuation

1) Motivated by $\mathbb{R}^{3,1} \to \mathbb{R}^{2,2}$ [Atanasov, Ball, Melton, Radu..., Strominger; Crawley, Miller, Narayanan, Strominger]

& on-shell S-matrix approach

2) Singularity structure of correlation functions is drastically different! [Simmons-Duffin & Kravchuk]

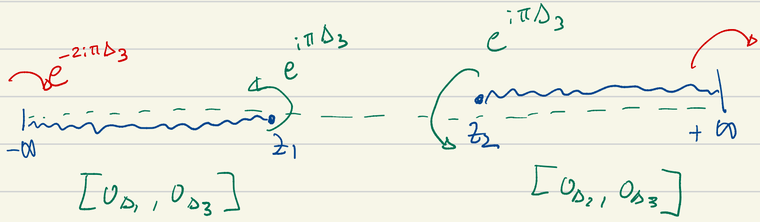
$$|z_1 - z_2|^2 \to z_{12}\bar{z}_{12} + i\epsilon \qquad \boxed{z, \bar{z} \in \mathbb{R}}$$

e.g. $$\langle O_\Delta O_\Delta O_{\Delta_3} \rangle = \frac{N}{(z_{12}\bar{z}_{12} + i\epsilon)^{\Delta - \Delta_3/2} (z_{13}\bar{z}_{13} + i\epsilon)^{\Delta_3/2} (z_{23}\bar{z}_{23} + i\epsilon)^{\Delta_3/2}}$$

$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

but does not factor into 1D + 1D, non trivial monodromies $\Rightarrow$ so does the OPE block.

z_3



$\bar{z}_1 < \bar{z}_3 < \bar{z}_2$

The monodromy @ $z=\infty$ is a consequence of conformal invariance

$$\langle \mathcal{O}_\Delta \cdots \rangle \to \frac{\#}{(\pm |z|)^\Delta}$$

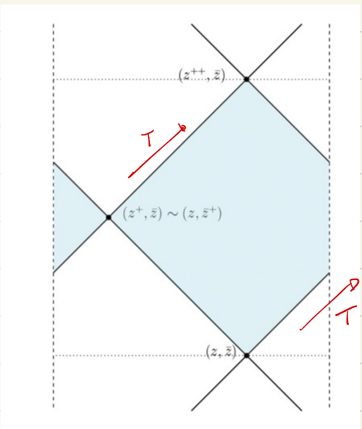
$z \to \pm\infty$ lie in different "Poincare patches"

or
$$T\, \mathcal{O}_\Delta |0\rangle = e^{i\pi\Delta}\, \mathcal{O}_\Delta |0\rangle$$

$\hookrightarrow$ translation along null geodesics (constant $\bar{z}$)

$T[z] = z^+$

Real slice: "Lorentzian cylinder"



$T^2 \uparrow \Delta$

Note that for $\Delta = k$

$$T^2\, \mathcal{O}_\Delta = e^{2i\pi\Delta}\, \mathcal{O}_\Delta = \mathcal{O}_\Delta$$

$\Rightarrow$ These primaries live in a torus!

[Atanasov, Ball, Melton, Raclariu, Strominger

but for generic Δ we still need to consider the cylinder.

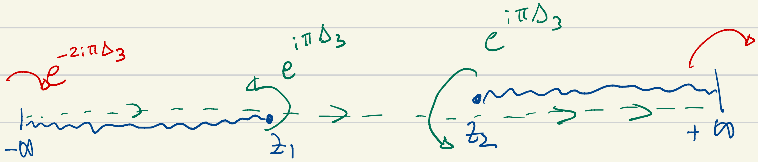
Light ray operators : For our scalar thy. they are given by

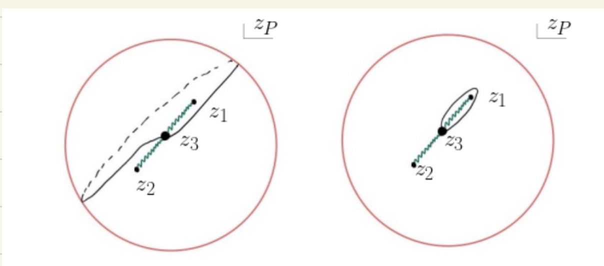
$$L[O_\Delta^{(m)}] = \int_z^{z^+} \frac{dz_p \; O_\Delta^{(m)}(z_p)}{(z_p - z)^{2-\Delta}}$$

$\hookrightarrow$ Poincare patch

$$\langle O_\Delta O_\Delta L[O_{\Delta_3}^{(m)}] \rangle = \int_z^{z^+} \frac{dz_3}{(z_3 - z)^{2-\Delta_3}} \langle O_\Delta O_\Delta O_{\Delta_3}^{(m)}(z_3) \rangle$$

The result is only non-trivial for $\bar{z}_1 < \bar{z}_3 < \bar{z}_2$





$$\langle O_\Delta O_\Delta L[O_{\Delta_3}^{m}] \rangle = \frac{2\pi i}{\Delta_3 - 1} \frac{\wedge}{\bar{z}_{23}^{\Delta_3/2} \bar{z}_{31}^{\Delta_3/2} \bar{z}_{21}^{\Delta - \Delta_3/2}}$$

[Atanasov et al]

$$\rightarrow \frac{\#}{\bar{z}_{21}^{\#} \bar{z}_{31}^{\Delta_3}} e^{i\pi\Delta_3/2} \rightarrow \text{monodromy in OPE limit}$$

Back to OPE block: Analytic continuation

Set $\int d^2 z_3 \longrightarrow \int -i\, dz_3\, d\bar{z}_3$

$$O_\Delta O_\Delta = \int_{\Delta \in 1+i\mathbb{R}} \frac{d\Delta_3}{2\pi i} \frac{C(\Delta_3)}{z_{12}\, \bar{z}_{12}^{\Delta}} \int_{\mathbb{R}} \frac{d\bar{z}_p\ \bar{z}_{21}^{1-\Delta_3/2}}{\bar{z}_{p1}^{1-\Delta_3/2}\ \bar{z}_{p2}^{1-\Delta_3/2}} \text{ ``}\times\text{'' } \int_{\mathbb{R}} \frac{dz_p\ z_{21}^{1-\Delta_3/2}}{z_{p1}^{1-\Delta_3/2}\, z_{p2}^{1-\Delta_3/2}}\ O_\Delta^m$$

The integral does not factor due to "$i\epsilon$" prescription

We handle it similarly to KLT construction

see also (Simons-Duffin, Stanford Witten)

& Tom's talk

$\boxed{\text{Output: complete factorization}}$

Extracting light-rays

$$O_\Delta O_\Delta = \int_{\Delta\in 1+i\mathbb{R}} \frac{d\Delta_3}{2\pi i}\, \frac{C(\Delta_3)}{z_{12}\,\bar z_{12}^{\,\Delta}} \int_{\mathbb{R}} \frac{d\bar z_p\, \bar z_{21}^{\,1-\Delta_3/2}}{\bar z_{p1}^{\,1-\Delta_3/2}\, \bar z_{p2}^{\,1-\Delta_3/2}} \text{ "}\times\text{" } \int_{\mathbb{R}} \frac{dz_p\, z_{21}^{1-\Delta_3/2}}{z_{p1}^{1-\Delta_3/2}\, z_{p2}^{1-\Delta_3/2}}\, O^m_{\Delta_3}$$

$$\overset{z_1\to z_2}{\supset} \int_{\Delta\in 1+i\mathbb{R}} \frac{d\Delta_3}{2\pi i}\, \frac{C(\Delta_3)}{z_{12}\,\bar z_{12}^{\,\Delta}} \int_{\mathbb{R}} \frac{d\bar z_p\, \bar z_{21}^{\,1-\Delta_3/2}}{\bar z_{p1}^{\,1-\Delta_3/2}\, \bar z_{p2}^{\,1-\Delta_3/2}} \text{ "}\times\text{" } \underbrace{\int_{\mathbb{R}} \frac{dz_p\, z_{21}^{1-\Delta_3/2}}{z_{p1}^{2-\Delta_3}}\, O^m_{\Delta_3}}_{L[O^m_{\Delta_3}]}$$

$$= \int d\Delta_3\, \frac{1-\Delta_3}{z_{21}^{\Delta+\Delta_3/2-1}}\, e^{i\pi\Delta_3/2}\, L[O^m_{\Delta_3}]$$

consistent with Atanasov et al!

Bonus Track : YM block

$$O^{+,a}_{\Delta_1}O^{+,b}_{\Delta_2} \supset -if^{abc}\frac{\Gamma(1-\Delta_1)\Gamma(1-\Delta_2)}{2\pi\Gamma(1-\Delta_1-\Delta_2)}\int \frac{d^2 z_3 O^{+,c}_{\Delta_1+\Delta_2-1}(z_3)}{z_{12}^{\Delta_1+\Delta_2} z_{32}^{1-\Delta_1} z_{31}^{1-\Delta_2} \bar{z}_{12}^{\Delta_1+\Delta_2-3}\bar{z}_{32}^{2-\Delta_1}\bar{z}_{31}^{2-\Delta_2}}\,, \tag{5.1}$$

introduced recently in [12], was indeed motivated[8] from higher-point amplitudes instead of three-point scattering as done here. It is then a pressing question to understand the relation of this block with the three-point functions $\langle O^{+,a}(\bar{z}_1)O^{+,b}(\bar{z}_2)O^{-,d}(\bar{z}_3)\rangle$, if any. A priori, this is complicated by the fact that the latter functions, as obtained in e.g. [51], are only defined in the region $\bar{z}_{23}\bar{z}_{31} \geq 0$, hence the OPE limit $\bar{z}_{12} \to 0$ is singular. Let us momentarily insist, however, on the relation (5.1) as providing an analytic continuation of the OPE in the complex $z, \bar{z}$ planes. In Lorentzian signature, following the procedure of the main text, we can easily extract the contribution from the gluon primary and its light-ray transform from (5.1). For $z_2 > z_1, \bar{z}_2 > \bar{z}_1$ we get

$$O^{+,a}_{\Delta_1}O^{+,b}_{\Delta_2} \supset \frac{f^{abc}}{\bar{z}_{21}^{\Delta_1+\Delta_2-3}z_{21}} L[O^{+,c}_{\Delta_1+\Delta_2-1}](z_2,\bar{z}_2)e^{i\pi(\Delta_1-1)} - iB(\Delta_1-1,\Delta_2-1)\frac{f^{abc}}{z_{21}}O^{+,c}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)\,, \tag{5.2}$$

We observe that the light-ray term does not involve the beta function corresponding to the OPE data of gluons (this is the OPE analog of (3.13)), precisely as occurs in the three-point function [37,42]. Indeed, conformal symmetry fixes the gluon light-ray pairing

$$\langle L[O^{+c}_{\Delta_1}](z_1,\bar{z}_1)O^{-d}_{\Delta_2}(z_2,\bar{z}_2)\rangle = C \times \delta^{cd}\delta(i(\Delta_1+\Delta_2-2))\delta(z_{12})\frac{1}{\bar{z}_{21}^{3-\Delta_1}}\,, \tag{5.3}$$

from which we obtain the colinear limit (for $\bar{z}_2 \neq \bar{z}_3$)

$$\langle O^{+,a}_{\Delta_1}O^{+,b}_{\Delta_2}O^{-,c}_{\Delta_3}\rangle \sim \delta\left(i\left(\sum_i \Delta_i - 3\right)\right)\frac{-C/z_{21}f^{abc}\delta(z_{23})}{\bar{z}_{21}^{-\Delta_3}\bar{z}_{32}^{2-\Delta_2}\bar{z}_{23}^{2-\Delta_1}} \tag{5.4}$$