Single valued Celestial Amplitudes

based on "Conformal Blocks from Celestial Amplitudes"
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Flowchart

shadow
Correlator

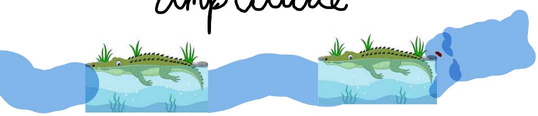
shadow
on 1

minimal
models

celestial (Mellin)
amplitude

conformal blocks
crossing symmetry
OPE bootstrap

single-valued
shadow corr.



celestial (SV)
amplitude

Coulomb gas

inverse
shadow

integral
representation

Related work on celestial conformal blocks:
Lam, Shao
Nandan, Schreiber, Volovich, Zlotnikov
Law, Zlotnikov
Atanasov, Melton, Raclariu, Strominger

Celestial Amplitude $\longrightarrow$ shadow correlator

MELLIN   shadow on 1

$$\left\langle \phi^{a_1}_{\Delta_1,-}(z_1,\bar z_1)\phi^{a_2}_{\Delta_2,-}(z_2,\bar z_2)\phi^{a_3}_{\Delta_3,+}(z_3,\bar z_3)\phi^{a_4}_{\Delta_4,+}(z_4,\bar z_4)\right\rangle = \delta\Big(\sum_{i=1}^{4}\lambda_i\Big)\prod_{i<j}(z_{ij}\bar z_{ij})^{-i\frac{\lambda_i}{2}-i\frac{\lambda_j}{2}}$$

$$\times\delta(z-\bar z)\Big(\frac{z_{12}}{z_{13}z_{24}z_{34}}\Big)\Big(\frac{\bar z_{34}^2}{\bar z_{13}\bar z_{24}\bar z_{14}\bar z_{23}}\Big)(f^{a_1a_2b}f^{a_3a_4b}-zf^{a_1a_3b}f^{a_2a_4b})\,,$$

with the cross-ratios

$$z=\frac{z_{12}z_{34}}{z_{13}z_{24}}\,,\quad \bar z=\frac{\bar z_{12}\bar z_{34}}{\bar z_{13}\bar z_{24}}\,.$$

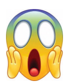
Pasterski, Shao, Strominger

$\Delta_i = 1+i\lambda_i$

$12\to 34$ planar, $z=\bar z>1$

CFT correlator defined on real axis only

distribution - valued



Shadow transform $z_1\to z_1'$ extends to $x=\frac{z_{1'2}\,z_{34}}{z_{1'3}\,z_{24}}\in\mathbb{C}$

$$\left\langle \tilde\phi^{a_1}_{\tilde\Delta_1,+}(z_{1'},\bar z_{1'})\phi^{a_2}_{\Delta_2,-}(z_2,\bar z_2)\phi^{a_3}_{\Delta_3,+}(z_3,\bar z_3)\phi^{a_4}_{\Delta_4,+}(z_4,\bar z_4)\right\rangle =$$

$$\int\frac{d^2z_1}{(z_1-z_1')^{2-i\lambda_1}(\bar z_1-\bar z_1')^{-i\lambda_1}}\left\langle \phi^{a_1}_{\Delta_1,-}(z_1,\bar z_1)\phi^{a_2}_{\Delta_2,-}(z_2,\bar z_2)\phi^{a_3}_{\Delta_3,+}(z_3,\bar z_3)\phi^{a_4}_{\Delta_4,+}(z_4,\bar z_4)\right\rangle$$

$|z|>1$ $(12\to 34)_{D=4}$

Pasterski, Crawley, Miller, Narayanan, A.S., Sharma

is it really a "good" correlator?

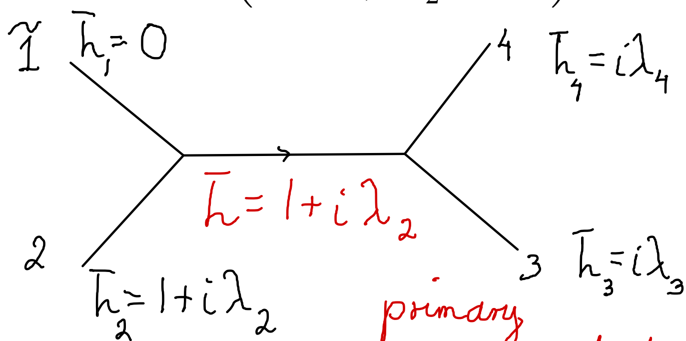
Shadow correlator $\tilde{G}(x,\bar{x})\big|_{\lambda_1=0}$

$$= f^{a_1a_2b}f^{a_3a_4b}S_1(x)\bar{I}_1(\bar{x}) + f^{a_1a_3b}f^{a_2a_4b}\tilde{S}_1(x)\bar{I}_1(\bar{x}),$$

$$\bar{I}_1(\bar{x}) = (1-\bar{x})^{-1+i\lambda_4}\bar{x}^{1+i\lambda_2} = \bar{x}^{1+i\lambda_2}\,{}_2F_1\left(\begin{matrix}2+i\lambda_2,\ 1-i\lambda_4\\ 2+i\lambda_2\end{matrix};\bar{x}\right)$$

Osborn

Conformal block



primary
+ all descendants

As $x = \frac{z_{1'2}\, z_{34}}{z_{1'3}\, z_{24}} \longrightarrow 0$

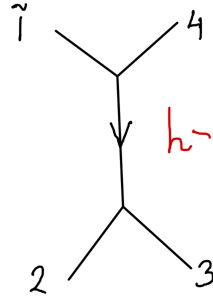
$$\tilde{G}(x,\bar{x}) \sim \frac{\bar{x}}{x^2}(x\bar{x})^{1+i\lambda_2}$$

but OPE $\phi^+_{\lambda_3}\phi^+_{\lambda_4} \sim \frac{1}{x}\phi^+_{\lambda_3+\lambda_4}$

Fan, Fotopoulos, T
Pate, Raclariu, Strominger, Yuan

"bad" OPE

Another problem:



$h - \bar{h} = J \sim i\lambda$

complex spin 

why is it bad? As $x \to 1$

$$\tilde{G}(x,\bar{x}) \sim \frac{1}{(1-x)(1-\bar{x})} (1-\bar{x})^{i\lambda_4}$$

branch point at $x = 1$

not surprising at all: hypergeometric fncts

Need a single-valued correlator SV

Hint: in minimal models with $\Phi_{(2,1)}$ — DiF Dotsenko in French

$$\tilde{G} \sim I_1 \bar{I}_1 + I_2 \bar{I}_2$$

shadow block

in our "heterotic" case

$$\tilde{G}_{SV} \sim S_1 \bar{I}_1 + S_2 \bar{I}_2$$

uniquely determined by SV requirements

$$\bar{I}_2(\bar{x}) = \bar{x}^{1-\bar{h}-\bar{h}_3-\bar{h}_4}\, {}_2F_1\left(\begin{matrix}1-\bar{h}-\bar{h}_{12},\ 1-\bar{h}+\bar{h}_{34}\\ 2-2\bar{h}\end{matrix};\bar{x}\right)\Bigg|_{\bar{h}=1+\frac{i\lambda_2}{2}} = {}_2F_1\left(\begin{matrix}1,\ i\lambda_3\\ -i\lambda_2\end{matrix};\bar{x}\right)$$

is the shadow of $\bar{I}_1$, $\bar{h} = -i\lambda_2$

Shadow Correlator $\longrightarrow$ SV shadow correlator

minimal models

$$\tilde{G}(x,\bar{x})_{SV} \begin{cases} 1.\ \text{no complex spin} \\ 2.\ \text{correct OPE} \\ 3.\ \text{crossing symmetry} \end{cases}$$

good CFT correlator

you will see details of 2 & 3 after shadow is removed

As $x \to 0$ $\quad \tilde{G}_{SV} \sim \frac{1}{x} \sim \frac{1}{z_{34}}$

$$\phi^{+}_{\lambda_3} \phi^{+}_{\lambda_4} \sim \frac{1}{z_{34}} \phi^{+}_{\lambda_3+\lambda_4}$$

with correct OPE coefficient $B(\Delta_3-1, \Delta_4-1)$

SV correlator $\longrightarrow$ integral representation

Coulomb gas: Dotsenko - Fateev (1984)

$$S_1 \, \bar{I}_1 + S_2 \, \bar{I}_2$$

$$\int_1^\infty \qquad \int_1^\infty \qquad \int_0^x \qquad \int_0^x$$

line integrals

power of complex plane

"inverse KLT"

aka "double copy" 40 years later

$$\tilde{G}(x,\bar{x})_{SV} = \frac{1}{\pi}\,(1+i\lambda_2)\;B(i\lambda_3,i\lambda_4)\frac{1}{x(1-x)}\times$$

$$\Big\{ f^{a_1a_2b}f^{a_3a_4b}\int d^2w\,w^{-1-i\lambda_4}\bar{w}^{-i\lambda_4}(w-1)^{1-i\lambda_2}(\bar{w}-1)^{-2-i\lambda_2}(w-x)^{-i\lambda_3}(\bar{w}-\bar{x})^{-i\lambda_3}$$

$$-\,x\,f^{a_1a_3b}f^{a_2a_4b}\int d^2w\,w^{-1-i\lambda_4}\bar{w}^{-i\lambda_4}(w-1)^{2-i\lambda_2}(\bar{w}-1)^{-2-i\lambda_2}(w-x)^{-1-i\lambda_3}(\bar{w}-\bar{x})^{-i\lambda_3}\Big\}.$$

Koba - Nielsen form of world-sheet integrals

$$\lambda_i \longleftrightarrow P_i \cdot P_j \quad \text{(kinematic inv.)}$$

Stieberger, T

$\rightarrow$ Connection to strings
& SV projection of Brown et al

Integral representation ⟶ SV amplitude

inverse shadow

$$\int d^2w \ldots \overset{?}{=} \int \frac{d^2 z_1}{(z_1 - z_1')^{2-i\lambda_1}(\bar{z}_1 - \bar{z}_1')^{-i\lambda_1}} \ldots$$

✓

in

out (Pate, Radariu, Strominger, Yuan)

$$\left\langle \tilde{\phi}^{a_1,-\epsilon}_{\tilde{\Delta}_1=1,+}(z_1',\bar{z}_1')\, \phi^{a_2,-\epsilon}_{\Delta_2,-}(z_2,\bar{z}_2)\, \phi^{a_3,+\epsilon}_{\Delta_3,+}(z_3,\bar{z}_3)\, \phi^{a_4,+\epsilon}_{\Delta_4,+}(z_4,\bar{z}_4)\right\rangle \leftarrow \text{SV shadow}$$

$$= \int \frac{d^2 z_1}{z_{11'}^2} \left\langle \phi^{a_1,-\epsilon}_{\Delta_1=1,-}(z_1,\bar{z}_1)\, \phi^{a_2,-\epsilon}_{\Delta_2,-}(z_2,\bar{z}_2)\, \phi^{a_3,+\epsilon}_{\Delta_3,+}(z_3,\bar{z}_3)\, \phi^{a_4,+\epsilon}_{\Delta_4,+}(z_4,\bar{z}_4)\right\rangle_{SV}$$

with the celestial single-valued amplitude given by

$$\left\langle \phi^{a_1,-\epsilon}_{\Delta_1=1,-}(z_1,\bar{z}_1)\, \phi^{a_2,-\epsilon}_{\Delta_2,-}(z_2,\bar{z}_2)\, \phi^{a_3,+\epsilon}_{\Delta_3,+}(z_3,\bar{z}_3)\, \phi^{a_4,+\epsilon}_{\Delta_4,+}(z_4,\bar{z}_4)\right\rangle_{SV}$$

$$\mathcal{A}^{SV}_{12\to 34} = -\frac{1}{\pi}(1+i\lambda_2)B(i\lambda_3,i\lambda_4)\, z_{12}^{2-i\lambda_2} z_{13}^{-i\lambda_3} z_{14}^{-i\lambda_4} \bar{z}_{12}^{-2-i\lambda_2} \bar{z}_{13}^{-i\lambda_3} \bar{z}_{14}^{-i\lambda_4}$$
$$\times \left\{ f^{a_1a_2b} f^{a_3a_4b} \frac{1}{z_{12}z_{23}z_{34}z_{41}} + f^{a_1a_3b} f^{a_2a_4b} \frac{1}{z_{13}z_{32}z_{24}z_{41}} \right\}.$$

PT denominators dressed with conformal factors

normalization factor fixed by SV

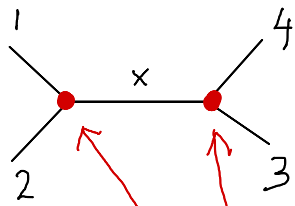
Let's check OPE, crossing symmetry & bootstrap....

Single-valued amplitude $\mathcal{A}_{sv}$

1. leading OPE
2. crossing/bootstrap
3. beyond leading OPE
4. conformal blocks

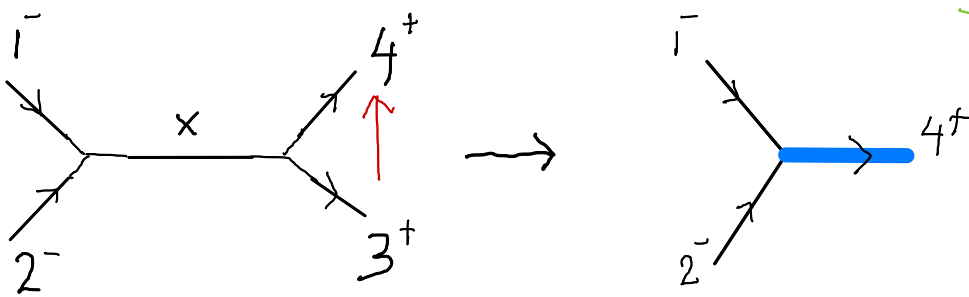
1. Leading OPE

overall factor of $\frac{1}{z_{34}}$:



$$(1+i\lambda_2)\, B(i\lambda_3, i\lambda_3)$$

$$= B(\Delta_2+1, \underbrace{2-\Delta_2-\Delta_3-\Delta_4}_{-1})\, B(\Delta_3-1, \Delta_4-1)\, \frac{f_{a_1 a_2 x}\, f_{a_3 a_4 x}}{\Gamma(\underbrace{2-\Delta_2-\Delta_3-\Delta_4}_{-1})}$$

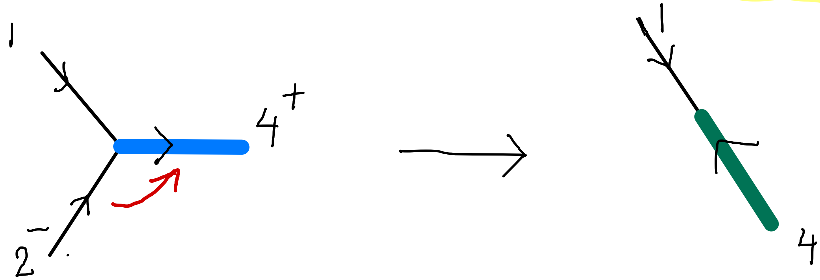


$$\phi^{a_3,+\epsilon}_{\Delta_3,+}(z_3,\bar z_3)\phi^{a_4,+\epsilon}_{\Delta_4,+}(z_4,\bar z_4) \sim \frac{-if^{a_3a_4x}}{z_{34}} B(\Delta_3-1,\Delta_4-1)\phi^{x,+\epsilon}_{\Delta_3+\Delta_4-1,+}(z_4,\bar z_4)\,.$$

Fan, Fotopoulos, T

Pate, Raclariu, Strominger, Yuan

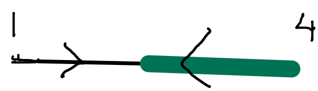
$$B(\Delta_2+1,\, 2-\Delta_2-\Delta_3-\Delta_4)\; B(\Delta_3-1,\, \Delta_4-1)\; \frac{f_{a_1a_2x}\, f_{a_3a_4x}}{\Gamma(2-\Delta_2-\Delta_3-\Delta_4)}$$



Pate, Raclariu, Strominger, Yuan

$$\begin{aligned}
&\phi^{a_2,-\epsilon}_{\Delta_2,-}(z_2,\bar z_2)\phi^{x,+\epsilon}_{\Delta_3+\Delta_4-1,+}(z_4,\bar z_4)\\
&\quad\sim \frac{if^{a_2xy}}{z_{24}}\Big[B(\Delta_2+1,2-\Delta_2-\Delta_3-\Delta_4)\phi^{y,+\epsilon}_{\Delta_2+\Delta_3+\Delta_4-2,-}(z_4,\bar z_4)\\
&\qquad - B(\Delta_3+\Delta_4-2,2-\Delta_2-\Delta_3-\Delta_4)\phi^{y,-\epsilon}_{\Delta_2+\Delta_3+\Delta_4-2,-}(z_4,\bar z_4)\Big]\\
&\quad + \frac{if^{a_2xy}}{\bar z_{24}}\Big[B(\Delta_2-1,2-\Delta_2-\Delta_3-\Delta_4)\phi^{y,+\epsilon}_{\Delta_2+\Delta_3+\Delta_4-2,+}(z_4,\bar z_4)\\
&\qquad - B(\Delta_3+\Delta_4,2-\Delta_2-\Delta_3-\Delta_4)\phi^{y,-\epsilon}_{\Delta_2+\Delta_3+\Delta_4-2,+}(z_4,\bar z_4)\Big].
\end{aligned}$$

Finally,

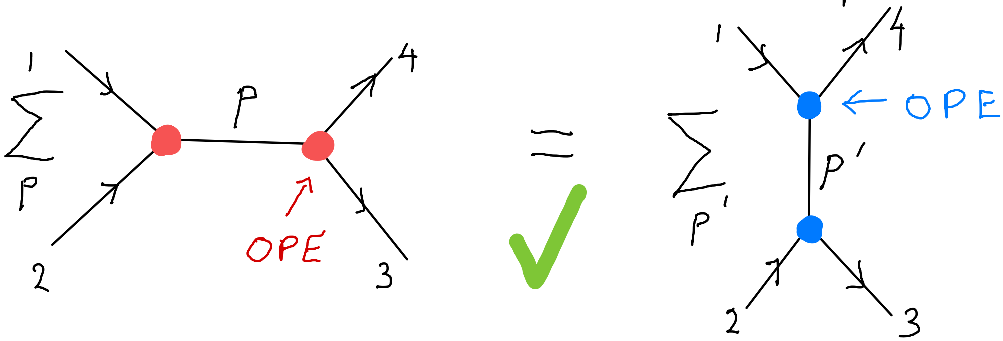


$$\left\langle \phi^{a_1,-\epsilon}_{\Delta_1,-}(z_1,\bar z_1)\phi^{y,+\epsilon}_{2-\Delta_1,-}(z_4,\bar z_4)\right\rangle = \frac{\delta^{a_1y}}{\pi\Gamma(\Delta_1-2)\,\bar z_{14}^2} \qquad (\Delta_1=1).$$

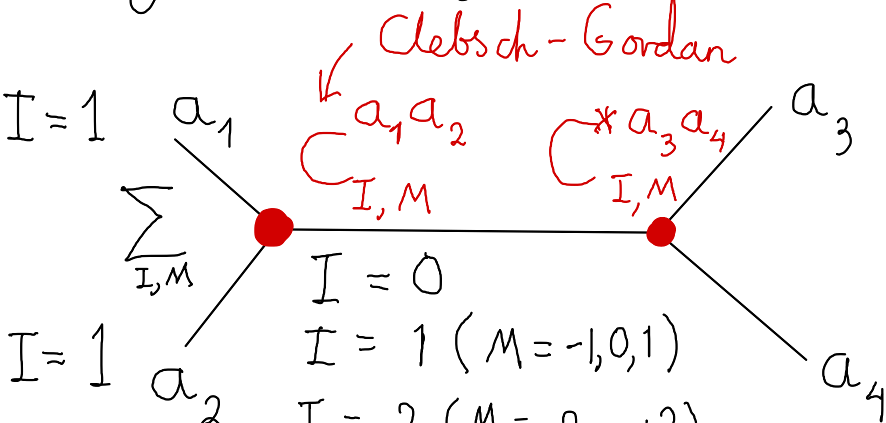
good factorization ✓

$$\Delta_1-2 = 2-\Delta_2-\Delta_3-\Delta_4$$

2. Crossing symmetry & bootstrap



3. Beyond leading OPEs [SU(2)]



In general, all reps in $\mathrm{Adj} \otimes \mathrm{Adj}$

4. Conformal blocks $\Delta = m + i\lambda$

always integer spin $\qquad m \geq 1$

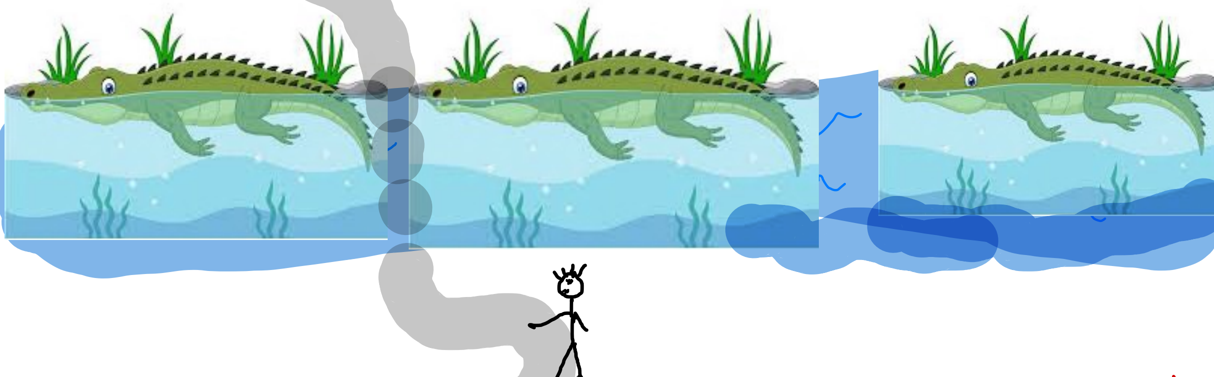
Summary

$$\mathcal{A}_{SV}(z,\bar{z}) = \frac{(1+i\lambda_2)B(i\lambda_3,i\lambda_4)}{\pi z(1-z)}\left(f^{a_1a_2b}f^{a_3a_4b} - z f^{a_1a_3b}f^{a_2a_4b}\right).$$

single-valued ✓
all known OPEs ✓
crossing symetry & bootstrap ✓

related to MHV, Nair's superamplitude

→ Witten's twistor strings?



$\mathcal{A}_{MELLIN}(z,\bar{z})$???

THANK YOU!