Geodesics

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- Straightest curves – longest proper times (locally)

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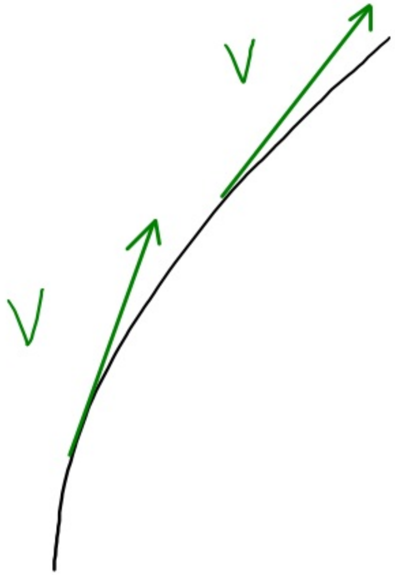
- The paths of the free ... (to fall)
- Straightest curves – longest proper times (locally)
- Curvature makes parallel geodesics deviate

– relative accelerations, "gravity"

* a curve is a geodesic if it parallel transports its tangent vector:

$$D_V V^\mu = 0 \Leftrightarrow V^\nu \nabla_\nu V^\mu = 0$$



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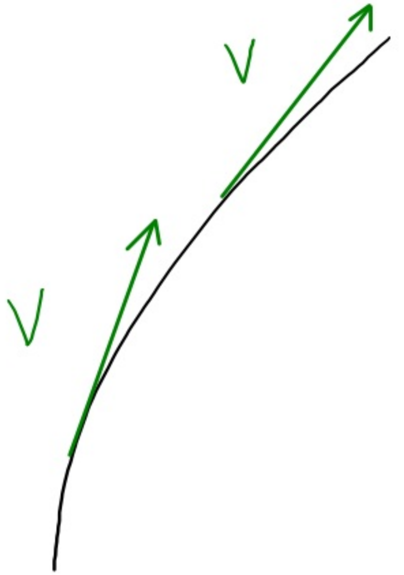
$$D_V V^\mu = 0 \Leftrightarrow V^\nu \nabla_\nu V^\mu = 0 \quad (1)$$



- a weaker condition is $D_V V^\mu = f V^\mu$, but with a reparametrization of the curve it can be recast to (1)

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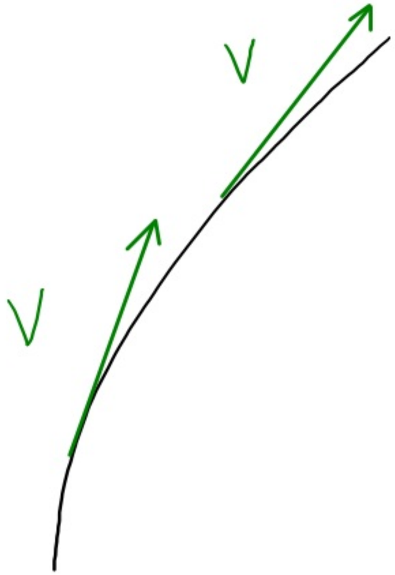

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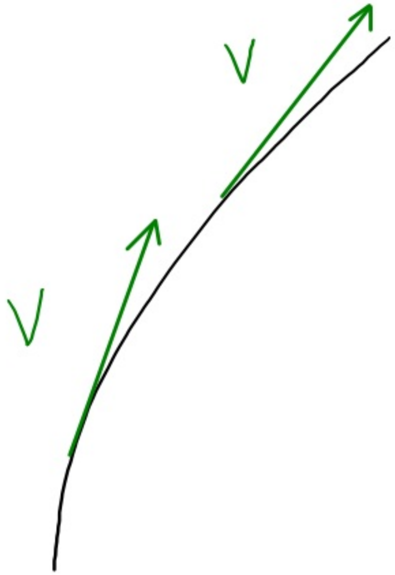

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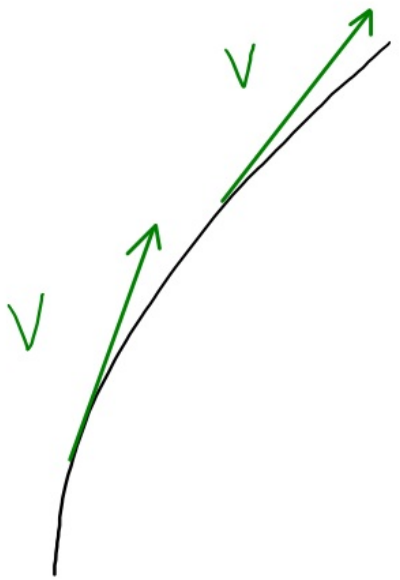
  $\tau' = \alpha \tau + \beta$, $\alpha, \beta \in \mathbb{R}$ is also an affine parameter

* a curve is a geodesic if it parallel transports its tangent vector

$$D_{\vec{V}} V^\mu = 0 \Leftrightarrow V^\nu \nabla_\nu V^\mu = 0$$

$$\Rightarrow V^\nu \partial_\nu V^\mu + \Gamma^\mu{}_{\nu\rho} V^\nu V^\rho = 0$$



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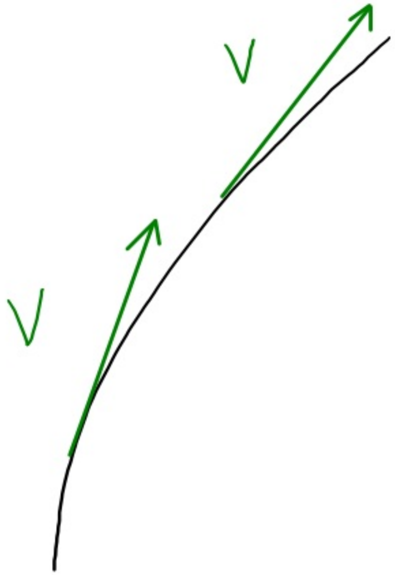

$$D_V V^\mu = 0 \iff V^\nu \nabla_\nu V^\mu = 0$$

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If $\{x^\mu\}$ are coordinates, $V^\mu = \dfrac{dx^\mu}{d\tau}$, and

$$\frac{dx^\nu}{d\tau} \frac{\partial}{\partial x^\nu}\left(\frac{dx^\mu}{d\tau}\right) + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

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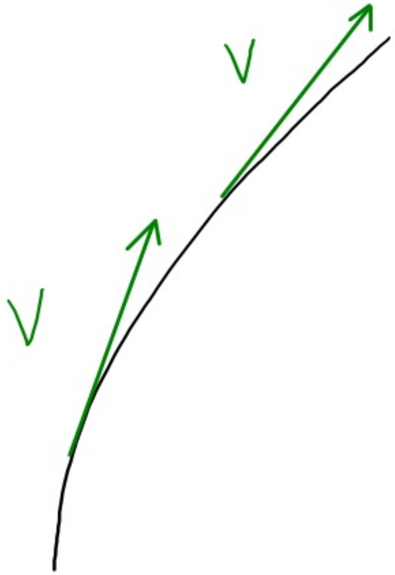

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$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

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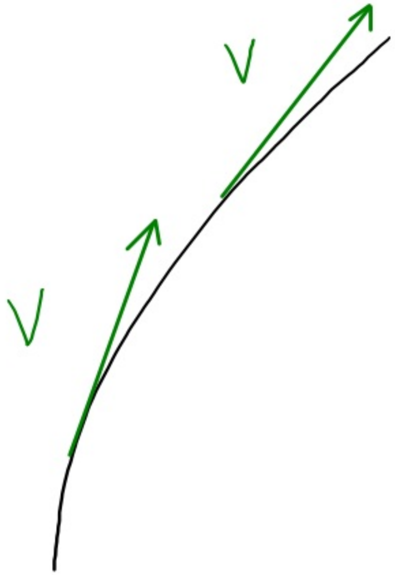

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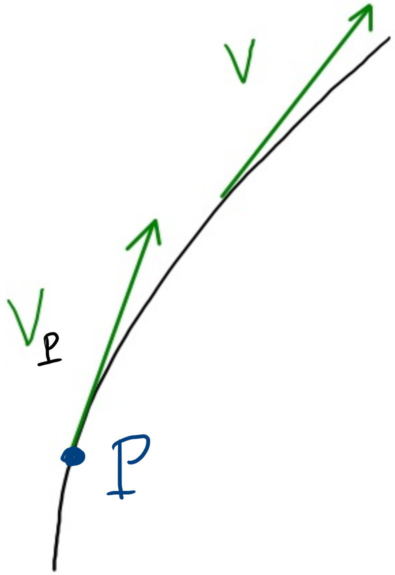
$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0 \Rightarrow \left(x^\mu(0), \frac{dx^\mu}{d\tau}(0)\right) \rightarrow \text{unique solution}$$

$\Rightarrow$ There is a unique geodesic through P with tangent vector V_P^μ



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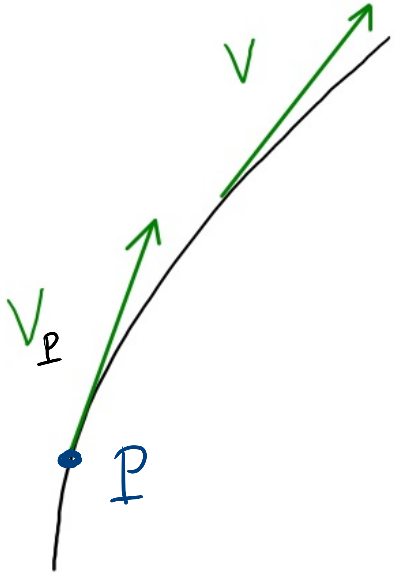
⇒ There is a unique geodesic through P with tangent vector V^μ_P



⇒ Geodesics depend only on symmetric part of $\Gamma^\mu{}_{\nu\rho}$

Adding $\Gamma^\mu{}_{[\nu\rho]} \neq 0$ would not affect geodesics (torsion)

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho}\overbrace{\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}}^{\text{symmetric term under } \nu \leftrightarrow} = 0 \Rightarrow \left(x^\mu(0), \frac{dx^\mu}{d\tau}(0)\right) \to \text{unique solution}$$

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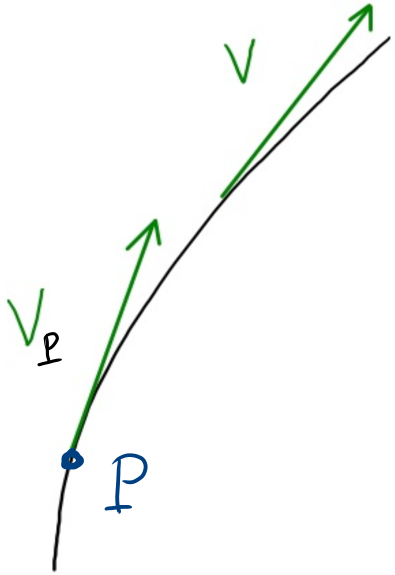

$\Rightarrow$ Geodesics depend only on symmetric part of $\Gamma^\mu{}_{\nu\rho}$

$\Rightarrow$ $\Gamma^\mu{}_{\nu\rho} = 0$ everywhere $\rightsquigarrow$ straight lines

$\hookrightarrow$ Flat spacetime has straight lines as geodesics!

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \Rightarrow \left(x^\mu(0), \frac{dx^\mu}{d\tau}(0)\right) \rightarrow \text{unique solution}$$

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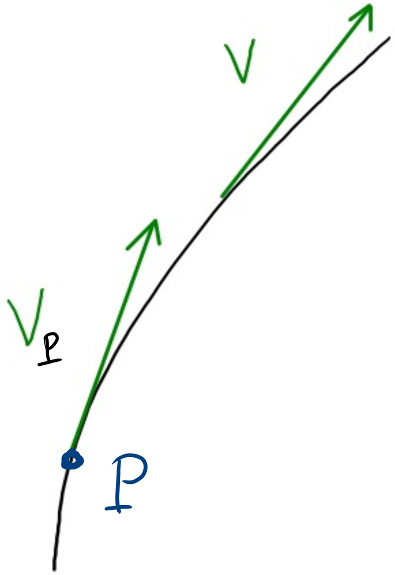

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$\Rightarrow$ $\Gamma^\mu{}_{\nu\rho} = 0$ everywhere $\rightsquigarrow$ straight lines

↳ we can set $\Gamma^\mu{}_{\nu\rho} = 0$ at a point
"almost" straight lines!

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \Rightarrow \left(x^\mu(0), \frac{dx^\mu(0)}{d\tau}\right) \to \text{unique solution}$$

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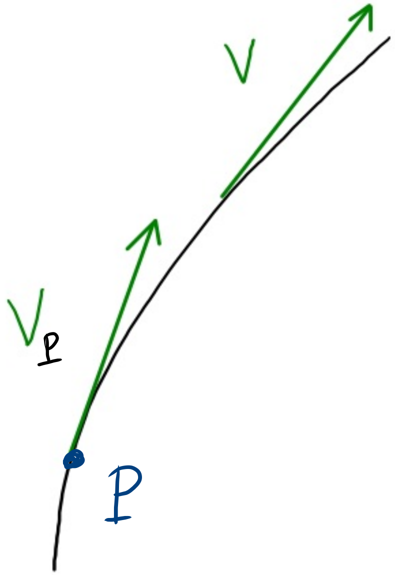

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- Free particles in GR move along geodesics

  $\hookrightarrow$ Equivalence principle: Free particles appear to move on straight lines in inertial frames

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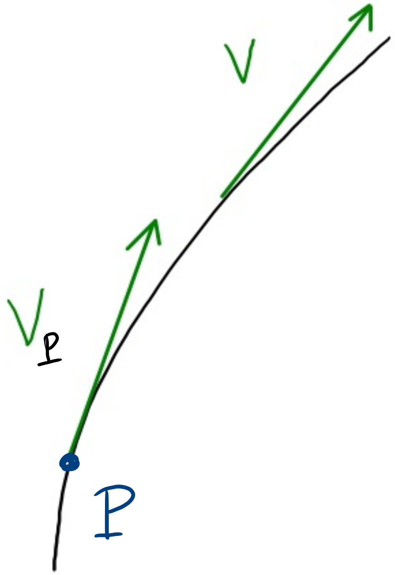

- Free particles in GR move along geodesics
- Character of geodesics (timelike/null/spacelike) does not change

Inner product $g_{\mu\nu} V^\mu V^\nu$ is constant: V^μ parallel transported

- Free massive particles: $U^\mu = \frac{dx^\mu}{d\tau}$ 4-velocity

$p^\mu = m U^\mu$

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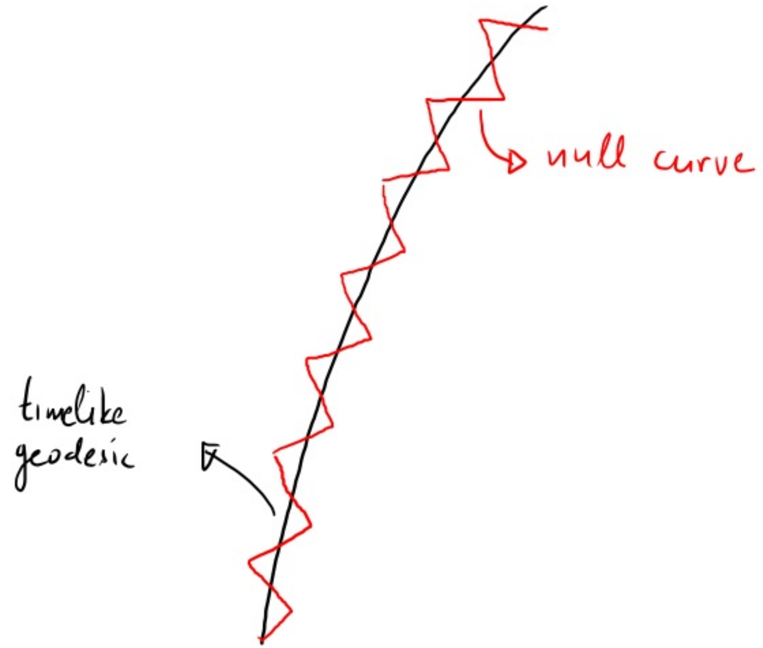
- Free massless particles:

We <u>choose</u> affine parameter λ, s.t. $p^\mu = \dfrac{dx^\mu}{d\lambda}$

$$p^\nu \nabla_\nu p^\mu = 0$$

- Timelike geodesics are local maxima of proper time

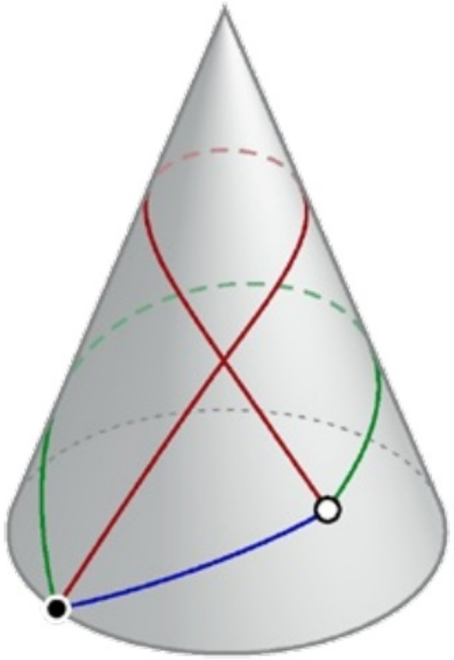
(can't be minima, always arbitrarily close to a null curve – zero length –)



- Timelike geodesics are local maxima of proper time

- Global topology may allow connecting two points with more than one geodesic

  – of different length –



http://www.rdrop.com/~half/Creations/Puzzles/cone.geodesics/index.html

- Timelike geodesics are local maxima of proper time

hence the "local"

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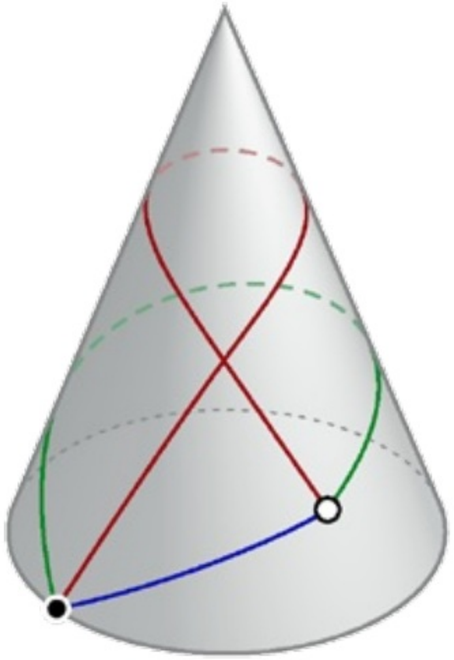

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Extremization of length/proper time → geodesics

Consider a timelike curve, $ds^2 = -d\tau^2 < 0$

$$\tau = \int d\tau \left\{ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{1/2}$$

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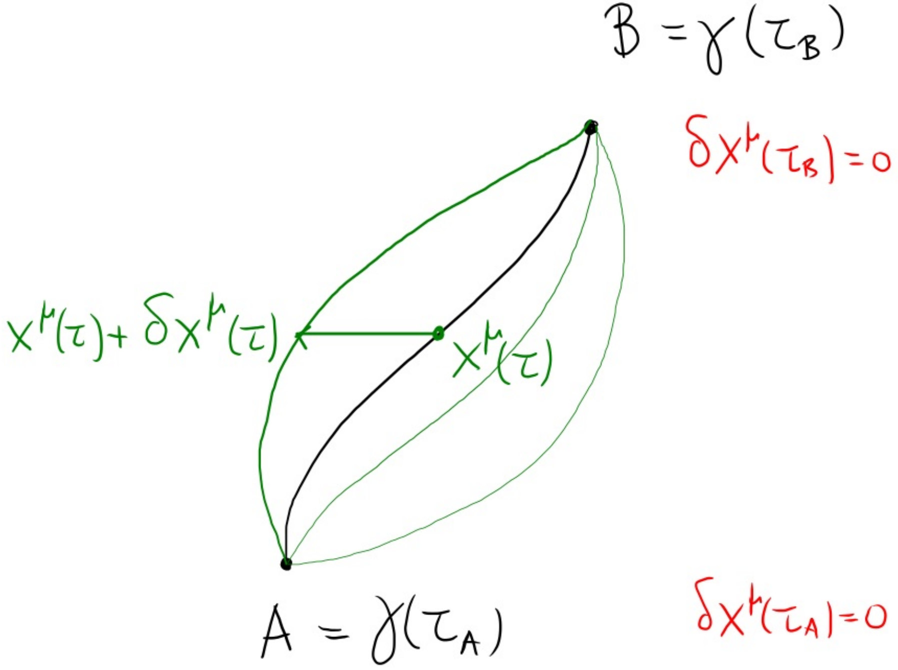

Vary $\gamma(\tau)$ s.t. $\gamma'(\tau)$ is set of points with $x^\mu(\tau) + \delta x^\mu(\tau)$. Then

$$\delta\tau = \int d\tau\, \delta \left\{ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{1/2}$$

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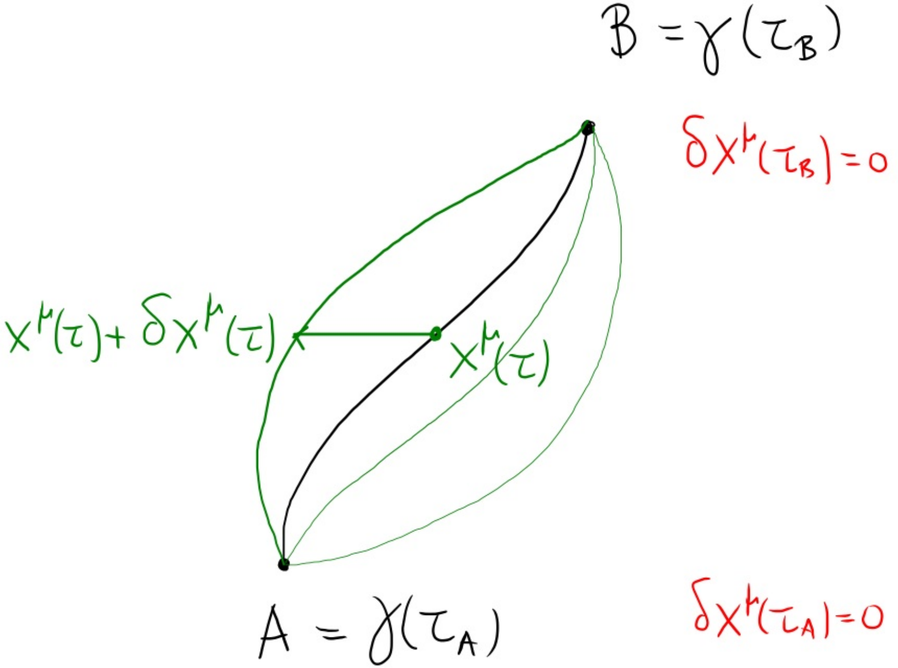

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$$\delta\tau = \int d\tau\, \delta \left\{ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{1/2}$$

$$= -\frac{1}{2} \int d\tau \left\{ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{-1/2} \delta \left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}$$

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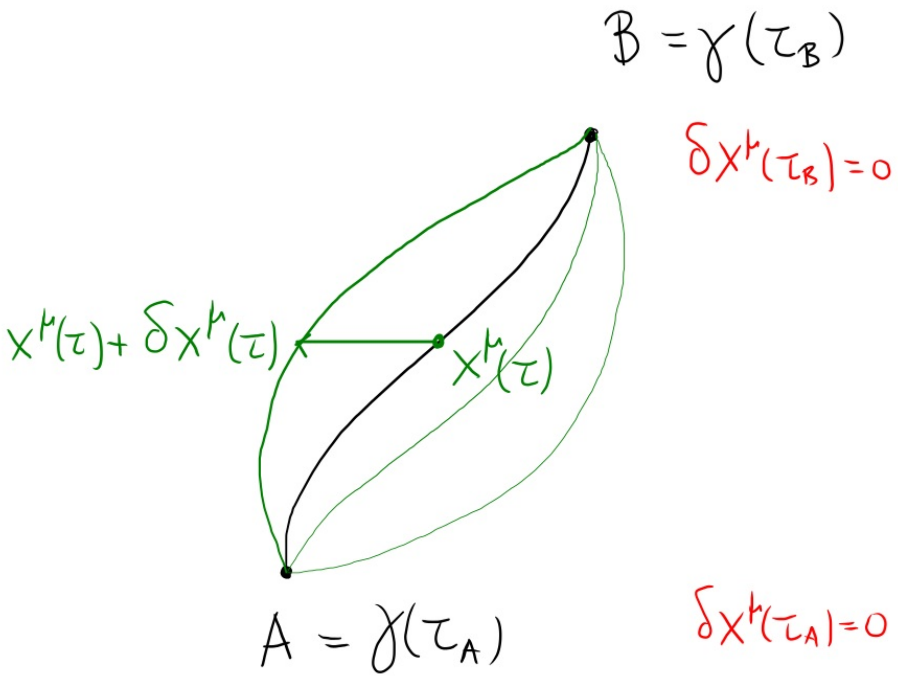

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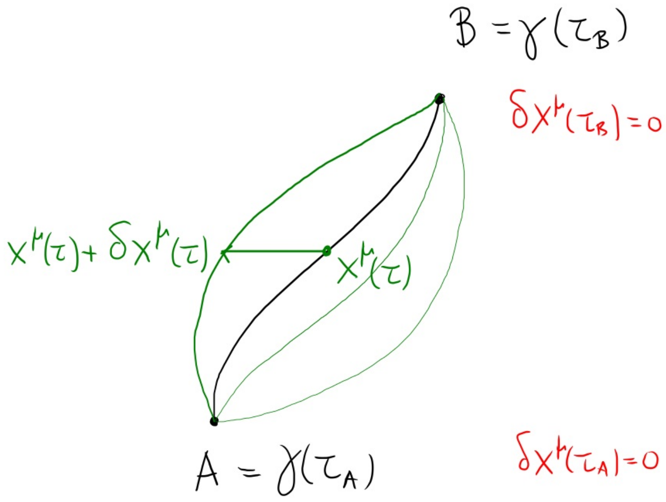
$$= -\frac{1}{2} \int d\tau \, \delta \left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}, \quad \text{since } g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = g_{\mu\nu} U^\mu U^\nu = -1$$

Extremization of length/proper time → geodesics

$$x^\mu \to x^\mu + \delta x^\mu$$



$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\lambda g_{\mu\nu}\, \delta x^\lambda + \dots$$ (Taylor series expansion)

$$\delta\tau = -\frac{1}{2}\int d\tau\, \delta\left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \right\},$$ since $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\mu\nu} U^\mu U^\nu = -1$

Extremization of length/proper time → geodesics

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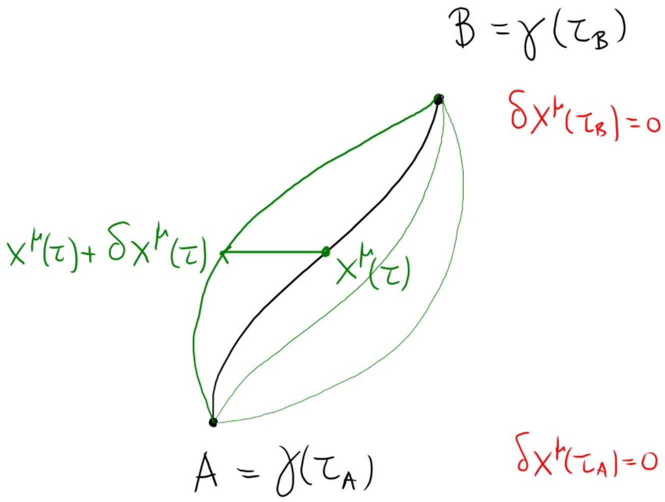

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$$\int d\tau \, \delta\left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\right) =$$

$$= \int d\tau \left\{ \delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \, \delta\left(\frac{dx^\mu}{d\tau}\right) \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau} \, \delta\left(\frac{dx^\nu}{d\tau}\right) \right\}$$

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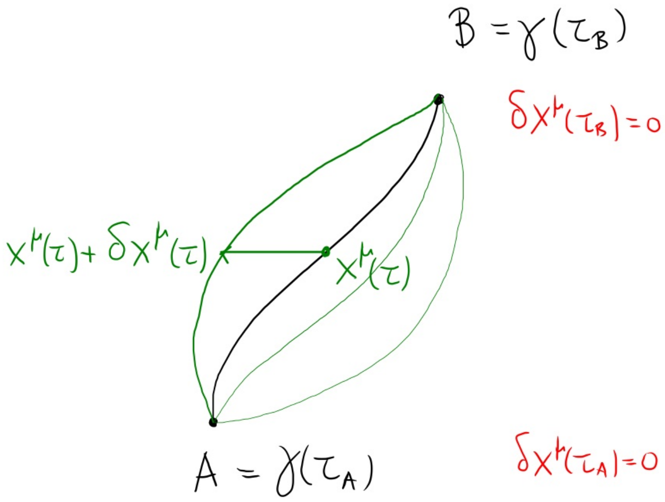

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- symmetric under $\mu \leftrightarrow \nu$
- $\delta\left(\frac{dx^\mu}{d\tau}\right) = \frac{d}{d\tau}\left(\delta x^\mu\right)$

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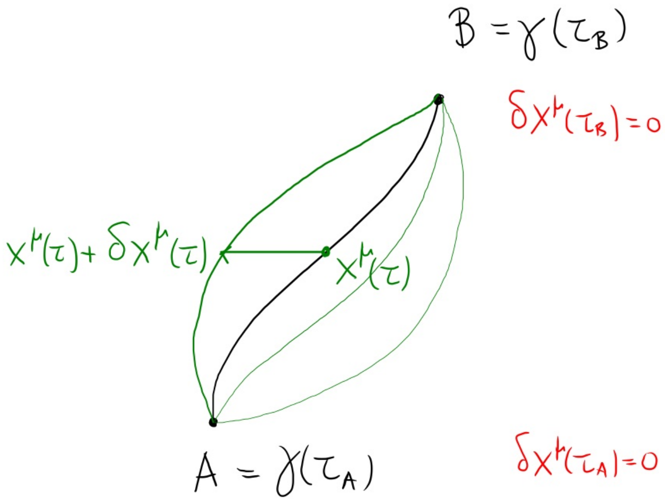

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$$= \int d\tau \left\{ \partial_\lambda g_{\mu\nu}\, \delta x^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + 2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}(\delta x^\nu) \right\}$$

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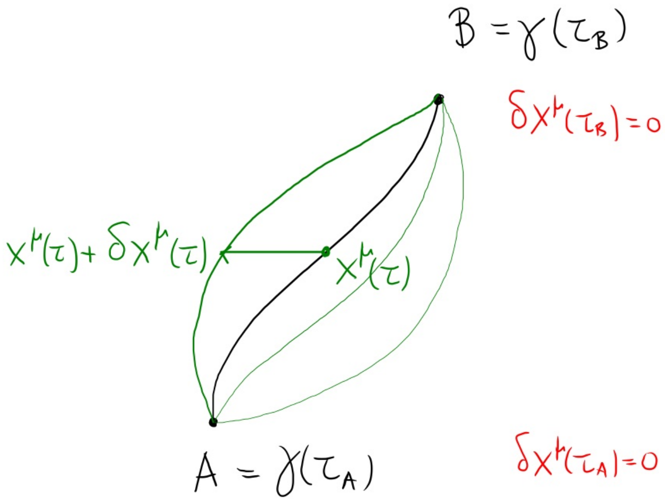

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But $2g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}\delta x^\nu = \frac{d}{d\tau}\left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau}\, \delta x^\nu\right] - \frac{d}{d\tau}\left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau}\right] \delta x^\nu$

Extremization of length/proper time → geodesics

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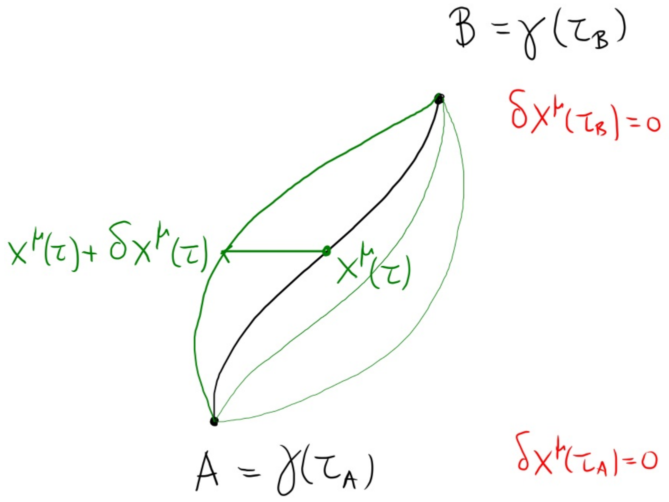

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$$\int_{\tau_A}^{\tau_B} d\tau\, 2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau} \delta x^\nu = \left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu \right]\Bigg|_{\tau_A}^{\tau_B} - \int_{\tau_A}^{\tau_B} \frac{d}{d\tau}\left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \right] \delta x^\nu\, d\tau$$

Extremization of length/proper time $\rightarrow$ geodesics

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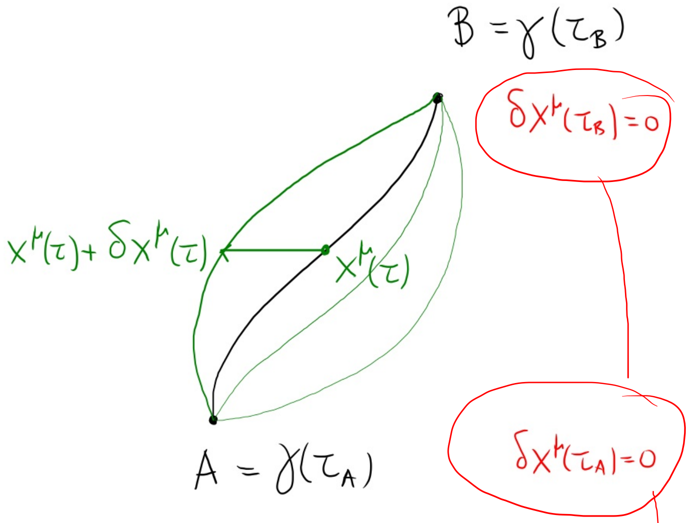

$$\int d\tau\, \delta\left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\right) =$$

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$$= \int d\tau \left\{ \partial_\lambda g_{\mu\nu}\, \delta x^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + 2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}(\delta x^\nu) \right\}$$

$$\int_{\tau_A}^{\tau_B} d\tau\, 2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau} \delta x^\nu = \left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \cancel{\delta x^\nu}\right]\Bigg|_{\tau_A}^{\tau_B} - \int_{\tau_A}^{\tau_B} \frac{d}{d\tau}\left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau}\right] \delta x^\nu\, d\tau$$

Extremization of length/proper time → geodesics

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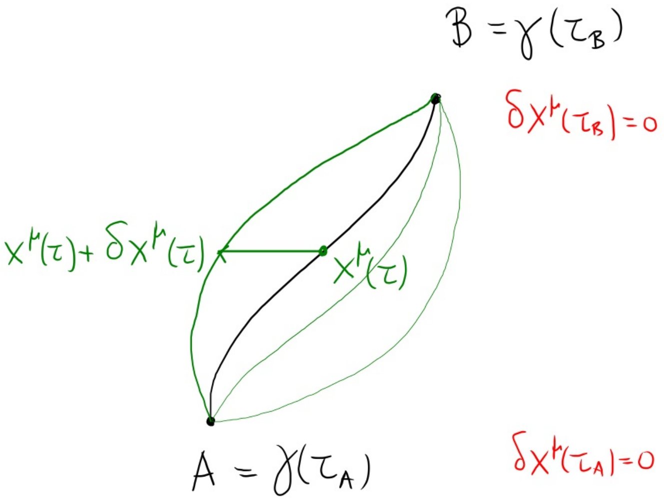

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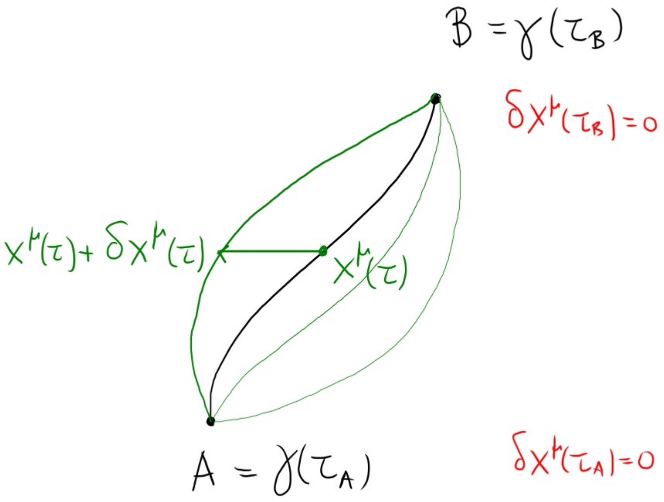
Extremization of length/proper time → geodesics



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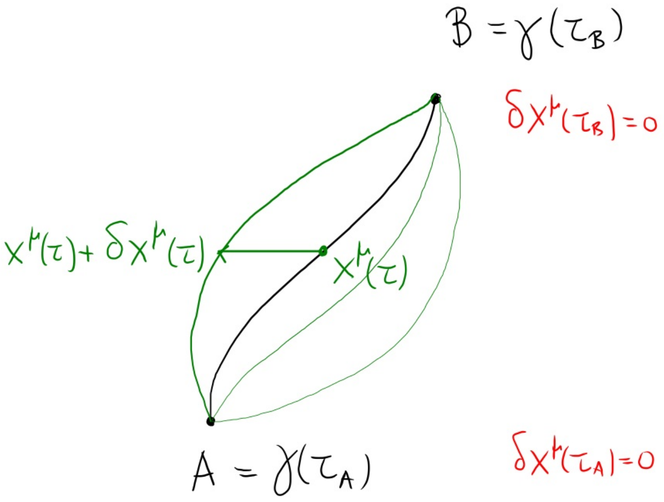
$\lambda \leftrightarrow \nu$

$$= \int d\tau \left\{ \partial_\lambda g_{\mu\nu}\, \delta x^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \frac{d}{d\tau}\left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau}\right] \delta x^\nu \right\}$$

Extremization of length/proper time → geodesics



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Extremization of length/proper time → geodesics

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Extremization of length/proper time → geodesics

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Extremization $\delta\tau = 0$ for any δx^μ implies the integrant is zero:

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Extremization of length/proper time → geodesics

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$$g^{\sigma\mu} g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} g^{\sigma\mu} \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda} \right) \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

Extremization of length/proper time → geodesics

$$\frac{d^2 x^\sigma}{d\tau^2} + \Gamma^\sigma{}_{\mu\lambda} \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

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Example: Flat space cosmology

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Example: Flat space cosmology

$$= \frac{1}{2}\int d\tau \left[+2\frac{d^2t}{d\tau^2}\,\delta t + 2a\dot{a}\,\delta t\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right]$$

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$$\Rightarrow \quad \begin{aligned} &\Gamma^0{}_{00} = \Gamma^0{}_{0i} = \Gamma^0{}_{i0} = 0 \\ &\Gamma^0{}_{ij} = a\dot{a}\,\delta_{ij} \end{aligned}$$

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$$I = \frac{1}{2}\int d\tau \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right] \qquad x^i \to x^i + \delta x^i$$

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$i \leftrightarrow j$ symmetric $i \leftrightarrow j$

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$$\delta x^j(\tau_A) = \delta x^j(\tau_B) = 0$$

$$= \int d\tau \; a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{d}{d\tau}\,\delta x^j$$

$$a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\delta x^j \Big|_{\tau_A}^{\tau_B}$$

$$//$$

$$= -\int d\tau \,\frac{d}{d\tau}\left[a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\right]\delta x^j + \int d\tau\,\frac{d}{d\tau}\left[a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\delta x^j\right]$$

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$$= -\int d\tau \frac{d}{d\tau}\left[a^2(t)\,\delta_{ij}\frac{dx^i}{d\tau}\right]\delta x^j$$

$$= -\int d\tau \left[2a\frac{da}{d\tau}\frac{dx^i}{d\tau} + a^2\frac{d^2x^i}{d\tau^2} \right]\delta_{ij}\,\delta x^j$$

Example: Flat space cosmology

But $\frac{da}{d\tau} = \frac{da}{dt}\frac{dt}{d\tau} = \dot{a}\frac{dt}{d\tau}$, so $\delta I = 0 \ \forall \delta x^j$ implies

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Example: Flat space cosmology

$$\Rightarrow \quad \Gamma^i{}_{00} = \Gamma^i{}_{jk} = 0$$

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Null geodesics: consider photons moving on +X axis

$$x^\mu(\lambda) = (\, t(\lambda), x(\lambda), 0, 0)$$

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solution: $\dfrac{dt}{d\lambda} = \dfrac{\omega_0}{a(t)}$

• consider comoving observers: $U^\mu = (1,0,0,0)$

$$E_{photon} = -p_\mu U^\mu = -g_{00}\, p^0\, U^0 = -g_{00}\frac{dx^0}{d\lambda}\cdot 1 = +\frac{\omega_0}{a}$$

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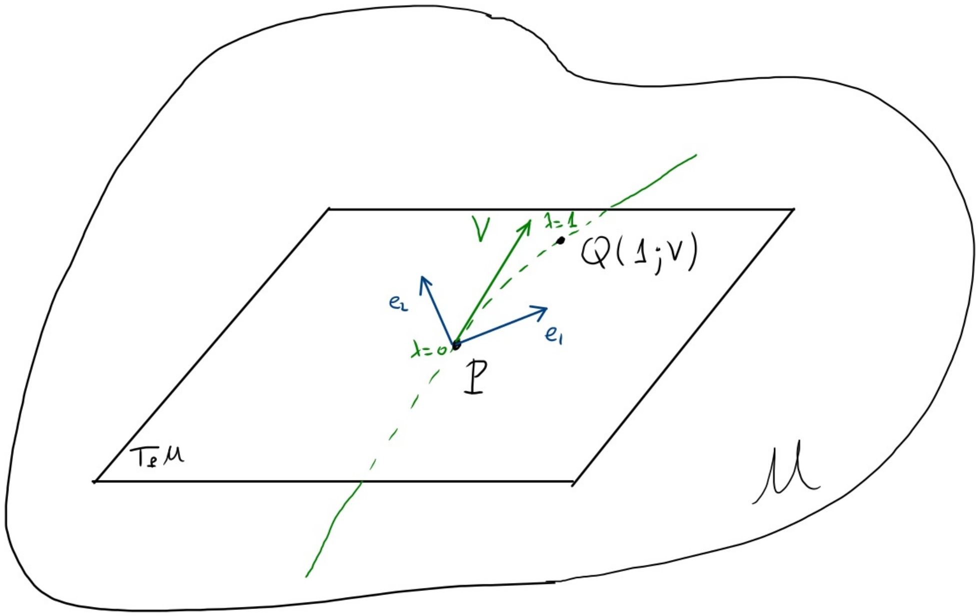
$$\frac{\omega_2}{\omega_1} = \frac{\omega_0/a_2}{\omega_0/a_1} = \frac{a_1}{a_2}$$ cosmological redshift

Note:
- it is the energy of the photon that is reduced
- photon is not "stretched"
- the relation of E and ω is quantum mechanical

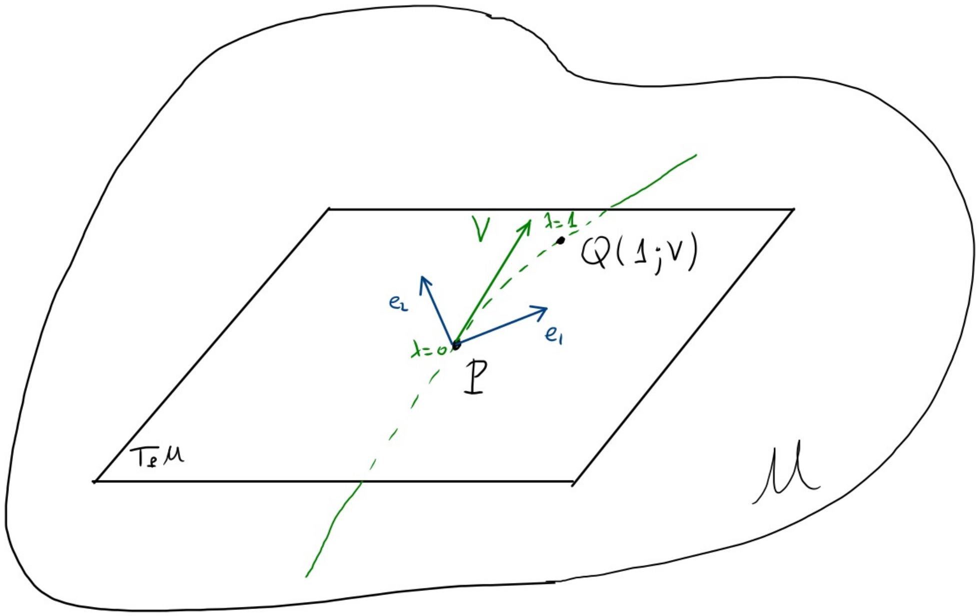
Riemann Normal Coordinates

Construct local inertial frame using geodesics:

- Pick an event P and consider all geodesics through P



Riemann Normal Coordinates



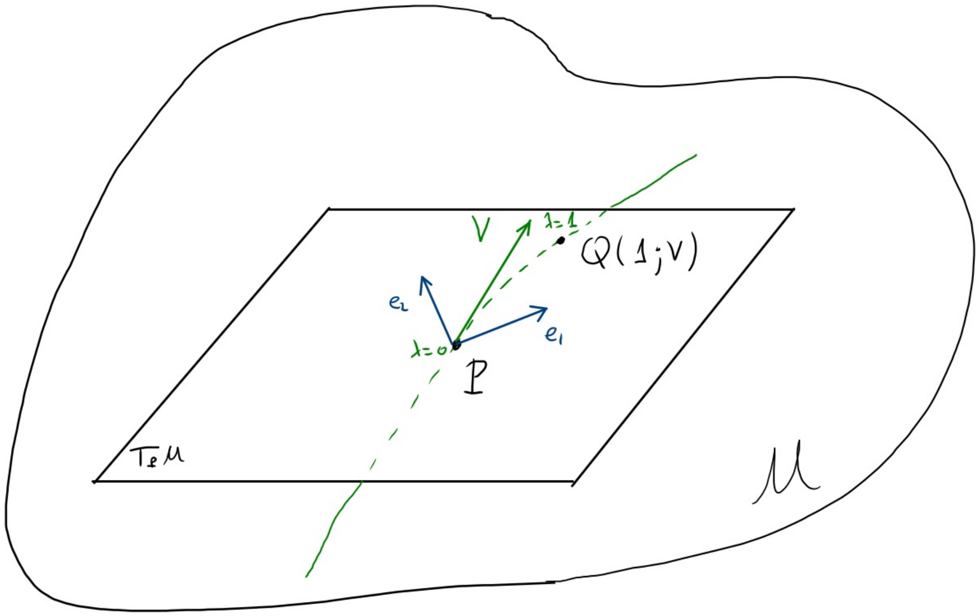
Construct local inertial frame using geodesics:

- Pick an event P and consider all geodesics through P
- Each vector V at P determines a geodesic w/affine parameter λ

Consider the point $Q = Q(\lambda; V)$

where on geodesic — which geodesic

Riemann Normal Coordinates



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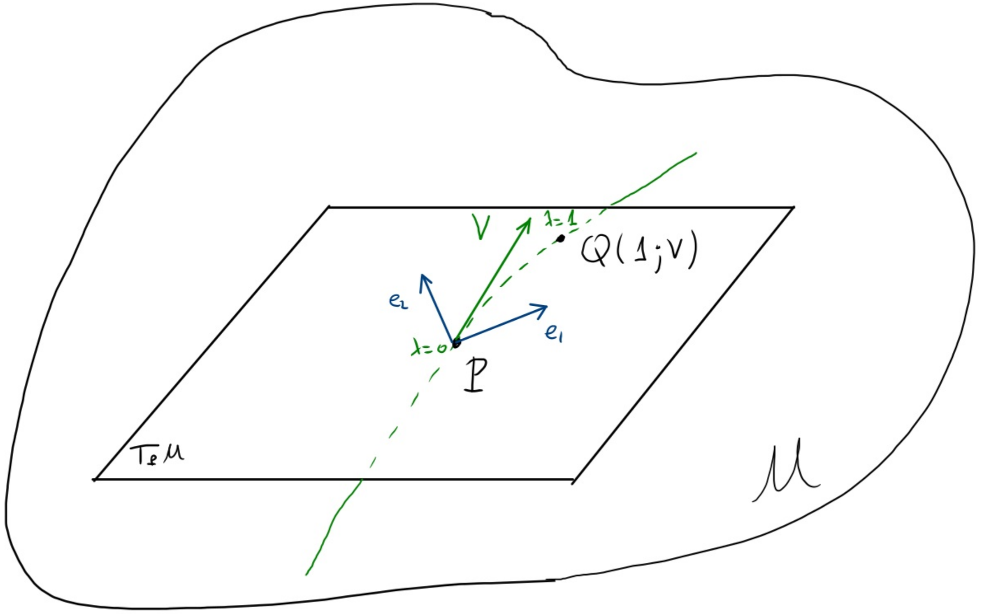
Riemann Normal Coordinates



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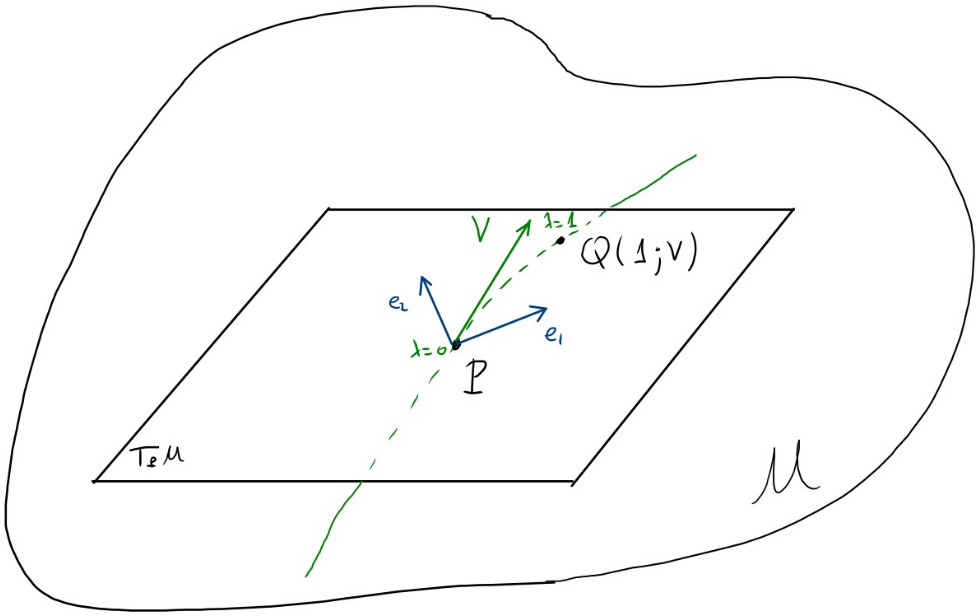
$\Rightarrow$ Choose orthonormal basis $\{e_\mu\}$ at $T_P\mathcal{M}$

If $V = x^\mu e_\mu$, define coordinates of $Q(1;V)$ to be $\{x^\mu\}$



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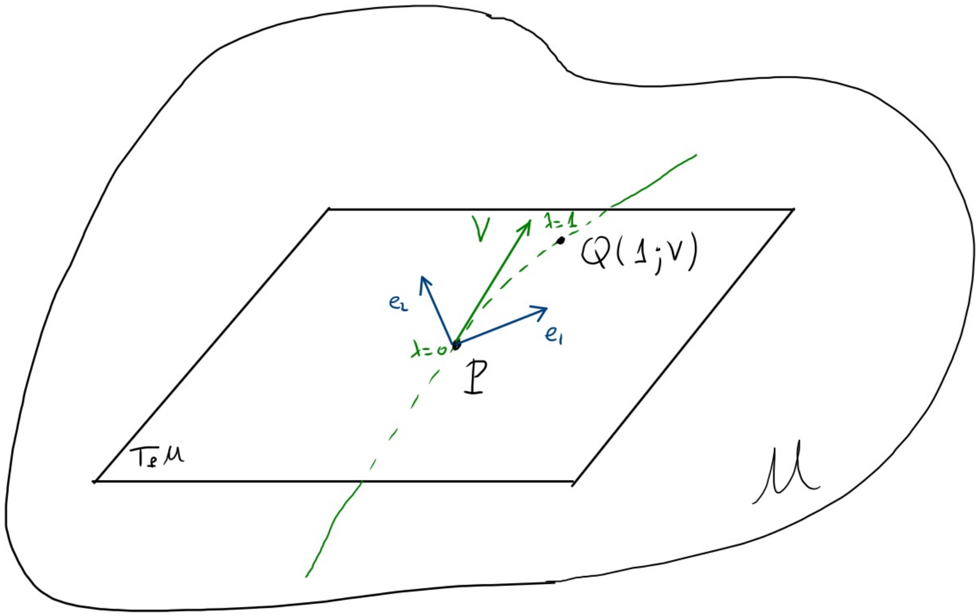

Then:

(a) $e_\mu = \partial_\mu \big|_P$

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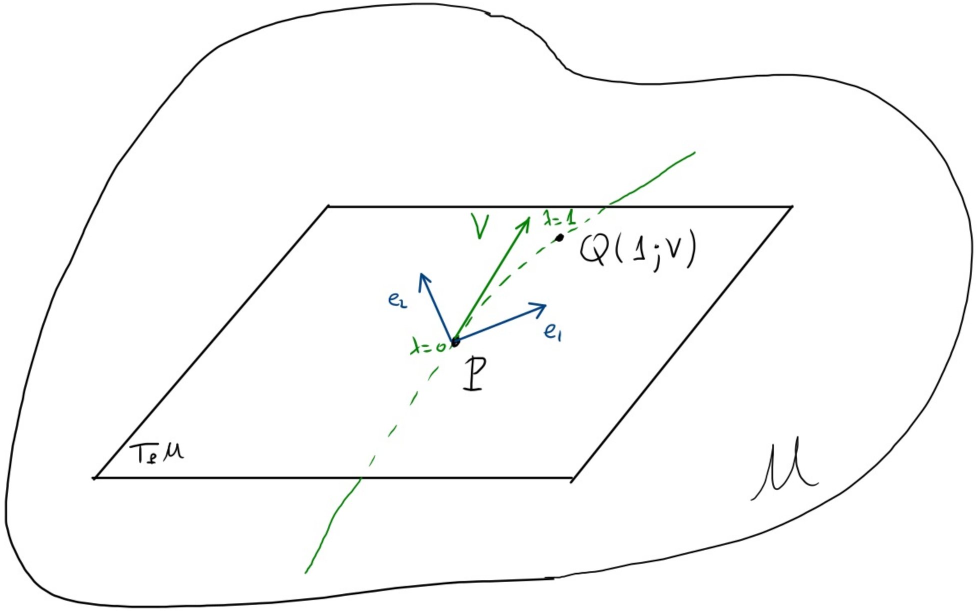

Then:

(a) $e_\mu = \partial_\mu \big|_P$

(b) $\Gamma^\mu{}_{\nu\rho}(P) = 0$

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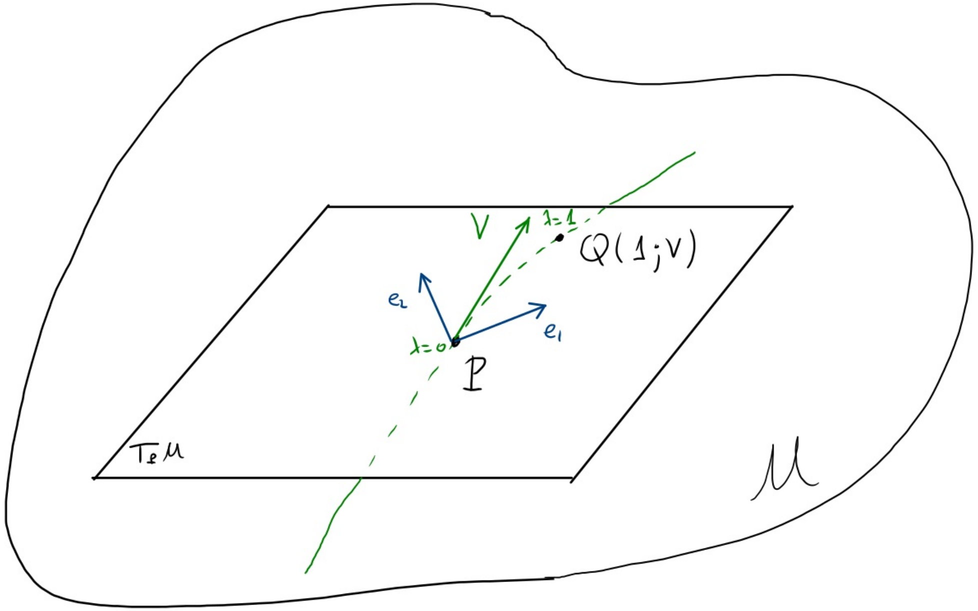

Then:

(a) $e_\mu = \partial_\mu \big|_P$

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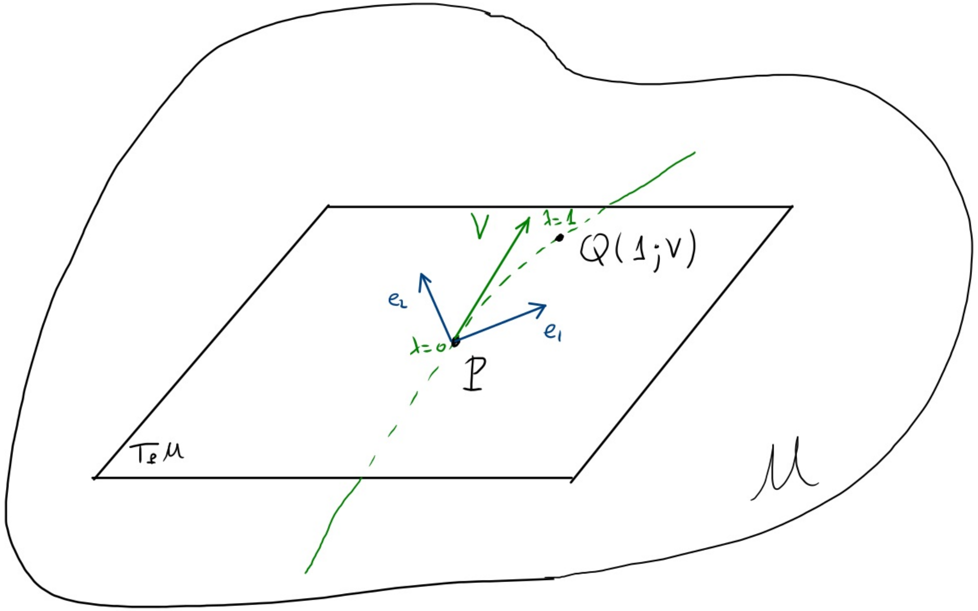
(c) $g_{\mu\nu}(P) = \eta_{\mu\nu}$

Riemann Normal Coordinates



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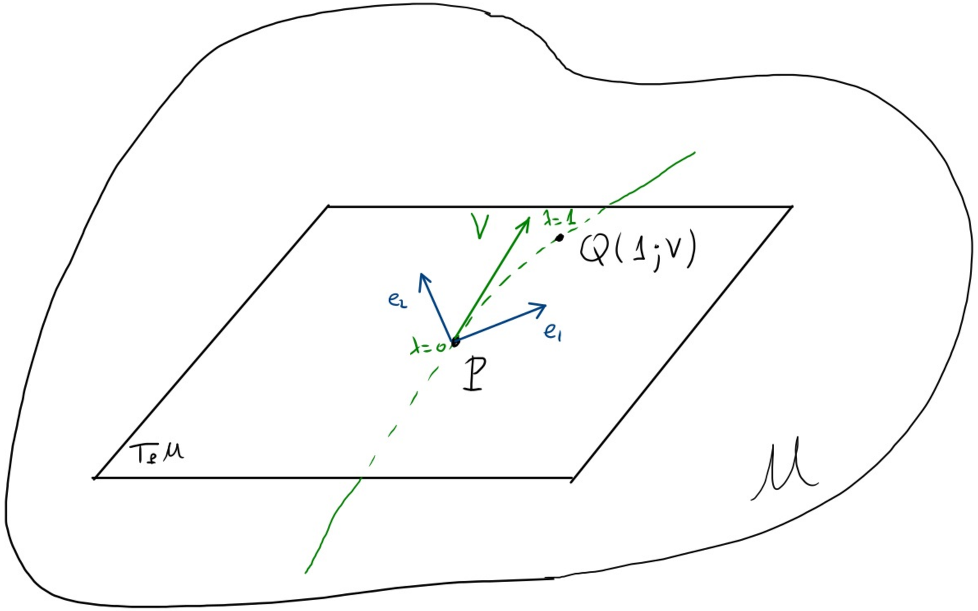

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Higher derivatives determined by curvature:

$$\partial_\rho \partial_\sigma g_{\mu\nu}(P) = -\frac{1}{3}\left(R_{\mu\sigma\nu\rho} + R_{\mu\rho\nu\sigma}\right)$$

$$\partial_\sigma \Gamma^\mu{}_{\nu\rho}(P) = -\frac{1}{3}\left(R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\nu\sigma}\right)$$

$$R_{\mu\nu\rho\sigma}(P) = \partial_\rho \partial_\nu g_{\mu\sigma}(P) - \partial_\sigma \partial_\nu g_{\mu\rho}(P)$$

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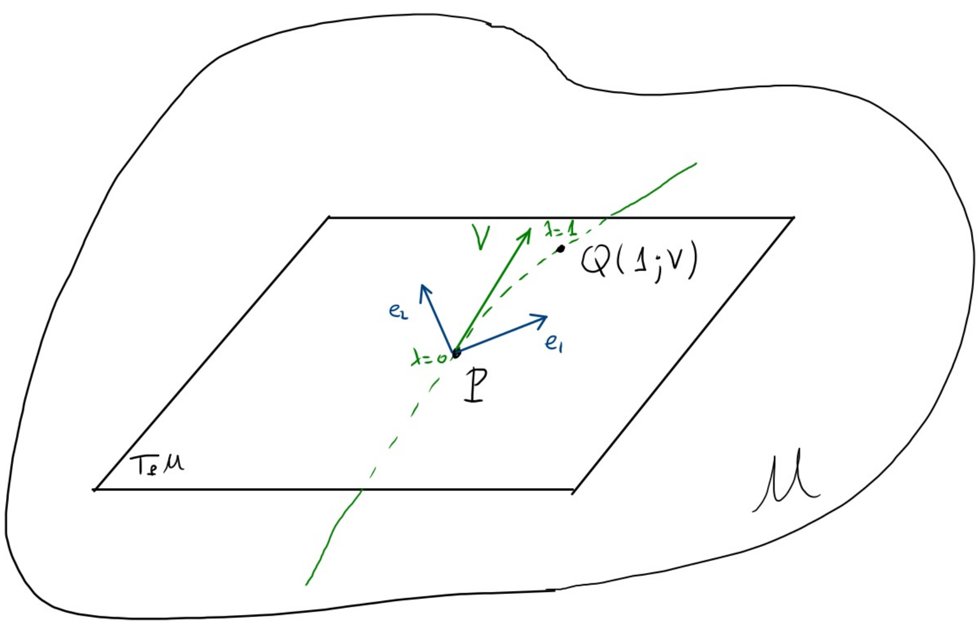

- Any other normal coordinate system same to 2^{nd} order

$$x^{\mu\prime}(P) = x^\mu(P) + \mathcal{O}((x^\mu)^3)$$

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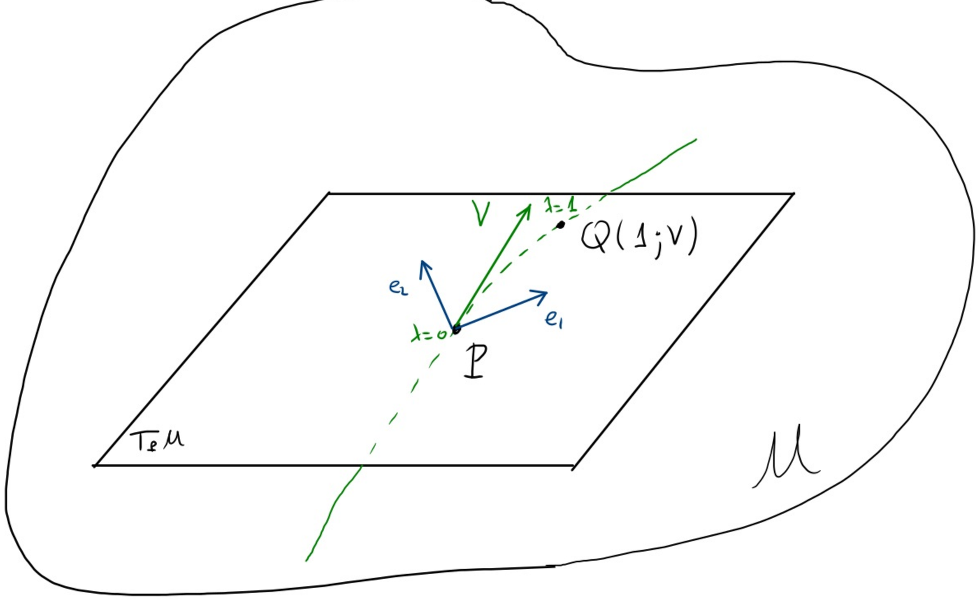

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- If same to 3rd order, Eqs (1)+(2) are preserved

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$$R_{\mu\nu\rho\sigma}(P) = \partial_\rho \partial_\nu g_{\mu\sigma}\cancel{(P)} - \partial_\sigma \partial_\nu g_{\mu\rho}(P) \quad (2)$$

Example: Wormhole geometry (Hartle, examples 8.2, 8.3, 8.5)

$$ds^2 = -dt^2 + dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$g_{tt} = -1 \qquad g_{rr} = 1$$

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Only non-vanishing derivatives:

$$\partial_r g_{\theta\theta} = 2r$$

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easier: only g_{tt}, g_{rr}, $g_{\theta\theta}$, $g_{\phi\phi}$

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$\mu\nu\rho$ + cyclic

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$\mu\nu\rho$ + cyclic

only permutations of $r\theta\theta$, $r\phi\phi$, $\theta\phi\phi$ may appear

$g_{rr} = 1 \qquad g_{\theta\theta} = b^2 + r^2 \qquad g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$

$\partial_r g_{\theta\theta} = 2r \qquad \partial_r g_{\phi\phi} = 2r\sin^2\theta \qquad \partial_\theta g_{\phi\phi} = 2(b^2+r^2)\sin\theta\cos\theta$

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$$g_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

$$g\,\Gamma = \frac{1}{2}\left(\partial g + \partial g - \partial g\right)$$

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$$g_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

permute cyclically: start from 3rd term, then 2nd, then 1st

$$g_r\Gamma_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right) \qquad r\to\theta\to\theta$$

$$g_\theta\Gamma_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right) \qquad \theta\to r\to\theta$$

$$g_r\Gamma_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right) \qquad r\to\phi\to\phi$$

$$g_\phi\Gamma_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right) \qquad \phi\to r\to\phi$$

$$g_\theta\Gamma_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right) \qquad \theta\to\phi\to\phi$$

$$g_\phi\Gamma_{\theta\phi} = \frac{1}{2}\left(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\phi\theta} - \partial_\phi g_{\theta\phi}\right) \qquad \phi\to\theta\to\phi$$

$g_{rr} = 1 \qquad g_{\theta\theta} = b^2 + r^2 \qquad g_{\phi\phi} = (b^2+r^2)\sin^2\theta$

$\partial_r g_{\theta\theta} = 2r \qquad \partial_r g_{\phi\phi} = 2r\sin^2\theta \qquad \partial_\theta g_{\phi\phi} = 2(b^2+r^2)\sin\theta\cos\theta$

$$g_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

no other choice: $g_{\mu\nu}$ is diagonal

$$g_{rr}\Gamma^{r}{}_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right)$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right)$$

$$g_{rr}\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right)$$

$$g_{\phi\phi}\Gamma^{\phi}{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right)$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right)$$

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$$g_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

$$g_{rr}\Gamma^{r}{}_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right)$$

Non-zero terms

$$g_{\theta\theta}\Gamma^{\theta}{}_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right)$$

$$g_{rr}\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right)$$

$$g_{\phi\phi}\Gamma^{\phi}{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right)$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right)$$

$$g_{\phi\phi}\Gamma^{\phi}{}_{\theta\phi} = \frac{1}{2}\left(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\phi\theta} - \partial_\phi g_{\theta\phi}\right)$$

$$g_{rr} = 1 \qquad g_{\theta\theta} = b^2 + r^2 \qquad g_{\phi\phi} = (b^2+r^2)\sin^2\theta$$

$$\partial_r g_{\theta\theta} = 2r \qquad \partial_r g_{\phi\phi} = 2r\sin^2\theta \qquad \partial_\theta g_{\phi\phi} = 2(b^2+r^2)\sin\theta\cos\theta$$

$$g_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

$$g_{rr}\Gamma^{r}{}_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right) \Rightarrow 1 \cdot \Gamma^{r}{}_{\theta\theta} = \frac{1}{2}(-2r) \Rightarrow \Gamma^{r}{}_{\theta\theta} = -r$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right)$$

$$g_{rr}\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right)$$

$$g_{\phi\phi}\Gamma^{\phi}{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right)$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right)$$

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$$g_{rr}\Gamma^{r}{}_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right) \Rightarrow 1\cdot\Gamma^{r}{}_{\theta\theta} = \frac{1}{2}(-2r) \Rightarrow \Gamma^{r}{}_{\theta\theta} = -r$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right) \Rightarrow (b^2+r^2)\Gamma^{\theta}{}_{r\theta} = \frac{1}{2}\cdot 2r \Rightarrow \Gamma^{\theta}{}_{r\theta} = \frac{r}{b^2+r^2}$$

$$g_{rr}\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right)$$

$$g_{\phi\phi}\Gamma^{\phi}{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right)$$

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$$g_{rr}\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right) \Rightarrow 1\cdot\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}(-2r\sin^2\theta) \Rightarrow \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta$$

$$g_{\phi\phi}\Gamma^{\phi}{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right)$$

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$$g_{\theta\theta}\Gamma^\theta{}_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right) \Rightarrow (b^2+r^2)\Gamma^\theta{}_{r\theta} = \frac{1}{2}\cdot 2r \Rightarrow \Gamma^\theta{}_{r\theta} = \frac{r}{b^2+r^2}$$

$$g_{rr}\Gamma^r{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right) \Rightarrow 1\cdot\Gamma^r{}_{\phi\phi} = \frac{1}{2}(-2r\sin^2\theta) \Rightarrow \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$g_{\phi\phi}\Gamma^\phi{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right) \Rightarrow (b^2+r^2)\sin^2\theta\cdot\Gamma^\phi{}_{r\phi} = \frac{1}{2}2r\sin^2\theta \Rightarrow \Gamma^\phi{}_{r\phi} = \frac{r}{b^2+r^2}$$

$$g_{\theta\theta}\Gamma^\theta{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right)$$

$$g_{\phi\phi}\Gamma^\phi{}_{\theta\phi} = \frac{1}{2}\left(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\phi\theta} - \partial_\phi g_{\theta\phi}\right)$$

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$$g_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

$$g_{rr}\Gamma^r{}_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right) \Rightarrow 1\cdot\Gamma^r{}_{\theta\theta} = \frac{1}{2}(-2r) \Rightarrow \Gamma^r{}_{\theta\theta} = -r$$

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$$g_{rr}\Gamma^r{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right) \Rightarrow 1\cdot\Gamma^r{}_{\phi\phi} = \frac{1}{2}(-2r\sin^2\theta) \Rightarrow \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$g_{\phi\phi}\Gamma^\phi{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right) \Rightarrow (b^2+r^2)\sin^2\theta\cdot\Gamma^\phi{}_{r\phi} = \frac{1}{2}2r\sin^2\theta \Rightarrow \Gamma^\phi{}_{r\phi} = \frac{r}{b^2+r^2}$$

$$g_{\theta\theta}\Gamma^\theta{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right) \Rightarrow (b^2+r^2)\Gamma^\theta{}_{\phi\phi} = \frac{1}{2}(-2(b^2+r^2)\sin\theta\cos\theta) \Rightarrow \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$g_{\phi\phi}\Gamma^\phi{}_{\theta\phi} = \frac{1}{2}\left(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\phi\theta} - \partial_\phi g_{\theta\phi}\right)$$

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$$\partial_r g_{\theta\theta}=2r \qquad \partial_r g_{\phi\phi}=2r\sin^2\theta \qquad \partial_\theta g_{\phi\phi}=2(b^2+r^2)\sin\theta\cos\theta$$

$$g_{\mu\sigma}\Gamma^\sigma_{\nu\rho}=\frac{1}{2}\left(\partial_\nu g_{\rho\mu}+\partial_\rho g_{\mu\nu}-\partial_\mu g_{\nu\rho}\right)$$

$$g_{rr}\Gamma^r_{\theta\theta}=\frac{1}{2}\left(\partial_\theta g_{\theta r}+\partial_\theta g_{r\theta}-\partial_r g_{\theta\theta}\right)\Rightarrow 1\cdot\Gamma^r_{\theta\theta}=\frac{1}{2}(-2r)\Rightarrow\Gamma^r_{\theta\theta}=-r$$

$$g_{\theta\theta}\Gamma^\theta_{r\theta}=\frac{1}{2}\left(\partial_r g_{\theta\theta}+\partial_\theta g_{\theta r}-\partial_\theta g_{r\theta}\right)\Rightarrow(b^2+r^2)\Gamma^\theta_{r\theta}=\frac{1}{2}\cdot 2r\Rightarrow\Gamma^\theta_{r\theta}=\frac{r}{b^2+r^2}$$

$$g_{rr}\Gamma^r_{\phi\phi}=\frac{1}{2}\left(\partial_\phi g_{\phi r}+\partial_\phi g_{r\phi}-\partial_r g_{\phi\phi}\right)\Rightarrow 1\cdot\Gamma^r_{\phi\phi}=\frac{1}{2}(-2r\sin^2\theta)\Rightarrow\Gamma^r_{\phi\phi}=-r\sin^2\theta$$

$$g_{\phi\phi}\Gamma^\phi_{r\phi}=\frac{1}{2}\left(\partial_r g_{\phi\phi}+\partial_\phi g_{\phi r}-\partial_\phi g_{r\phi}\right)\Rightarrow(b^2+r^2)\sin^2\theta\cdot\Gamma^\phi_{r\phi}=\frac{1}{2}2r\sin^2\theta\Rightarrow\Gamma^\phi_{r\phi}=\frac{r}{b^2+r^2}$$

$$g_{\theta\theta}\Gamma^\theta_{\phi\phi}=\frac{1}{2}\left(\partial_\phi g_{\phi\theta}+\partial_\phi g_{\theta\phi}-\partial_\theta g_{\phi\phi}\right)\Rightarrow(b^2+r^2)\Gamma^\theta_{\phi\phi}=\frac{1}{2}(-2(b^2+r^2)\sin\theta\cos\theta)\Rightarrow\Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta$$

$$g_{\phi\phi}\Gamma^\phi_{\theta\phi}=\frac{1}{2}\left(\partial_\theta g_{\phi\phi}+\partial_\phi g_{\phi\theta}-\partial_\phi g_{\theta\phi}\right)\Rightarrow(b^2+r^2)\sin^2\theta\,\Gamma^\phi_{\theta\phi}=\frac{1}{2}2(b^2+r^2)\sin\theta\cos\theta\Rightarrow\Gamma^\phi_{\theta\phi}=\cot\theta$$

$$\Gamma^{r}{}_{\theta\theta} = -r \qquad \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^{\phi}{}_{\phi\theta} = \Gamma^{\phi}{}_{\theta\phi} = \cot\theta$$

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

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$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^t{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow \ddot{t} = 0$$

$$\Gamma^{r}{}_{\theta\theta} = -r \qquad \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^{\phi}{}_{\phi\theta} = \Gamma^{\phi}{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^{t}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \Rightarrow \ddot{t} = 0$$

$$\ddot{r} + \Gamma^{r}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$$

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^t{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow \ddot{t} = 0$$

$$\ddot{r} + \Gamma^r{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow$$

$$\ddot{r} + \Gamma^r{}_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^r{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0$$

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^t{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow \ddot{t} = 0$$

$$\ddot{r} + \Gamma^r{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow$$

$$\ddot{r} + \Gamma^r{}_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^r{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0 \Rightarrow$$

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 = 0$$

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^t{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow \ddot{t} = 0$$

$$\ddot{r} + \Gamma^r{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow$$

$$\ddot{r} + \Gamma^r{}_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^r{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0 \Rightarrow$$

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 = 0$$

$$\ddot{\theta} + \Gamma^\theta{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^t{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow \ddot{t} = 0$$

$$\ddot{r} + \Gamma^r{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow$$

$$\ddot{r} + \Gamma^r{}_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^r{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0 \Rightarrow$$

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 = 0$$

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$$\ddot{\theta} + \Gamma^\theta{}_{r\theta}\dot{r}\dot{\theta} + \Gamma^\theta{}_{\theta r}\dot{\theta}\dot{r} + \Gamma^\theta{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0$$

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Radially falling particle through wormhole:

$\dot{\theta} = \dot{\phi} = 0$

(1) $\Rightarrow$ $t = \alpha\tau + \beta$ (proper time too)

(2) $\Rightarrow$ $\ddot{r} = 0$

$$\ddot{t} = 0 \qquad (1)$$

$$\ddot{r} - r\dot{\theta}^2 - r\sin\theta\,\dot{\phi}^2 = 0 \qquad (2)$$

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Consider 4-velocity $(m>0)$

$u^\mu = (u^0, u^r, 0, 0)$

$$\ddot{t} = 0 \qquad (1)$$

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Radially falling particle through wormhole:

$\dot\theta = \dot\phi = 0$

(1) $\Rightarrow t = \alpha\tau + \beta$

(2) $\Rightarrow \ddot r = 0$ (3)

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$u^0 = \sqrt{1+(u^r)^2}$

take $u^\mu = (\sqrt{1+U^2}, U, 0, 0)$, then

(3) $\Rightarrow r(\tau) = U \cdot \tau$

if $r(0)=0$ (i.e. passing the throat)

$\ddot t = 0$ (1)

$\ddot r - r\dot\theta^2 - r\sin\theta\,\dot\phi^2 = 0$ (2)

$\ddot\theta + \frac{2r}{b^2+r^2}\dot r\dot\theta - \sin\theta\cos\theta\,\dot\phi^2 = 0$

$\ddot\phi + \frac{2r}{b^2+r^2}\dot r\dot\phi + 2\cot\theta\,\dot\phi\dot\theta = 0$

Radially falling particle through wormhole:

$\dot{\theta} = \dot{\phi} = 0$

$(1) \Rightarrow t = \alpha \tau + \beta$

$(2) \Rightarrow \ddot{r} = 0 \qquad (3)$

Consider 4-velocity $(u>0)$

$u^\mu = (u^0, u^r, 0, 0)$

$u^\mu u_\mu = -1 \Rightarrow g_{00}(u^0)^2 + g_{rr}(u^r)^2 = -1 \Rightarrow$

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So, going from $r(\tau_1) = R$ to $r(\tau_2) = -R$, the proper time for the particle is

$$\Delta\tau = \tau_2 - \tau_1 = \frac{2R}{U}$$

Radially falling particle through wormhole:

The spatial distance it travelled was ($dt = d\theta = d\phi = 0$)

$$s = \int ds = \int |dr| = 2R$$

take $u^\mu = (\sqrt{1+U^2}, U, 0, 0)$, then

(3) $\Rightarrow$ $r(\tau) = U \cdot \tau$

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