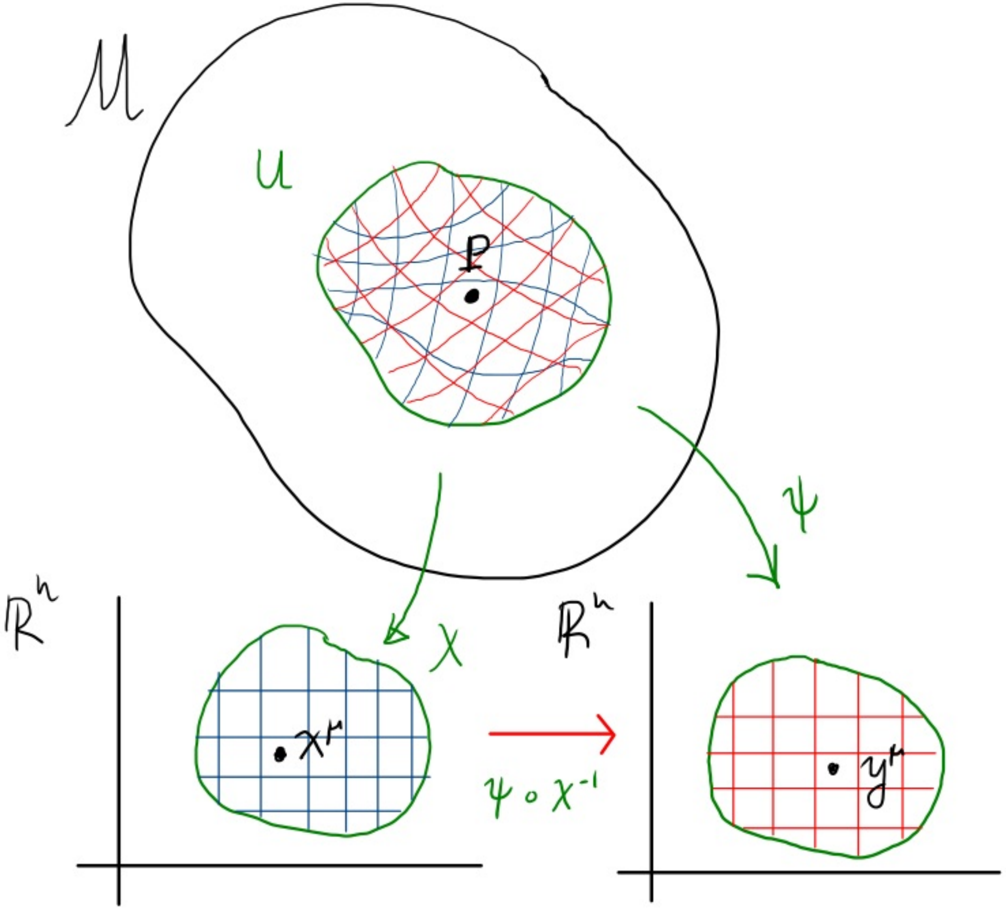
Differentiable Manifords

- look locally like $\mathbb{R}^n$
- coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$

- differentiable structure

$$\frac{\partial y^\mu}{\partial x^\nu}$$



Inherited from $\mathbb{R}^n$ $\Rightarrow$ derivatives, integrals, analysis, all geometric!

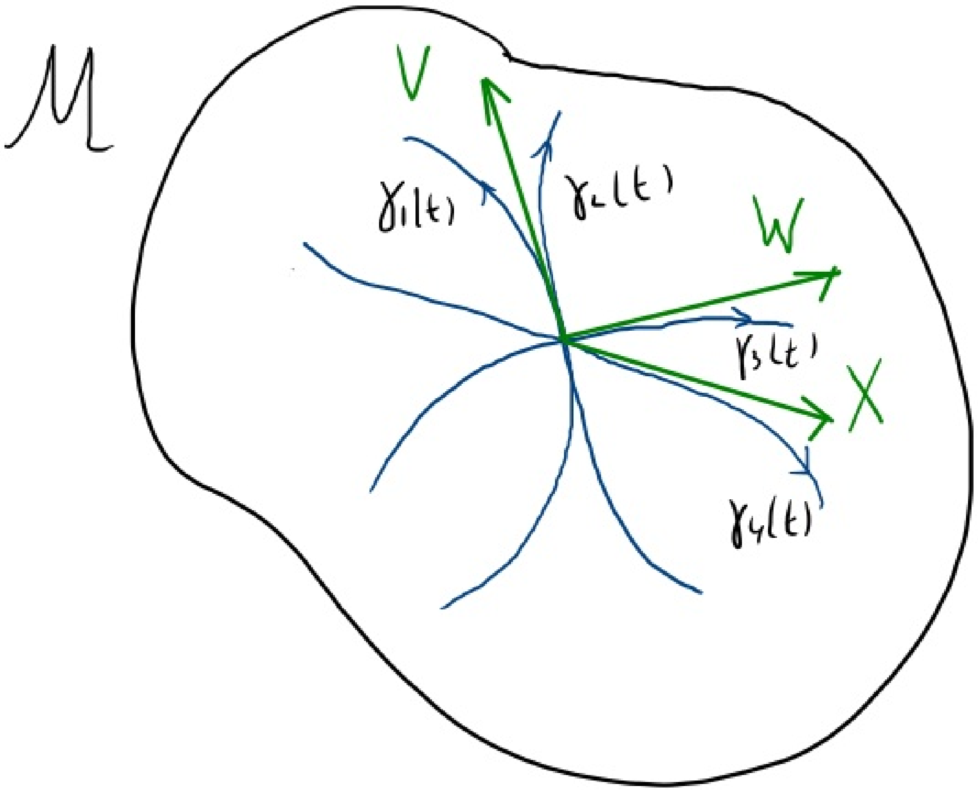
Differentiable Manifords

- look locally like $\mathbb{R}^n$
- coordinate transformations

$$y^\mu = y^\mu(x^\nu)$$

- differentiable structure
- differentiable fields

curves → vectors → tensor fields



Geometric Objects
(can do physics!)

Differentiable Manifolds

The map:

Topological spaces, locally $\mathbb{R}^n$

$\downarrow$

Local coordinate systems

$\downarrow$

Differentiable coordinate transformations

$\downarrow$

Differential Structure

Differentiable Manifords

The map:

$$\begin{tikzcd}[column sep=small]
 & \text{Differential Structure} \arrow[d] & & \\
 & \text{Fields} \arrow[dl] & & \\
\text{vector} \arrow[r] \arrow[d] & \text{1-form} \arrow[r] \arrow[d] & \text{n-form} \\
\text{Lie Derivatives} & \text{tensor} &
\end{tikzcd}$$

n-form
exterior derivative
integrals

Only using the maniford structure!

Differentiable Manifords

Additional Structure:

- **Affine Connection**
  - covariant derivative
  - parallel transport
  - geodesics
  - curvature

- **Metric**
  - distances
  - angles, inner product
  - causal stucture (GR)
  - singles out affine connection and curvature

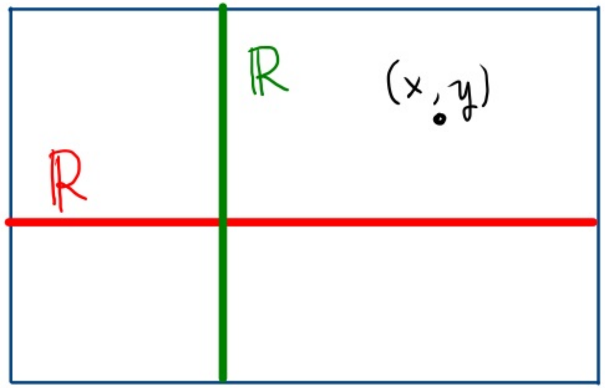
→ Now we can have spacetime and do GR!

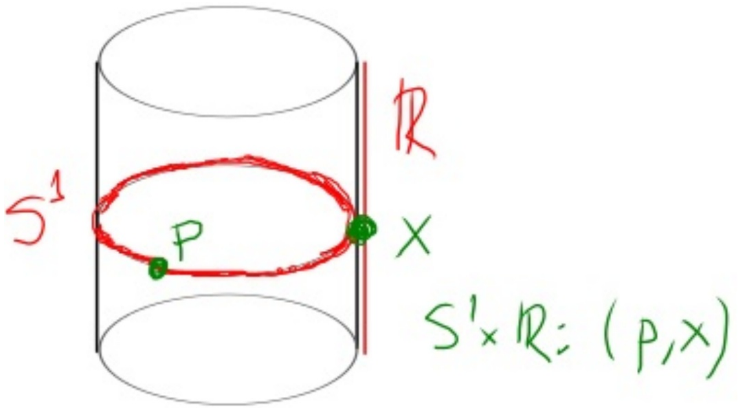
→ Infinite metrics, choose using dynamics of GR

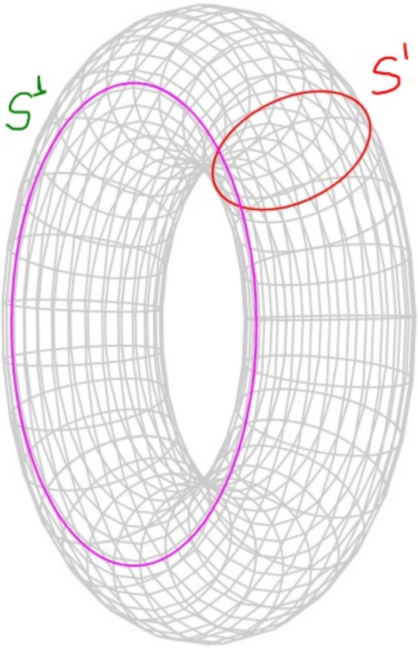
Examples of Manifolds

- $\mathbb{R}^n$: $\mathbb{R}$ (line), $\mathbb{R}^2$ (plane), $\mathbb{R}^3$ (space), ...
- S^n : S^0 (2 points), S^1 (circle), S^2 (sphere), ...
- T^n: T^2 (torus)
- Lie groups: rotations, Lorentz transformations
  $U(N)$, $SU(N)$, ...
- $M = M_1 \times M_2 = \{ P = (P_1, P_2) \mid P_1 \in M_1, P_2 \in M_2 \}$

Examples of Manifolds







$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

(x, y)

$x \in \mathbb{R} \quad y \in \mathbb{R}$

$S^1 \times \mathbb{R}$

(p, x)

$p \in S^1 \quad x \in \mathbb{R}$

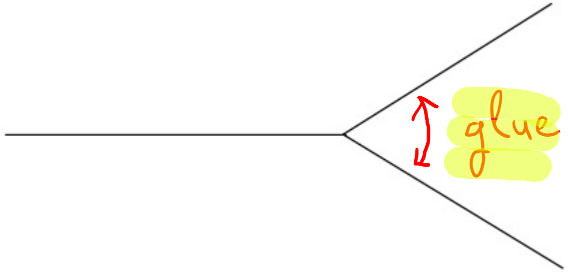
$T^2 = S^1 \times S^1$

(p_1, p_2)

$p_1 \in S^1 \quad p_2 \in S^1$

- $M = M_1 \times M_2 = \{ P = (P_1, P_2) \mid P_1 \in M_1, P_2 \in M_2 \}$

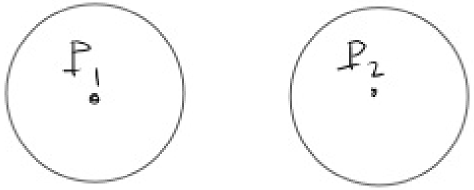
Examples of non-Manifolds



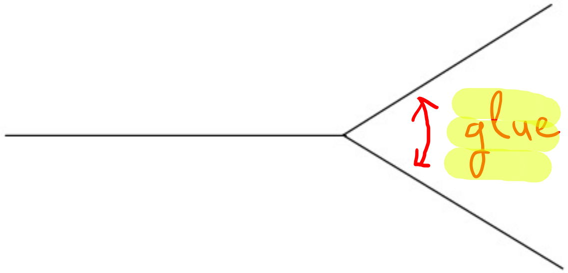
non-Hausdorff

Hausdorff:

$\forall p_1, p_2\ \exists$ disjoint neighborhoods of p_1, p_2



Examples of non-Manifolds



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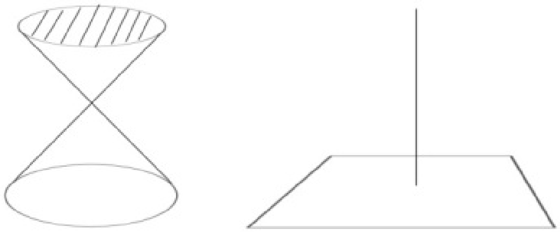
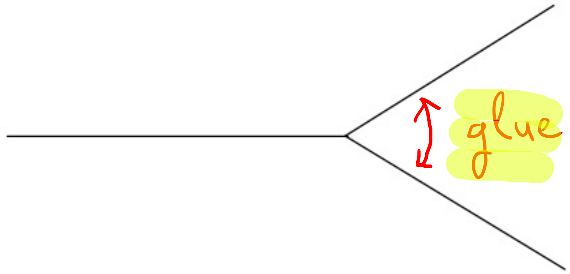


not locally everywhere $\cong \mathbb{R}^n$

Examples of non-Manifolds



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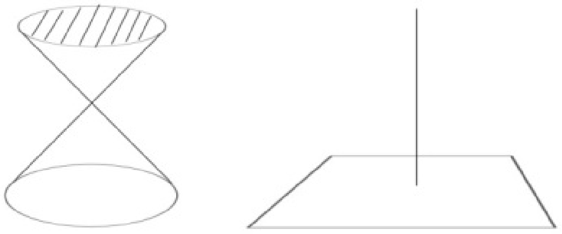

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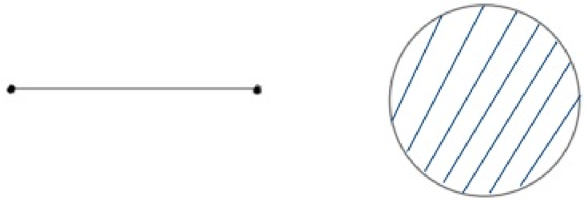
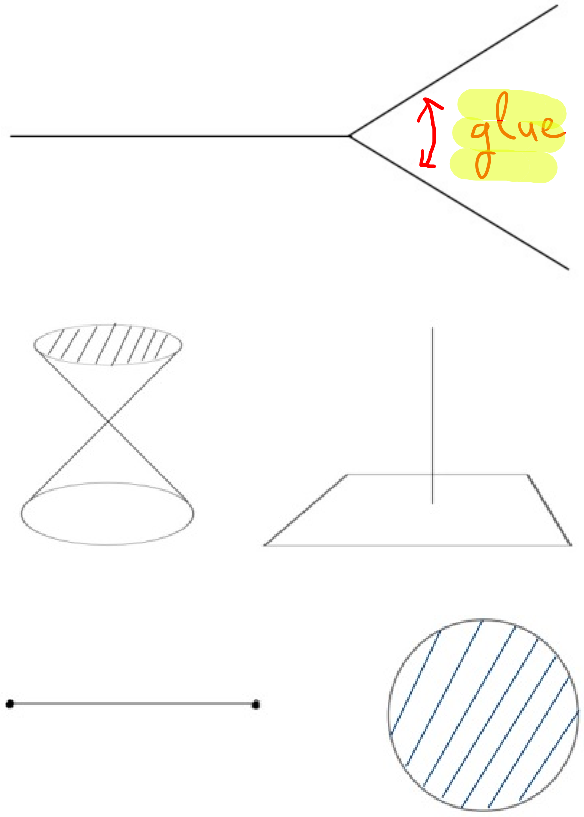
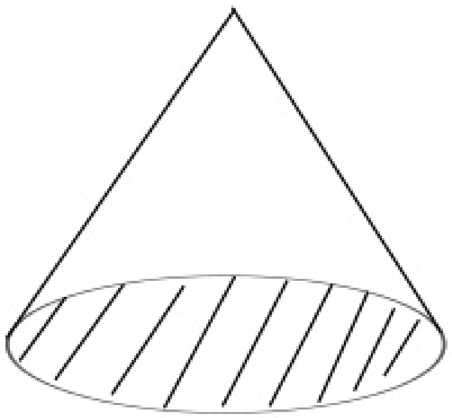


Manifolds with boundary

Examples of non-Manifolds



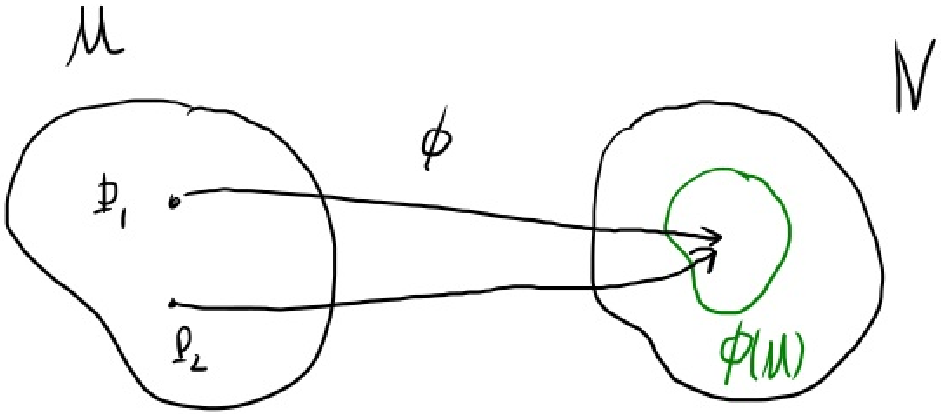


Quiz:

Is the cone a manifold?

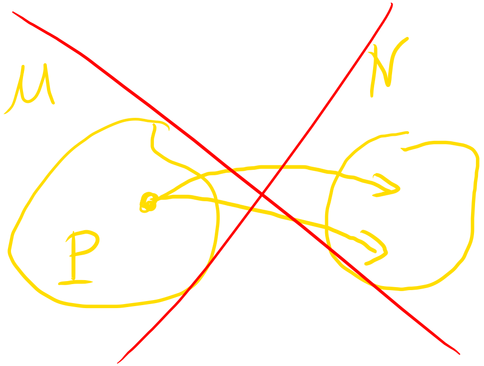
Maps:

$\phi : M \to N$

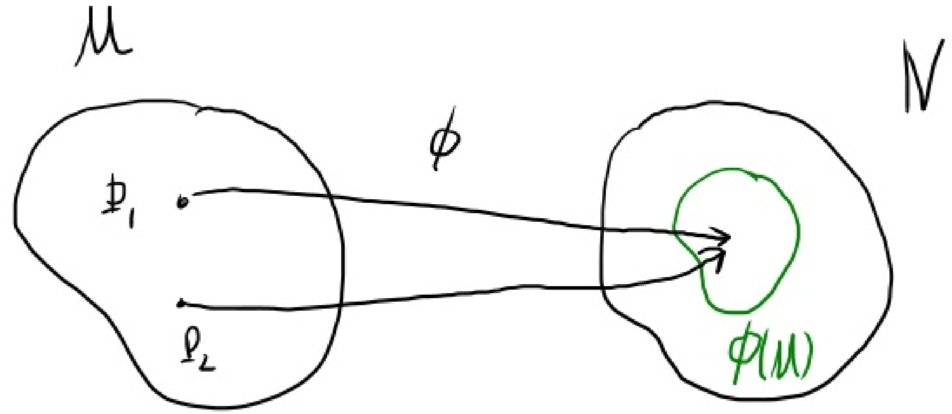


- M: domain
- N: codomain
- $\phi(M)$: image of M or codomain

$\phi(p_1) \neq \phi(p_2) \Rightarrow p_1 \neq p_2$



Maps: $\phi : M \to N$



- M : domain
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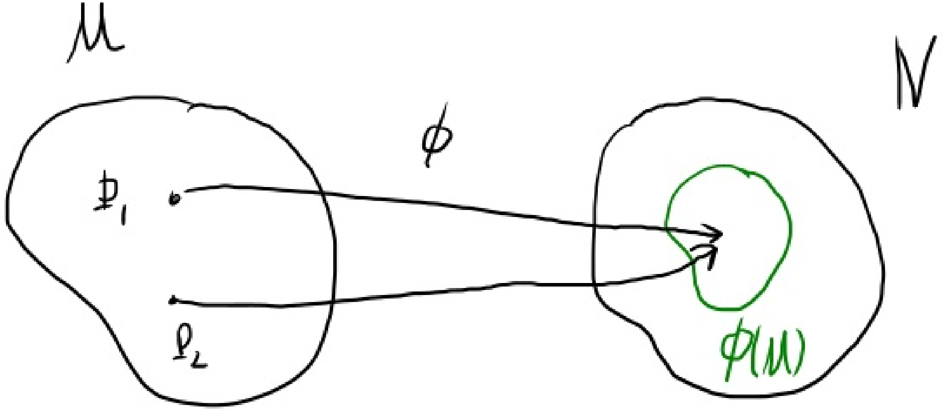
onto: $\phi(M) = N$ (surjective)

Maps: $\phi : M \to N$



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- $\phi(M)$: image of M or codomain

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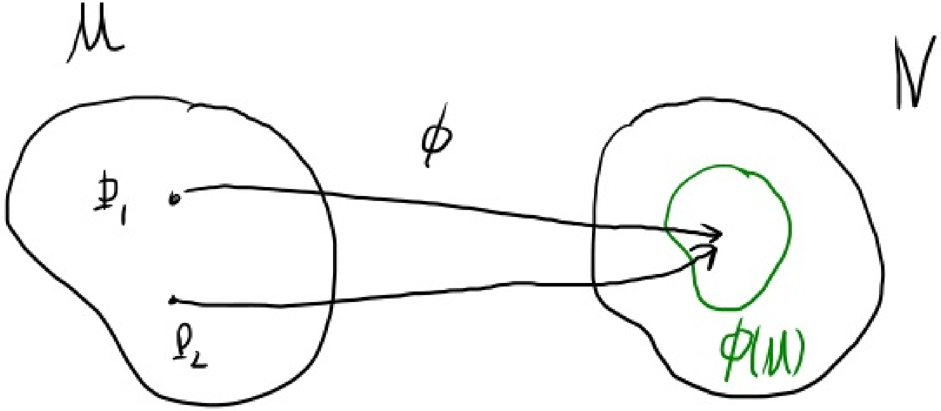
1-1: $P_1 \neq P_2 \Rightarrow \phi(P_1) \neq \phi(P_2)$ (injective)

Maps: $\phi: M \to N$



- M: domain
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onto: $\phi(M) = N$ (surjective)

1-1: $p_1 \neq p_2 \Rightarrow \phi(p_1) \neq \phi(p_2)$ (injective)

invertible: 1-1 and onto $\Rightarrow \phi^{-1}: N \to M$ a map

Topology


topology
[top-ol-uh-jee]
noun
1. The study of adding tasty toppings.
CHOOSE FROM 7 BREAKFAST SANDWICHES
buy
PRO-GO
PROTEIN PACK
any Coke product

Topology

M is a topological space if it is covered by open sets $\{U_\alpha\}$ such that

$U_\alpha \cap U_\beta$ and $\bigcup_\gamma U_\gamma$ are open sets.

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only finite intersections required to be open

possibly infinite

Topology

M is a topological space if it is covered by open sets $\{U_\alpha\}$ such that

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$\emptyset$ and M are open sets

Topology

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$\{U_\alpha\}$ define a topology on M

Topology

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$\{U_\alpha\}$ define $\underbrace{\text{a}}$ topology on M

there are many & inequivalent

Topology

$\mathcal{M}$ is a topological space if it is covered by open sets $\{U_\alpha\}$ such that

$U_\alpha \cap U_\beta$ and $\bigcup_\gamma U_\gamma$ are open sets.

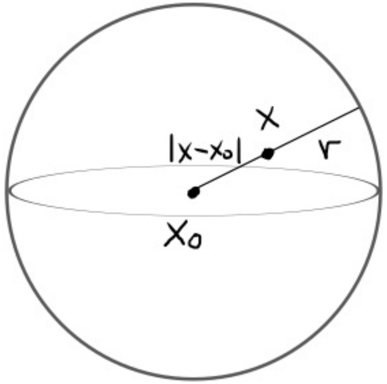
$\{U_\alpha\}$ define a topology on $\mathcal{M}$

e.g $\{\phi, \mathcal{M}\}$ is a trivial topology on $\mathcal{M}$

Topology

$\mathbb{R}^n$ is assumed with the topology generated by open balls

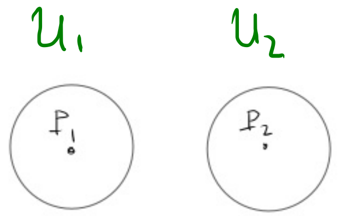
$$B_{x_0}(r) = \{ x \mid x \in \mathbb{R}^n,\ \underbrace{|x - x_0| < r}_{\text{strictly less}} \}$$



Topology

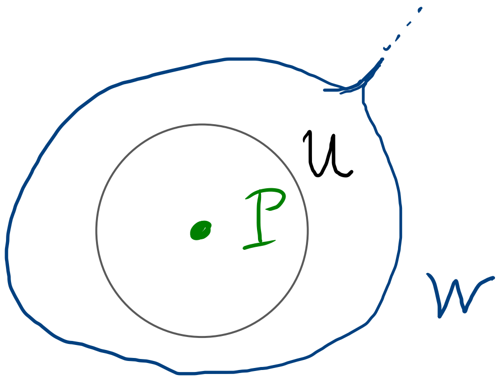
- $\mathcal{M}$ is Hausdorff iff

$\forall\, p_1 \neq p_2$, $\exists$ open U_1, U_2 s.t. $U_1 \cap U_2 = \emptyset$



Topology

- $\mathcal{M}$ is Hausdorff iff
  $\forall P_1 \neq P_2$, $\exists$ open U_1, U_2 s.t. $U_1 \cap U_2 = \emptyset$
- W is a neighborhood of P:
  $P \in W \subseteq \mathcal{M}$, and $\exists$ open $U \subseteq W$ s.t. $P \in U$



W not necessarily open

Topology

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- W is closed:
  $W \subseteq \mathcal{M}$, and $\mathcal{M} \setminus W$ is open

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- $\overline{W}$ is the closure of W:
  $\overline{W} \subseteq \mathcal{M}$ is the smallest closed superset of W

Topology

- W° is the interior of W:
  $W^\circ \subseteq M$ is the largest open subset of W

- W is closed:
  $W \subseteq M$, and $M \setminus W$ is open

- $\overline{W}$ is the closure of W:
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Topology

- W° is the interior of W:
  $W^\circ \subseteq \mathcal{M}$ is the largest open subset of W
- ∂W is the boundary of W:
  $\partial W \subseteq \mathcal{M}$, and $\partial W = \overline{W} \setminus W^\circ$

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Topology

- W° is the interior of W:
  $W^\circ \subseteq \mathcal{M}$ is the largest open subset of W
- ∂W is the boundary of W:
  $\partial W \subseteq \mathcal{M}$, and $\partial W = \bar{W} \setminus W^\circ$
- W is compact:
  $W \subseteq \mathcal{M}$, and $\forall \{U_\alpha\}$ open covering of W,
  $\exists$ finite $\{U_{\alpha'}\} \subset \{U_\alpha\}$ which is also an open covering of W

Topology

For $\mathbb{R}^n$ with the usual topology:

W compact $\Longleftrightarrow$ W is closed + bounded

(theorem)

- W is compact:

$W \subseteq \mathcal{U}$, and $\forall \{U_\alpha\}$ open covering of W,

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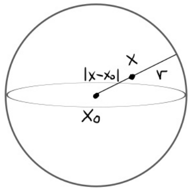
Topology

For $\mathbb{R}^n$ with the usual topology:

W compact $\Longleftrightarrow$ W is closed + bounded

(theorem)

examples: $\bar{B}_{x_0}(r) = \{ x \mid |x - x_0| \leq r \}$



S^n

T^n

} because closed + bounded in $\mathbb{R}^{n+1}$

Topology

- W is connected:

  $W \subseteq M$, and $\nexists$ U_1, U_2 s.t. $U_1 \cup U_2 = W$ and $U_1 \cap U_2 = \emptyset$

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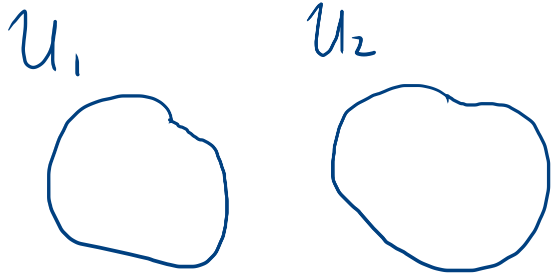
  $W \subseteq \mathcal{M}$, and $\nexists\ U_1, U_2$ s.t. $U_1 \cup U_2 = W$ and $U_1 \cap U_2 = \emptyset$
- W is disconnected:
  If W is not connected



$W = U_1 \cup U_2$

disconnected

Topology

- W is connected:
  $W \subseteq M$, and $\nexists\ U_1, U_2$ s.t. $U_1 \cup U_2 = W$ and $U_1 \cap U_2 = \emptyset$
- W is disconnected:
  If W is not connected
- W is simply connected:
  If all loops are contractible to a point

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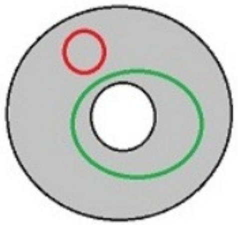

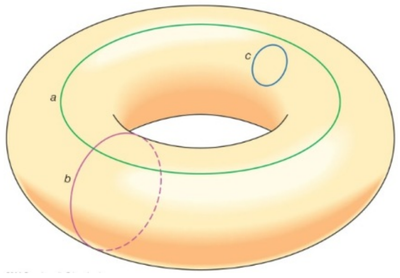

Topology

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$\mathbb{R}^2, S^2$ are simply connected

$\mathbb{R}^2 - \{0\}, T^2, S^1$ are not simply connected

Topology

- Continuity:

$\phi : M \to N$ is continuous iff

$\forall$ open $V \subseteq N \Rightarrow \underbrace{\phi^{-1}(V)}_{\text{inverse image}} \subseteq M$ is open

ϕ^{-1} may not be a map

Topology

- Continuity:

$\phi: M \to N$ is continuous iff

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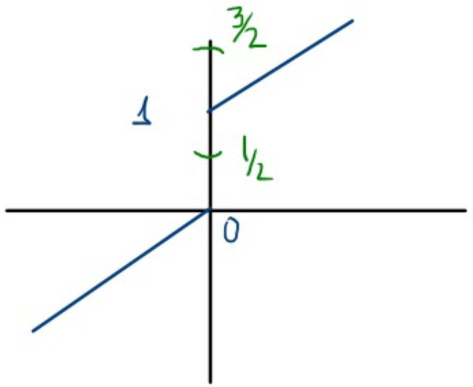

e.g. $f(x) = \begin{cases} x & , x<0 \\ x+1 & , x \geq 0 \end{cases}$

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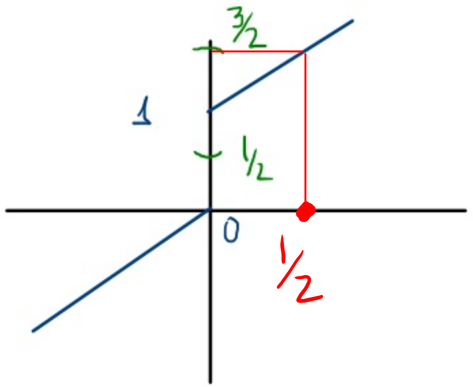

e.g. $f(x) = \begin{cases} x & , x<0 \\ x+1 & , x \geq 0 \end{cases}$

$f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right) = \left[0, \frac{1}{2}\right)$ not open!

Topology

- Continuity:

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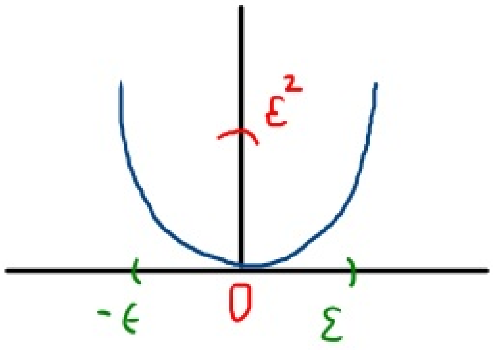
$f(x) = x^2$ continuous



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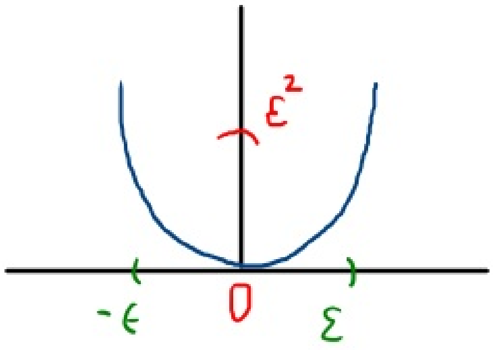

$f(x) = x^2$ continuous

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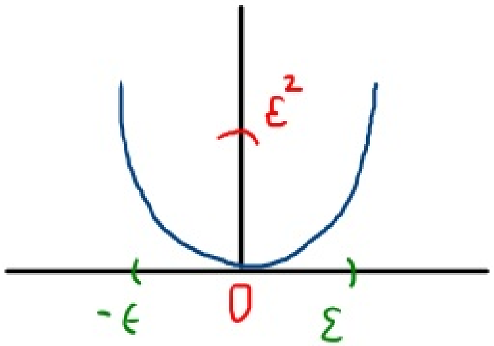


$f(x) = x^2$ continuous

$f((-\epsilon, \epsilon)) = [0, \epsilon^2)$ not open

Continuous maps do not necessarily map open to open

Homeomorphisms

$\phi : M \to N$ is a homeomorphism iff

ϕ is continuous and invertible

<u>and</u> ϕ^{-1} is continuous

Homeomorphisms

$\phi : M \to N$ is a homeomorphism iff

ϕ is continuous and invertible

and ϕ^{-1} is continuous

- M and N are homeomorphic :

$\exists$ homeomorphism $\phi : M \to N$

Homeomorphisms

$\phi : M \to N$ is a homeomorphism iff

ϕ is continuous and invertible

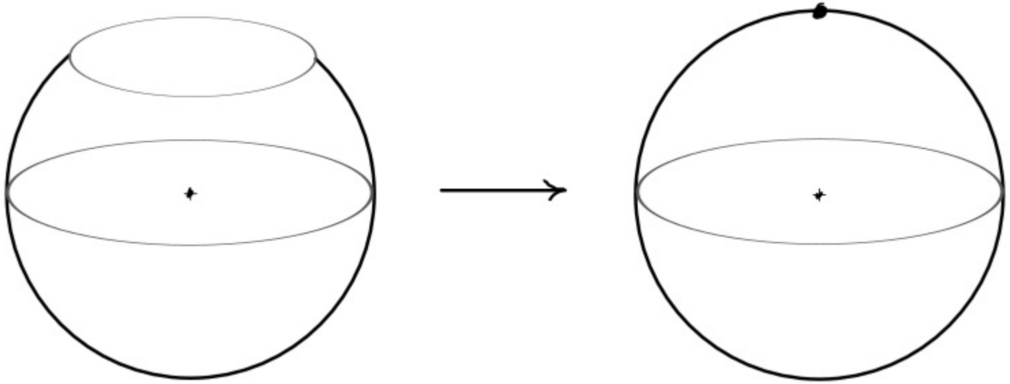
<u>and</u> ϕ^{-1} is continuous

- M and N are homeomorphic:

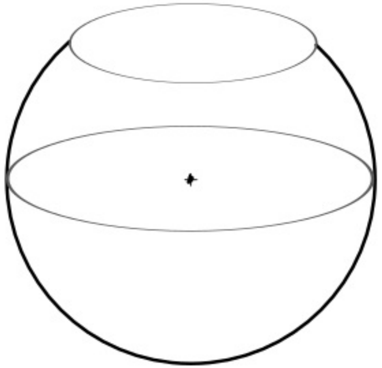
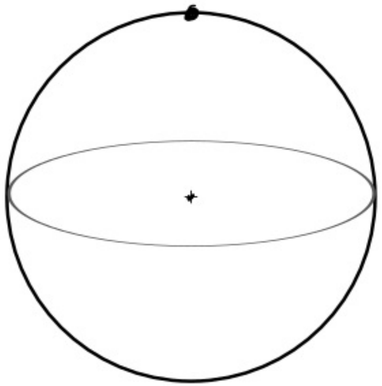
  $\exists$ homeomorphism $\phi : M \to N$

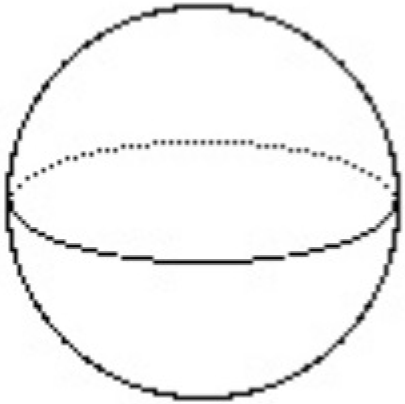
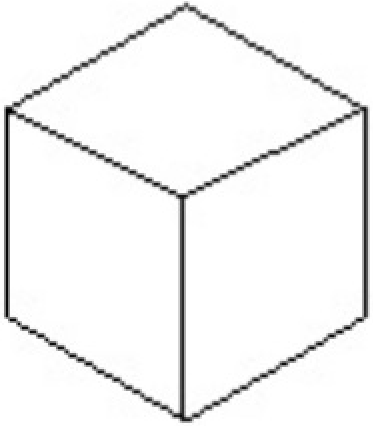
– we say that $M \cong N$ are topologically equivalent

Examples:

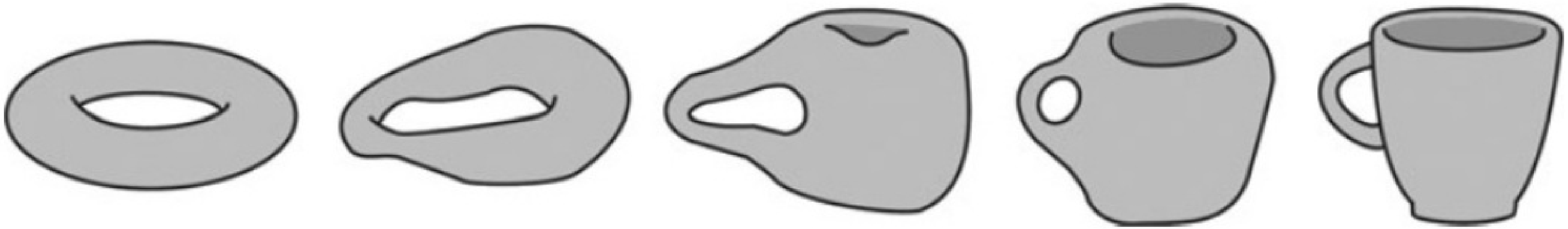


Examples:

 $\longrightarrow$ 

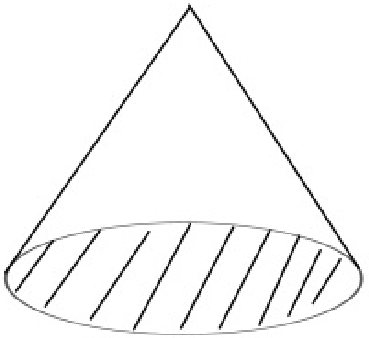
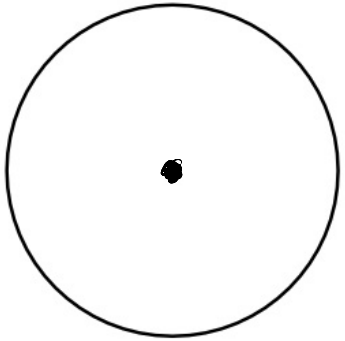
 $\cong$ 

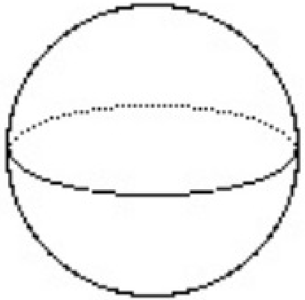
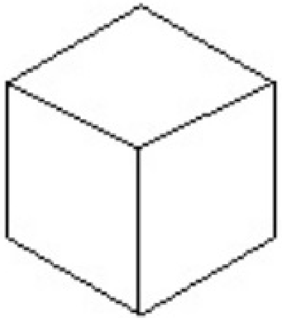
Examples:



punctures, cuts, ... prohibited

Examples:

 $\cong$ 

 $\cong$ 

Creasing, crumpling, ... allowed!

(singularities in geometry)

Differentiable Manifold

- M is a differentiable manifold of $\dim M = n$ with maximal atlas if:

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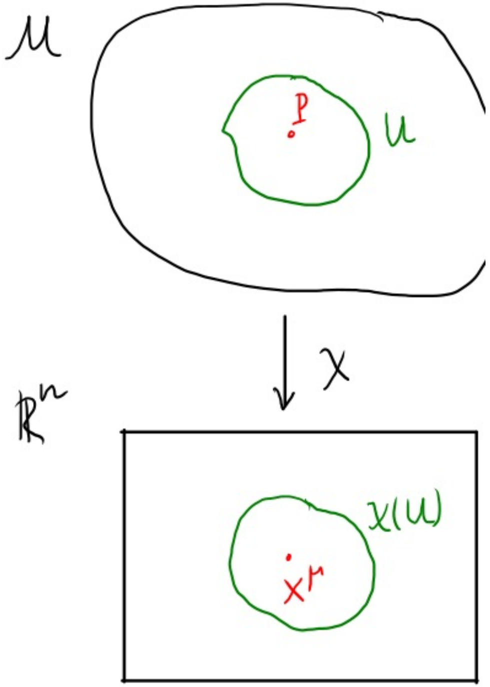
  _ M is locally like $\mathbb{R}^n$

  $\forall P \in M$ is in a chart (U, x) s.t.

  × U is an open neighborhood of P

  × $x : U \to x(U) \subseteq \mathbb{R}^n$ a homeomorphism

  $P \to x^\mu(P) \equiv x(P)$



Differentiable Manifold

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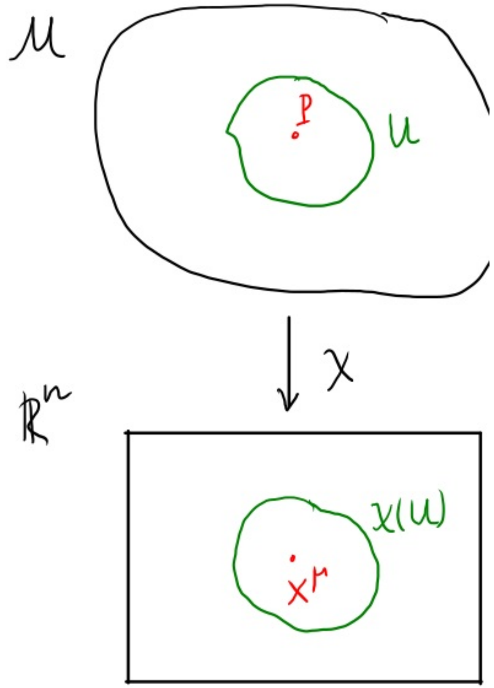
a chart (U, x)



$x^\mu(P)$ are the coordinates of P in chart

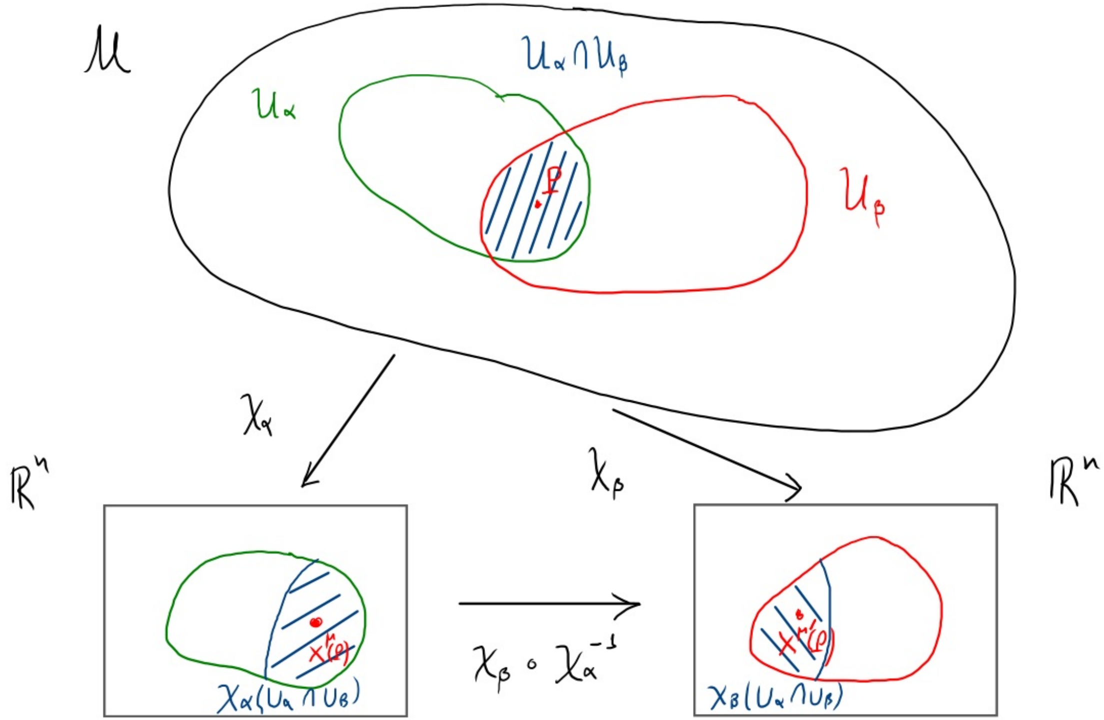
$U \cong x(U)$ are topologically equivalent

$P \to x^\mu(P) \equiv x(P)$

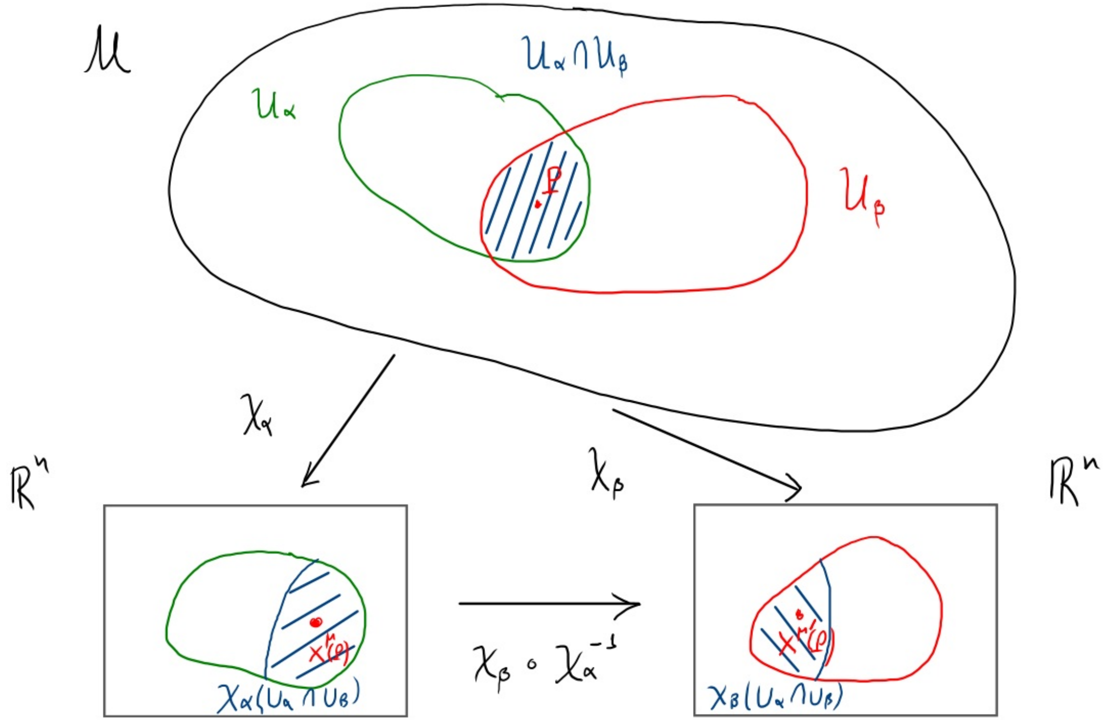
Differentiable Manifold

Coordinate transformations:

$$\chi_\beta \circ \chi_\alpha^{-1}$$



Differentiable Manifold



Coordinate transformations:

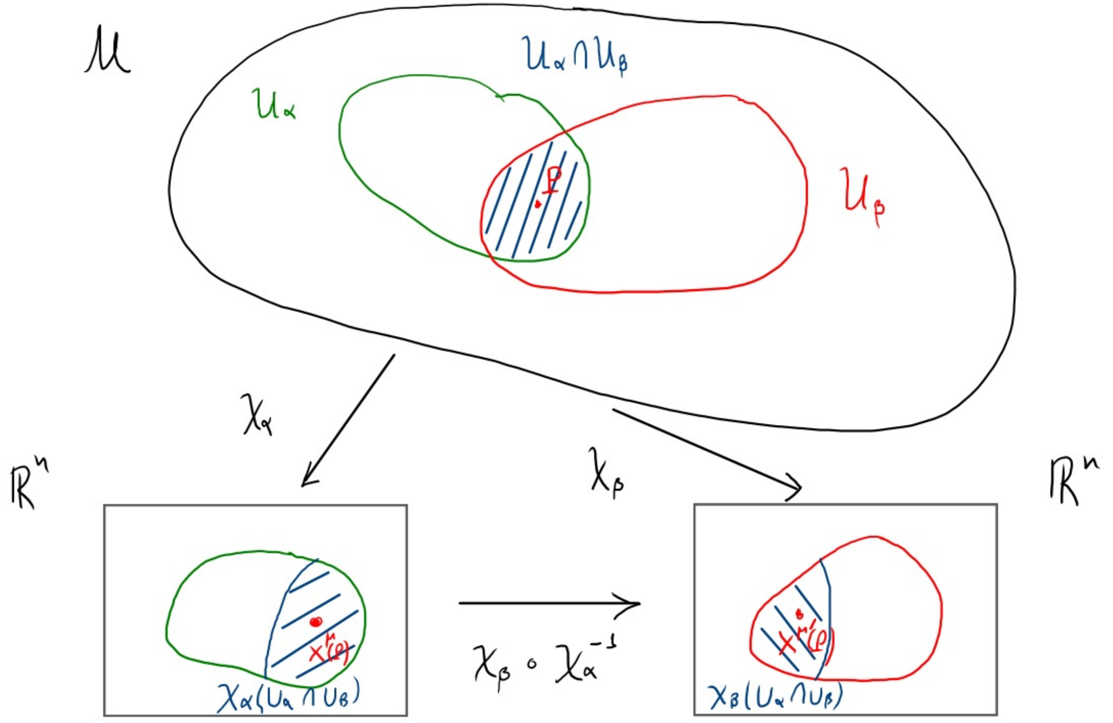
$$\chi_\beta \circ \chi_\alpha^{-1}$$

transition functions

$$\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \chi_\beta(U_\alpha \cap U_\beta)$$

$$x^\nu \quad \longrightarrow \quad x^{\mu'}(x^\nu)$$

Differentiable Manifold



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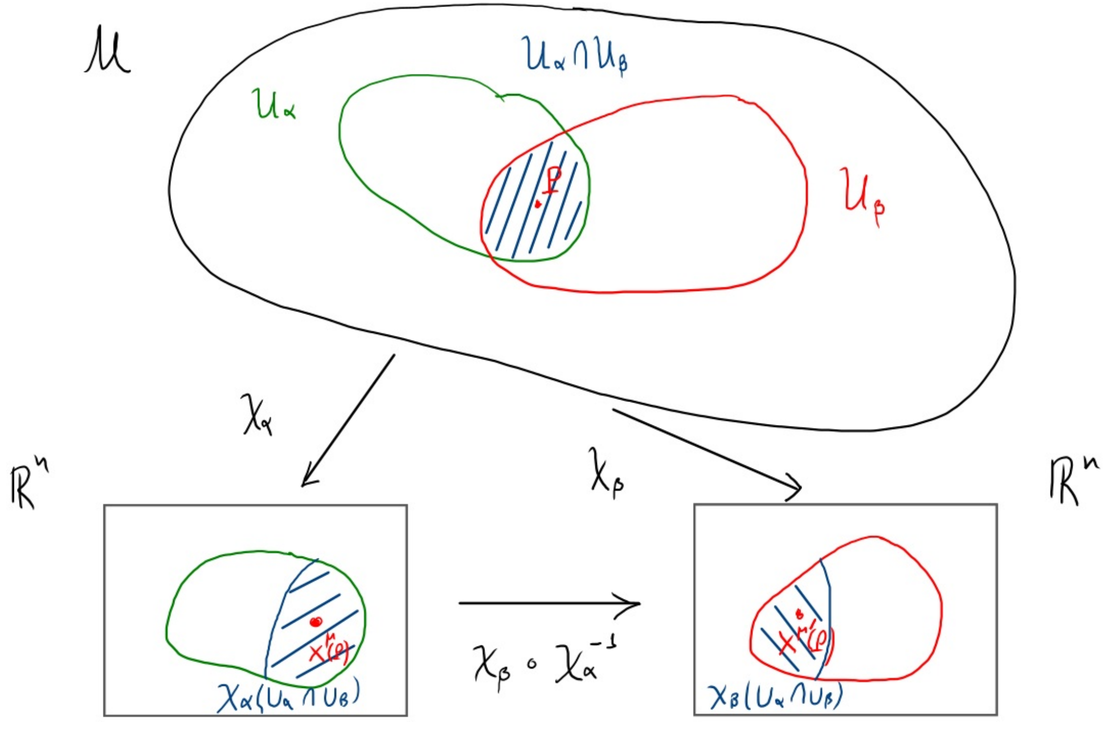
Differentiable!

Differentiable Manifold



Coordinate transformations:

$$\chi_\beta \circ \chi_\alpha^{-1} :$$

$$x^{\mu'} = x^{\mu'}(x^\nu) \equiv \chi_\beta \circ \chi_\alpha^{-1}(x^\nu)$$

- an ordinary function on $\mathbb{R}^n$
- borrow calculus from $\mathbb{R}^n$-functions

Differentiable Manifold

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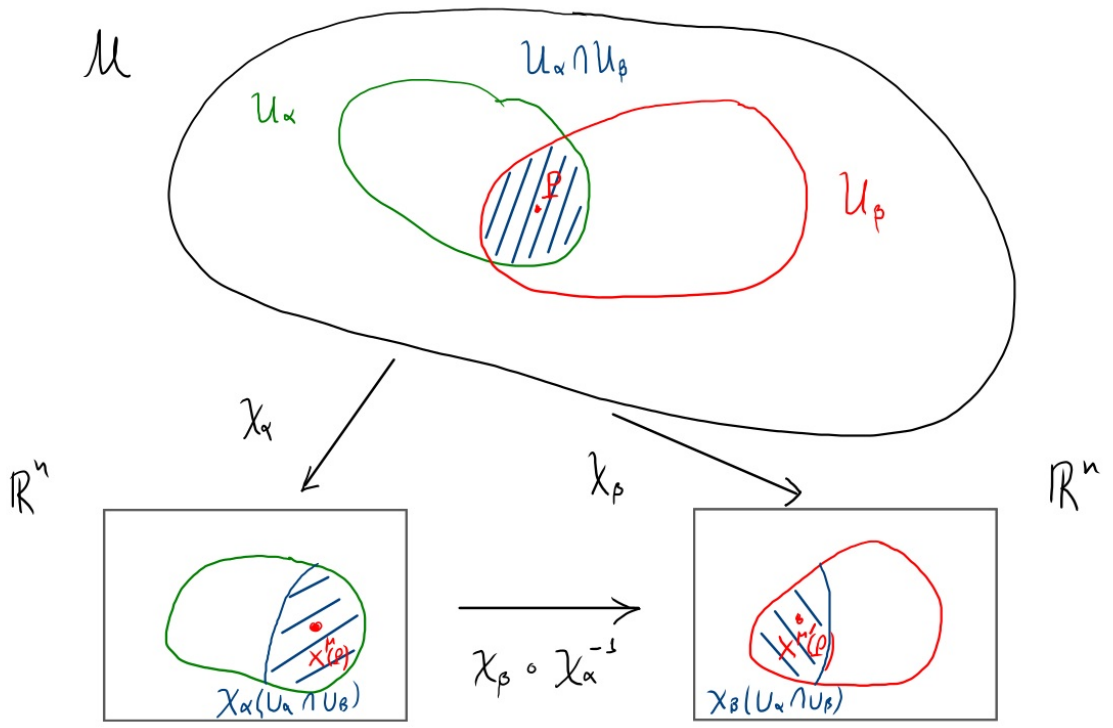
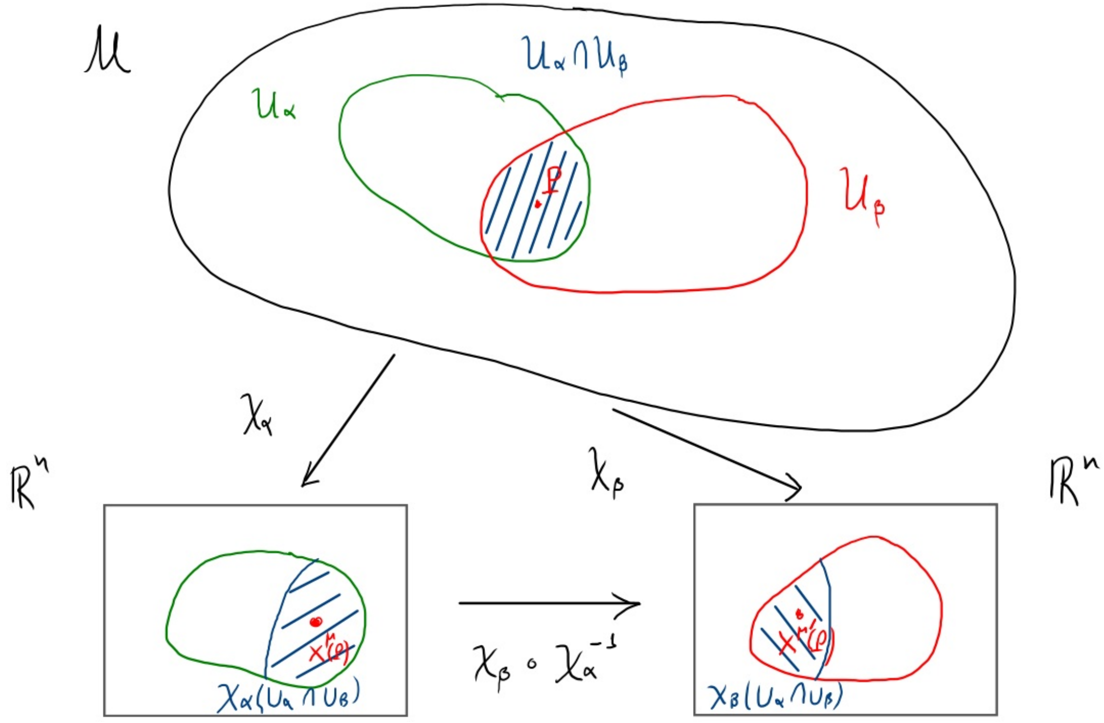


assume $\dfrac{\partial x^{\mu'}}{\partial x^\nu}$ is continuous, then C^1

$\dfrac{\partial^p x^{\mu'}}{\partial x^{\nu_1} \partial x^{\nu_2} \cdots \partial x^{\nu_p}}$ " " C^p

Differentiable Manifold



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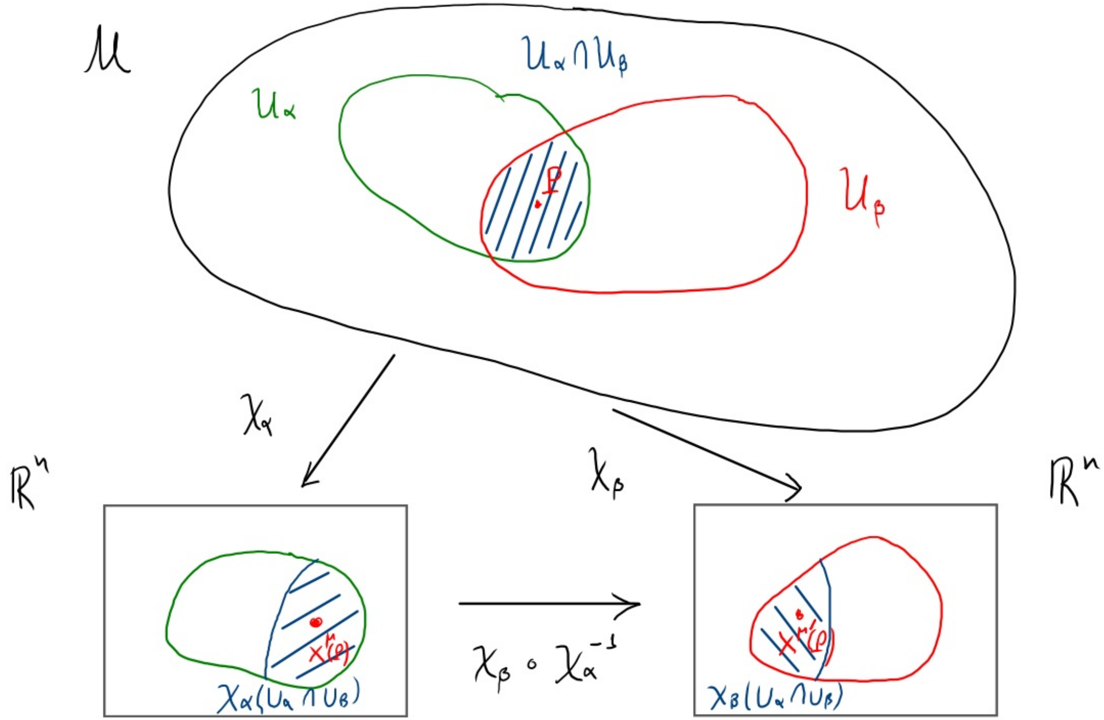
If analytic we (usually) write C^∞

Differentiable Manifold

Coordinate transformations introduce a differentiable structure



Differentiable Manifold

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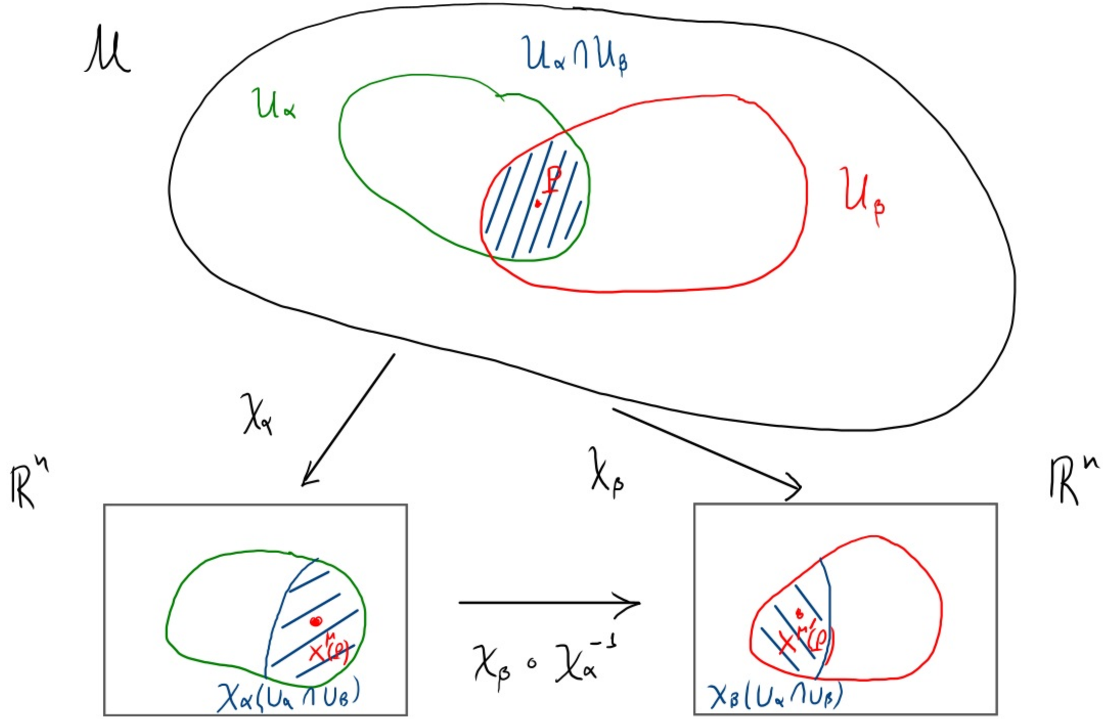
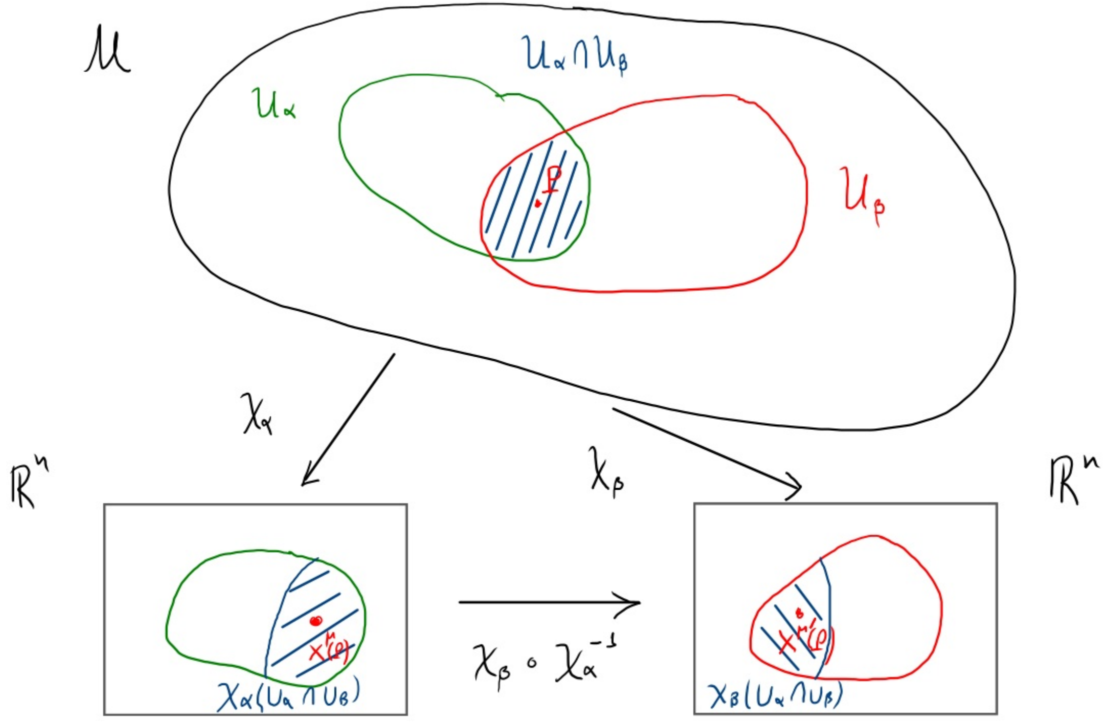


A set of $\{(U_\alpha, \chi_\alpha)\}$

with $\{U_\alpha\}$ an open covering of $\mathcal{M}$, and $\chi_\beta \circ \chi_\alpha^{-1}$ C^p-differentiable when $U_\alpha \cap U_\beta$ open sets, constitute a C^p atlas of $\mathcal{M}$ (we physicists may assume C^∞)

Differentiable Manifold

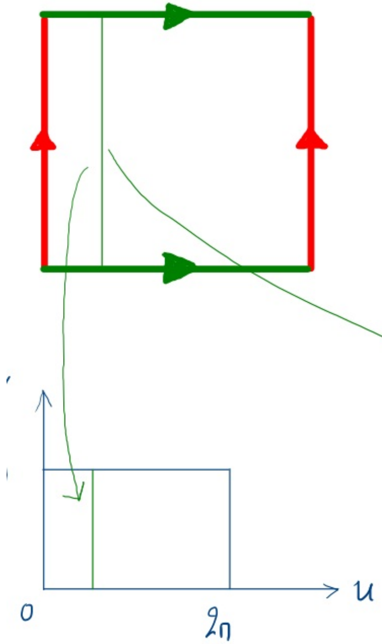


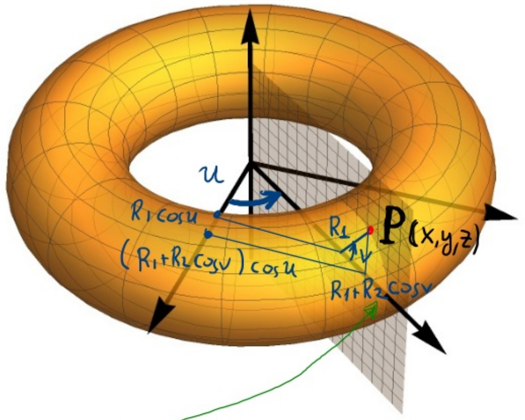
Maximal atlas:
contains all compatible charts

A set of $\{(U_\alpha, \chi_\alpha)\}$ with $\{U_\alpha\}$ an open covering of $\mathcal{M}$, and $\chi_\beta \circ \chi_\alpha^{-1}$ C^p-differentiable when $U_\alpha \cap U_\beta$ open sets, constitute a C^p **atlas** of $\mathcal{M}$ (we physicists may assume C^∞)

Many times we define manifolds using embeddings

the manifold T^2





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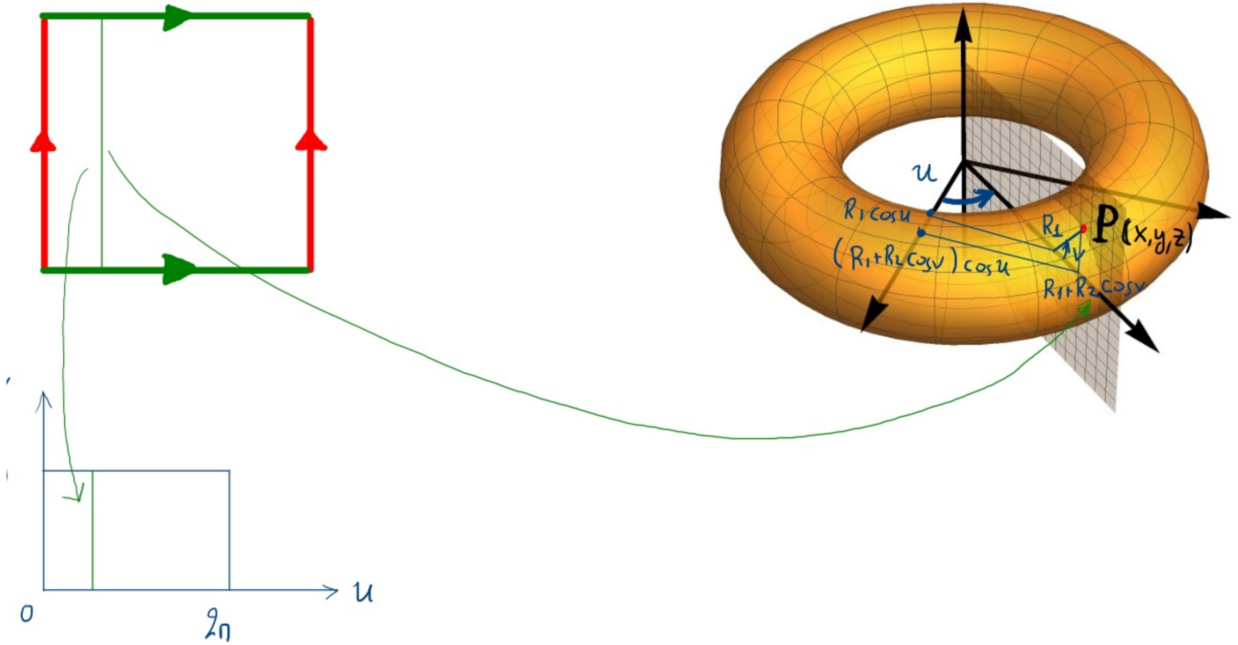

a coordinate system

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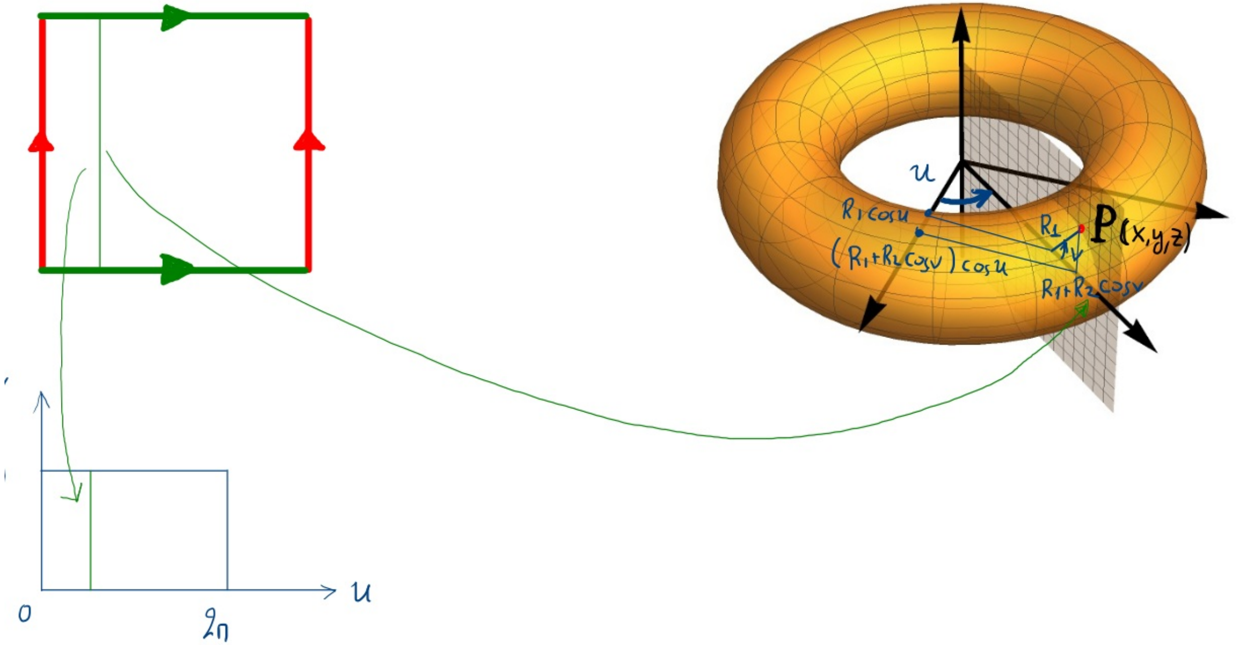

the embedding

$$x=(R_1+R_2\cos v)\cos u$$
$$y=(R_1+R_2\cos v)\sin u$$
$$z=R_2\sin v$$

a coordinate system

Many times we define manifolds using embeddings

- manifolds don't need the embeddings to exist

e.g. spacetime in GR is not embedded anywhere

(beware of string theorists...)

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  a n-dimensional manifold is always embeddable in $\mathbb{R}^{2n}$
  (Whitney's embedding theorem)

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- embeddings of a n-dim manifold may not be possible
  in any $\mathbb{R}^m$ $n < m < 2n$

  e.g. Klein bottle in $\mathbb{R}^3$

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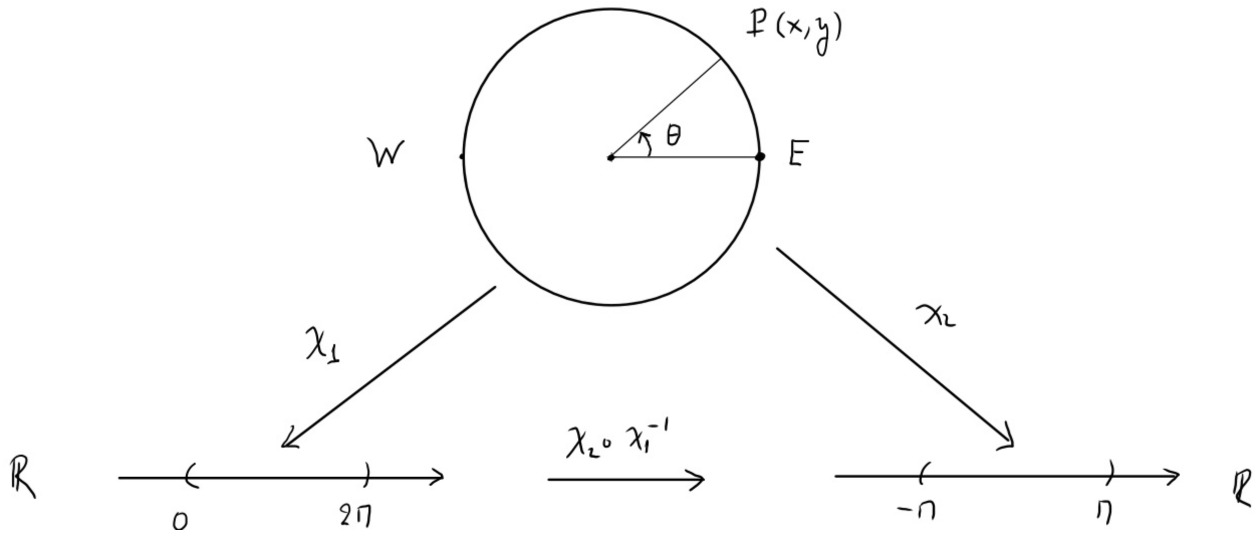
- Manifolds may need more than one charts to be covered

Examples: S^1

(U^1, χ_1):

$U^1 = S^1 \setminus \{E\}$



Examples: S^1

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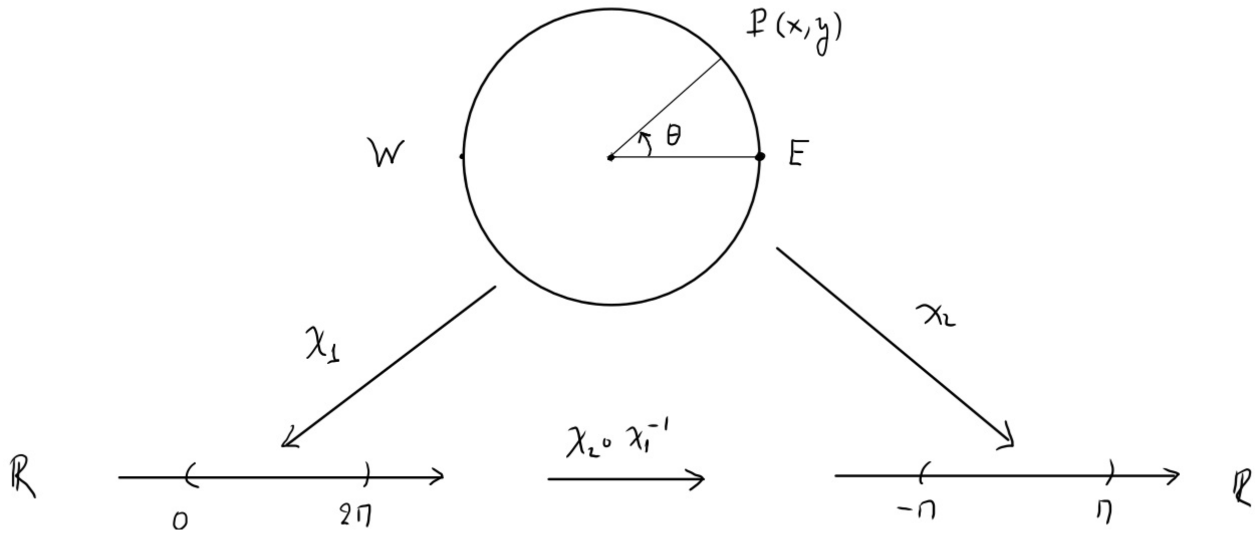
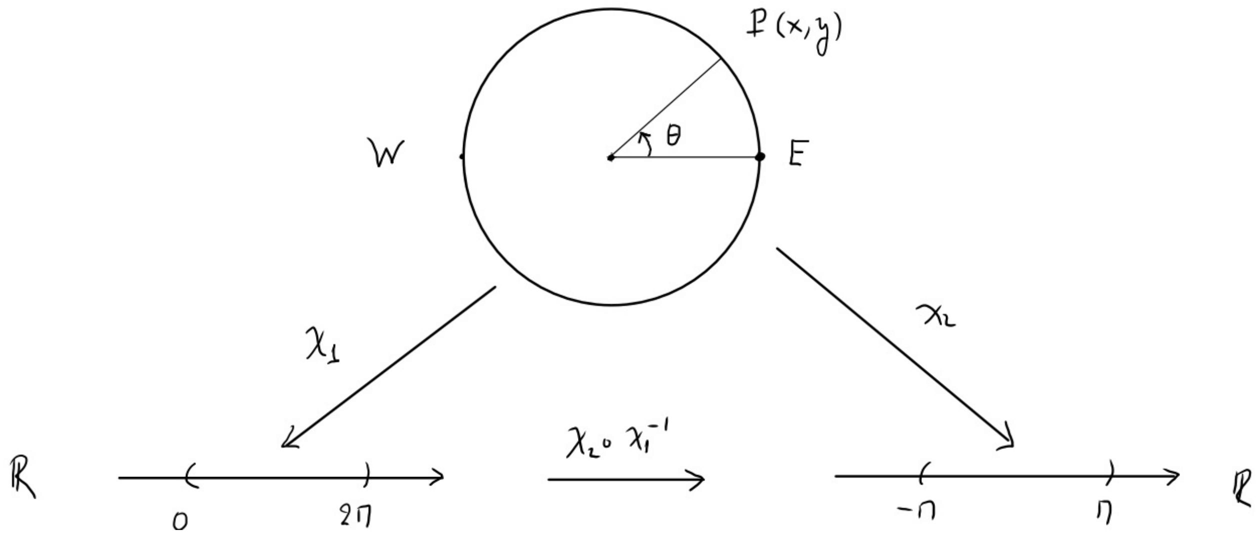
$\chi_1(P) = \theta \qquad 0 < \theta < 2\pi$



Examples: S^1



(U^1, χ_1):

$U^1 = S^1 \setminus \{E\}$

$\chi_1(P) = \theta \qquad 0 < \theta < 2\pi$

$\chi_1 : \underbrace{(x, y)}_{P} \mapsto \theta$

(x,y) is a label for P, not manifold coordinates!

Examples: S^1

(U^1, x_1):

$U^1 = S^1 \setminus \{E\}$

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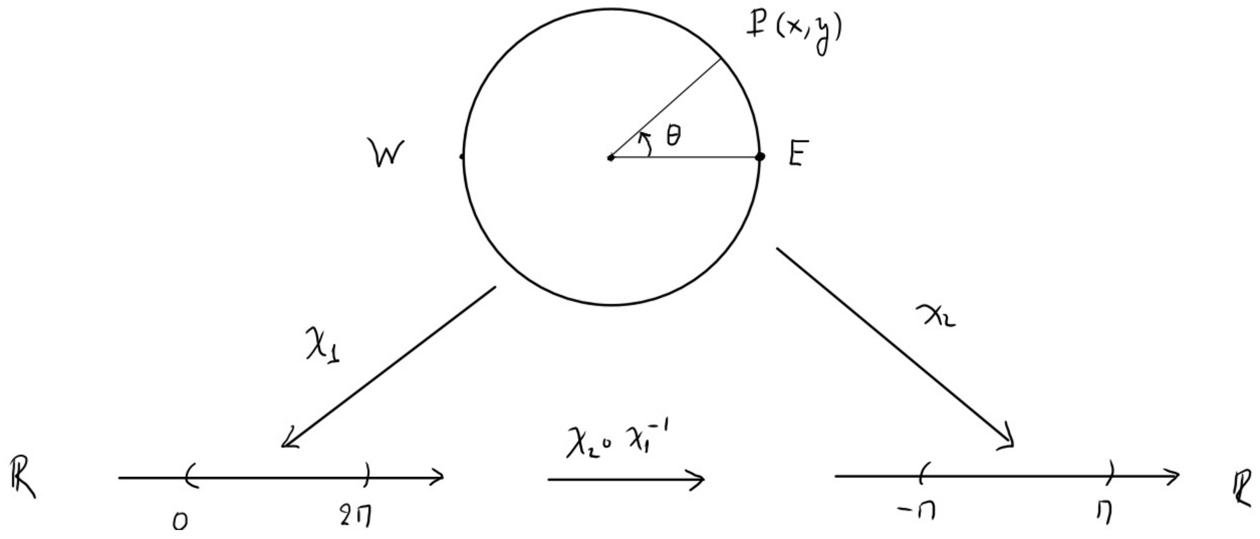
$x_1 : (x, y) \mapsto \theta$

(U^2, x_2): $U^2 = S^1 \setminus \{W\}$

$x_2(P) = \theta \qquad -\pi < \theta < \pi$



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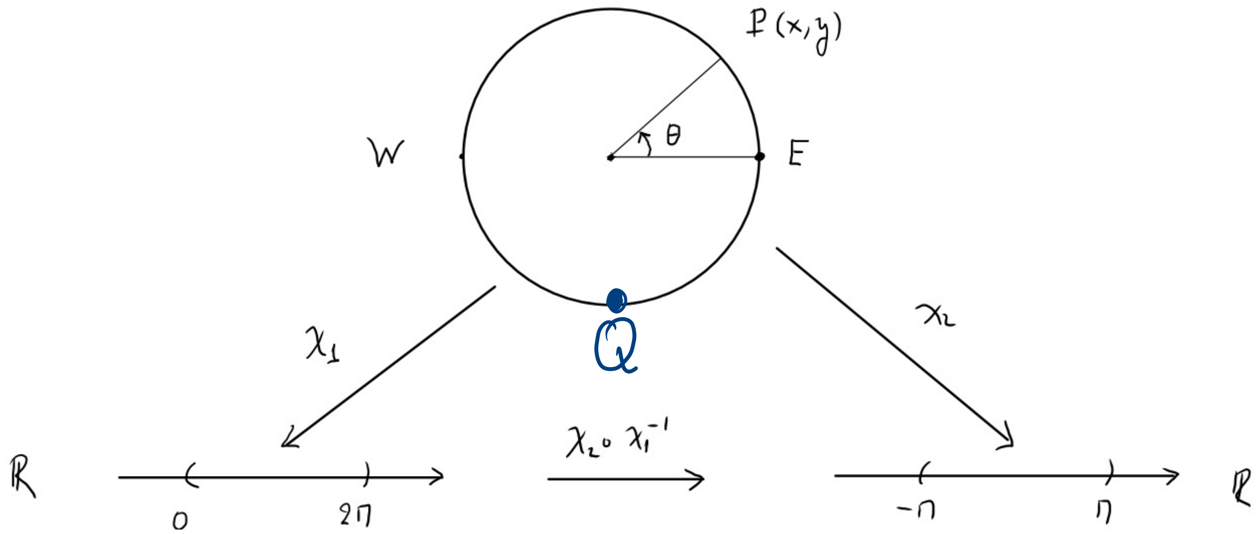

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Transition function:

$$\chi_2 \circ \chi_1^{-1}(\theta) = \begin{cases} \theta & 0 < \theta < \pi \\ \theta - 2\pi & \pi < \theta < 2\pi \end{cases}$$

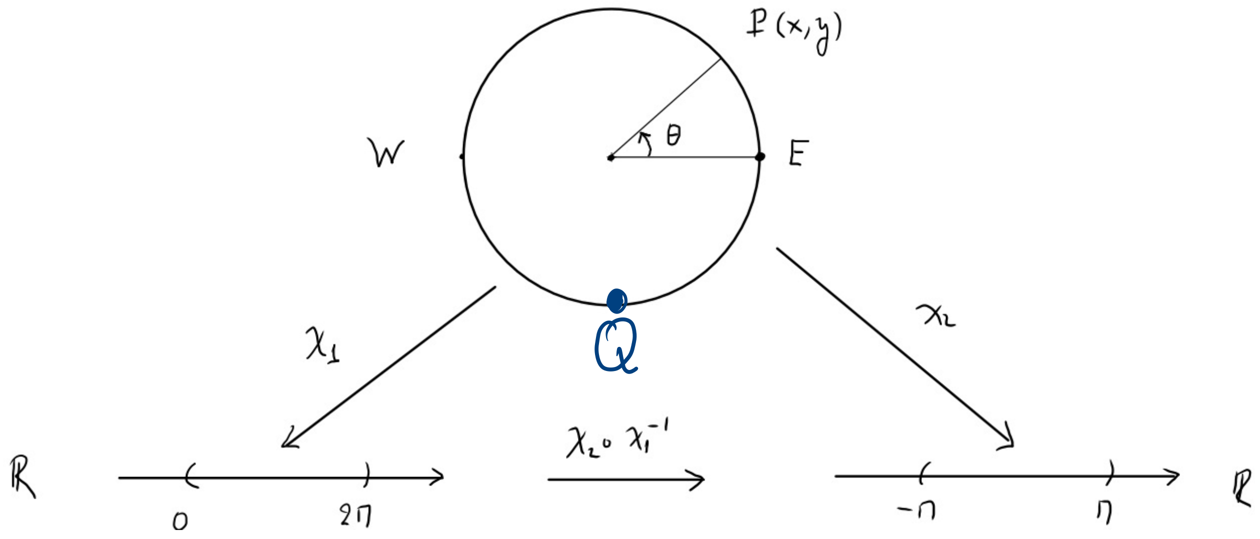
e.g.

$\chi_1(Q) = \frac{3\pi}{2}$

$\frac{3\pi}{2} \to -\frac{\pi}{2} = \frac{3\pi}{2} - 2\pi$

$\chi_2(Q) = -\frac{\pi}{2}$

Examples: S^1



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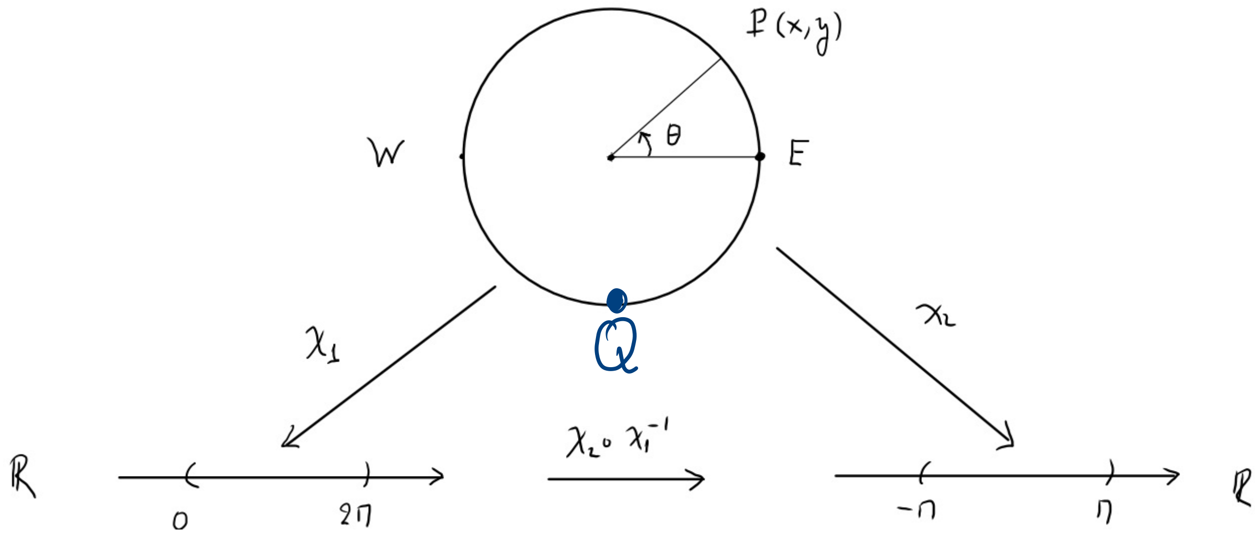
Differentiable!

Examples: S^1



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$S^1 = U^1 \cup U^2$, so

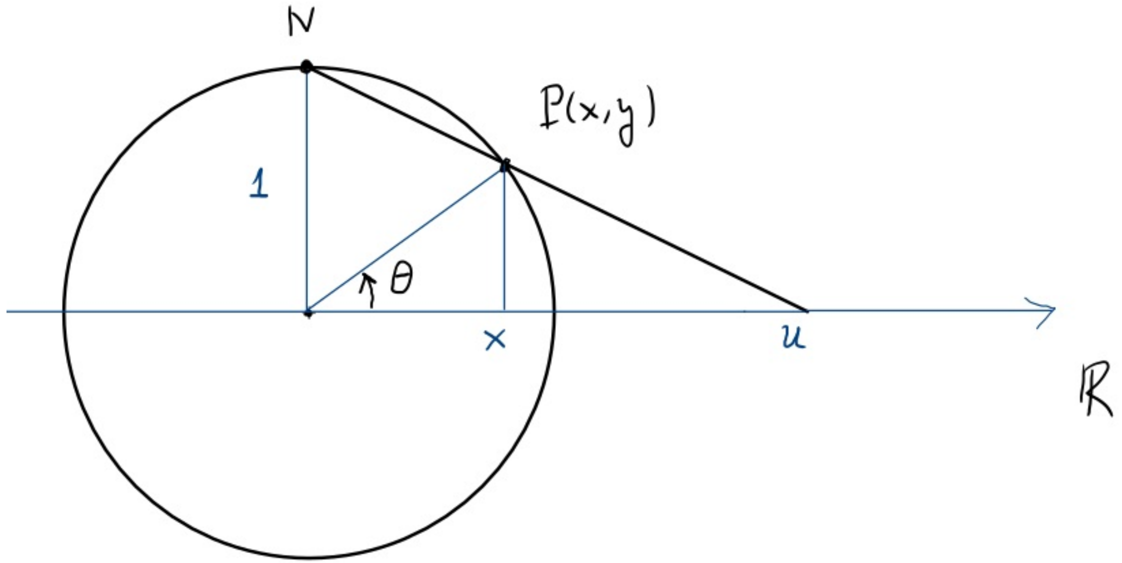
$\{(U^1, \chi_1), (U^2, \chi_2)\}$ an atlas of S^1

Examples: S^1

(U^3, χ_3) : $U^3 = S^1 \setminus \{N\}$

$\chi_3 : (x,y) \mapsto u = \dfrac{x}{1-y}$ $\quad -\infty < u < +\infty$

another chart!



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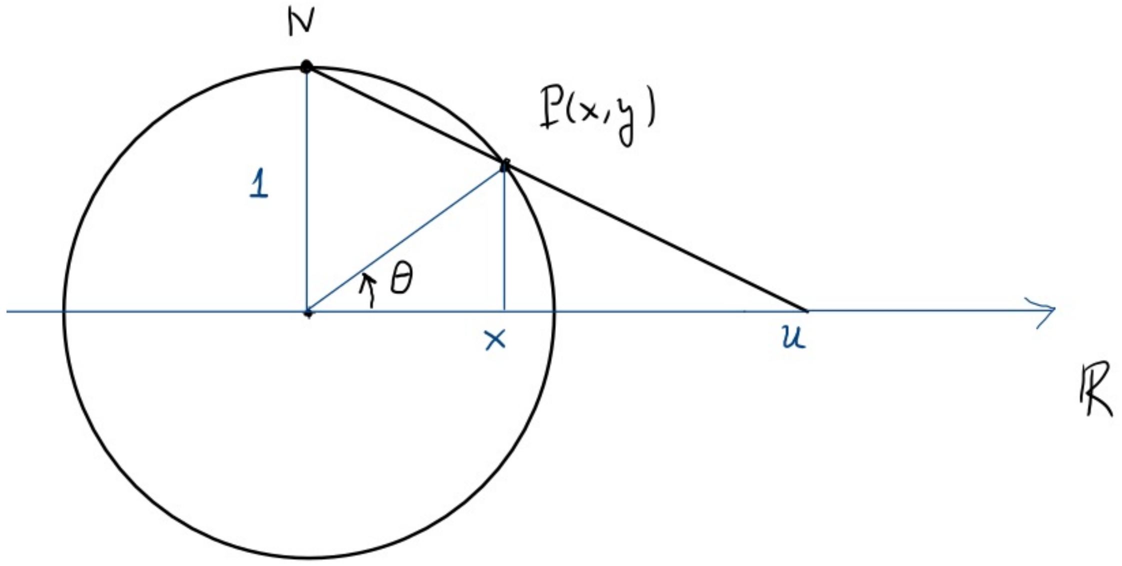

(U^3, χ_3): $U^3 = S^1 \setminus \{N\}$

$\chi_3 : (x,y) \longmapsto u = \dfrac{x}{1-y}$ $\quad -\infty < u < +\infty$

another chart!

From triangle similarity: $\dfrac{u}{1} = \dfrac{u-x}{y} \Rightarrow u = \dfrac{x}{1-y}$

Examples: S^1

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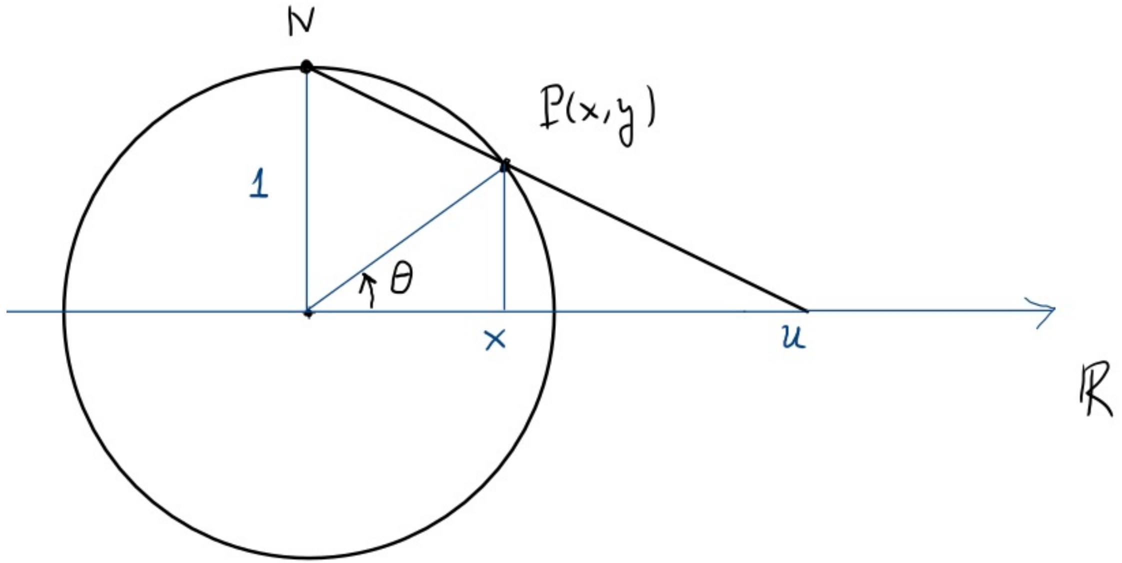
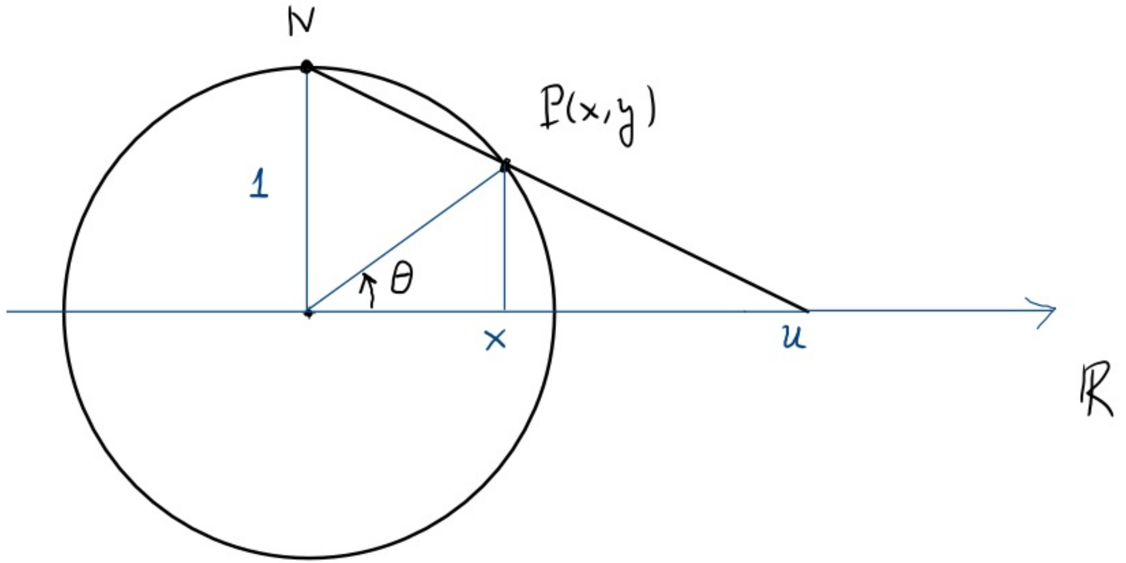


$$\left.\begin{array}{l} x = \cos\theta \\ y = \sin\theta \end{array}\right\} \Rightarrow \quad u = \frac{\cos\theta}{1-\sin\theta}$$

Examples: S^1



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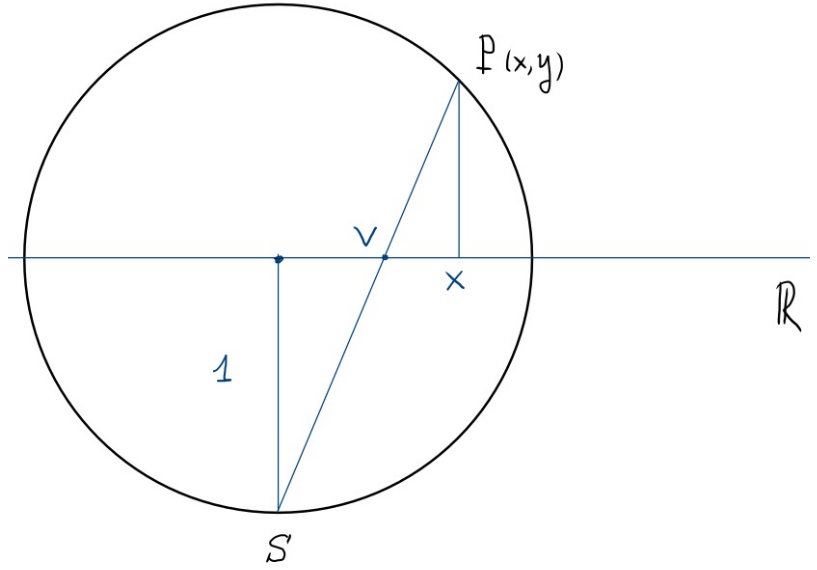
This is $\chi_3 \circ \chi_1^{-1}(\theta)$

Differentiable

Examples: S^1

(U^4, χ_4) : $U^4 = S^1 \setminus \{S\}$



$\chi_4 : (x, y) \mapsto v = \dfrac{x}{1+y}$ $\quad -\infty < v < +\infty$

$$\left.\begin{array}{l} x = \cos\theta \\ y = \sin\theta \end{array}\right\} \Rightarrow v = \frac{\cos\theta}{1+\sin\theta}$$

This is $\chi_4 \circ \chi_1^{-1}(\theta)$

Differentiable

Examples: S^1

(U^1, χ_1): $U_1 = S^1 \setminus \{E\}$

$$\chi_1(x,y) = \theta \quad 0 < \theta < 2\pi$$

(U^2, χ_2): $U^2 = S^1 \setminus \{W\}$

$$\chi_2(x,y) = \theta \quad -\pi < \theta < \pi$$

(U^3, χ_3): $U^3 = S^1 \setminus \{N\}$

$$\chi_3(x,y) = \frac{x}{1-y}$$

(U^4, χ_4): $U^4 = S^1 \setminus \{S\}$

$$\chi_4(x,y) = \frac{x}{1+y}$$

Transition maps:

$$u = \chi_3 \circ \chi_1^{-1}(\theta) = \frac{\cos\theta}{1-\sin\theta}, \quad v = \chi_4 \circ \chi_1^{-1}(\theta) = \frac{\cos\theta}{1+\sin\theta},$$

$$v = \chi_4 \circ \chi_3^{-1}(u) = \frac{1}{u}, \ldots$$

Differentiable

Examples: S^1

(U^1, χ_1): $U_1 = S^1 \setminus \{E\}$

$\chi_1(x,y) = \theta \quad 0 < \theta < 2\pi$

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$\chi_3(x,y) = \dfrac{x}{1-y}$

(U^4, χ_4): $U^4 = S^1 \setminus \{S\}$

$\chi_4(x,y) = \dfrac{x}{1+y}$

Transition maps:

$$\left.\begin{array}{l} u \cdot v = \dfrac{x}{1-y} \, \dfrac{x}{1+y} = \dfrac{x^2}{1-y^2} \\ x^2 + y^2 = 1 \Rightarrow 1 - y^2 = x^2 \end{array}\right\} \Rightarrow u \cdot v = \frac{x^2}{x^2} = 1$$

$$\Rightarrow v = \frac{1}{u}$$

$$v = \chi_4 \circ \chi_3^{-1}(u) = \frac{1}{u}$$

Examples: S^1

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---

atlases:

$A_1 = \{(U^1, \chi_1), (U^2, \chi_2)\}$

$A_2 = \{(U^3, \chi_3), (U^4, \chi_4)\}$

Examples: S^1

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---

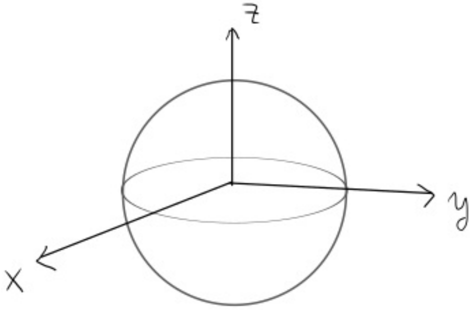
atlases: $A_1 = \{(U^1, x_1), (U^2, x_2)\}$

$A_2 = \{(U^3, x_3), (U^4, x_4)\}$

$\nexists$ atlas with only one chart!

$\mathbb{R} \not\cong S^1$

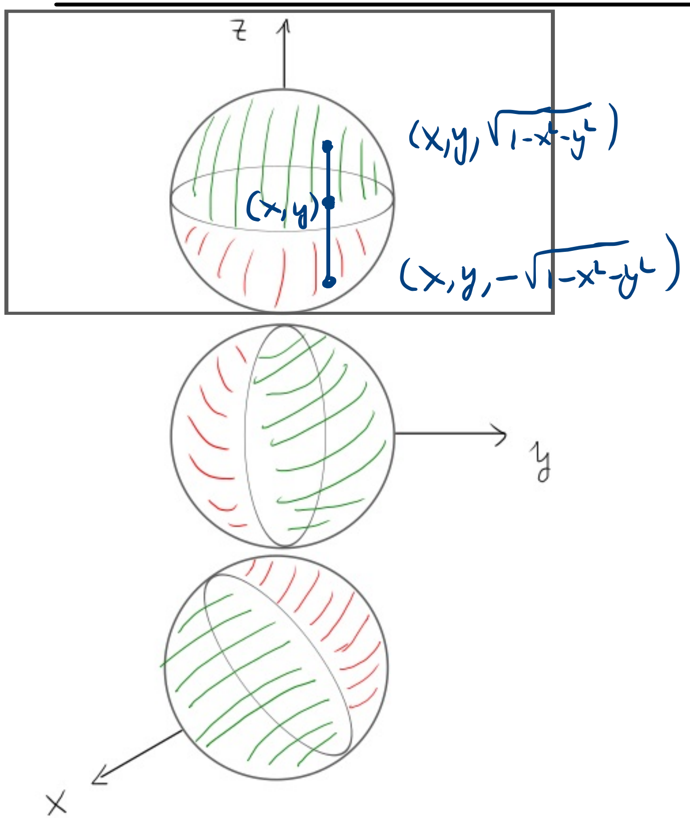
S^2: The sphere



Define S^2 by (x, y, z) s.t.

$$x^2 + y^2 + z^2 = 1$$

S^2: The sphere Define 6 charts-hemispheres



$$U_{z_+} = \{(x,y,z) \in S^2 \mid z>0\}$$

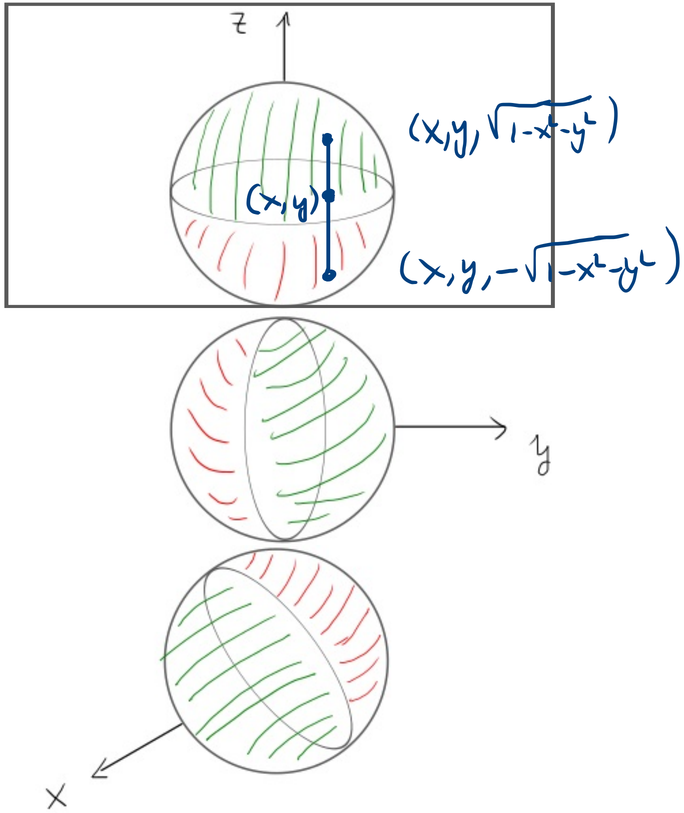
$$\chi_{z_+}: (x,y,+\sqrt{1-x^2-y^2}) \mapsto (x,y)$$

$$U_{z_-} = \{(x,y,z) \in S^2 \mid z<0\}$$

$$\chi_{z_-}: (x,y,-\sqrt{1-x^2-y^2}) \mapsto (x,y)$$

project on the same point on plane

S^2: The sphere Define 6 charts-hemispheres

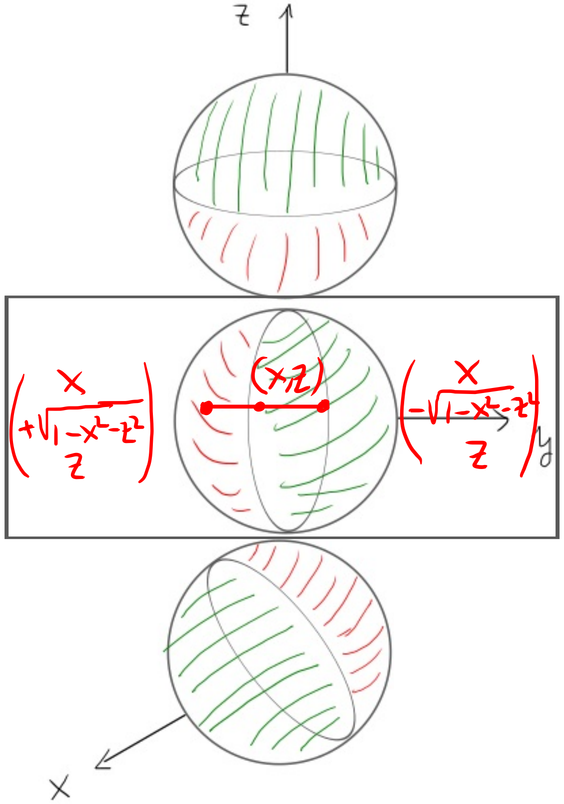


$$\chi_{z_+}: (x,y,+\sqrt{1-x^2-y^2}) \mapsto (x,y)$$

$$\chi_{z_-}: (x,y,-\sqrt{1-x^2-y^2}) \mapsto (x,y)$$

projection on (x,y) plane

S^2: The sphere Define 6 charts-hemispheres



$$\chi_{z_+}: (x, y, +\sqrt{1-x^2-y^2}) \mapsto (x, y)$$

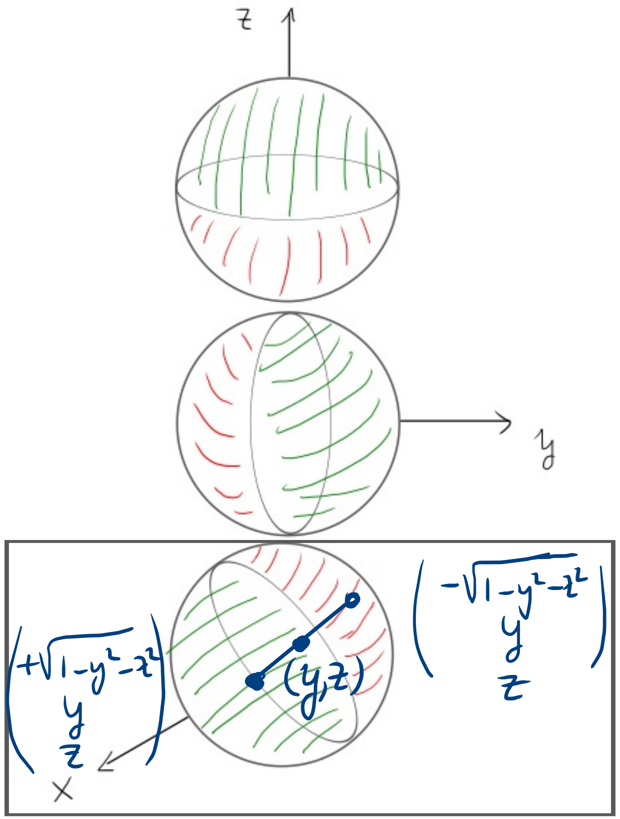
$$\chi_{z_-}: (x, y, -\sqrt{1-x^2-y^2}) \mapsto (x, y)$$

$$\chi_{y_+}: (x, +\sqrt{1-x^2-z^2}, z) \mapsto (x, z)$$

$$\chi_{y_-}: (x, -\sqrt{1-x^2-z^2}, z) \mapsto (x, z)$$

Projection on (x,z) plane

S^2: The sphere Define 6 charts–hemispheres



$$\chi_{z_+} : (x, y, +\sqrt{1-x^2-y^2}) \mapsto (x, y)$$

$$\chi_{z_-} : (x, y, -\sqrt{1-x^2-y^2}) \mapsto (x, y)$$

$$\chi_{y_+} : (x, +\sqrt{1-x^2-z^2}, z) \mapsto (x, z)$$

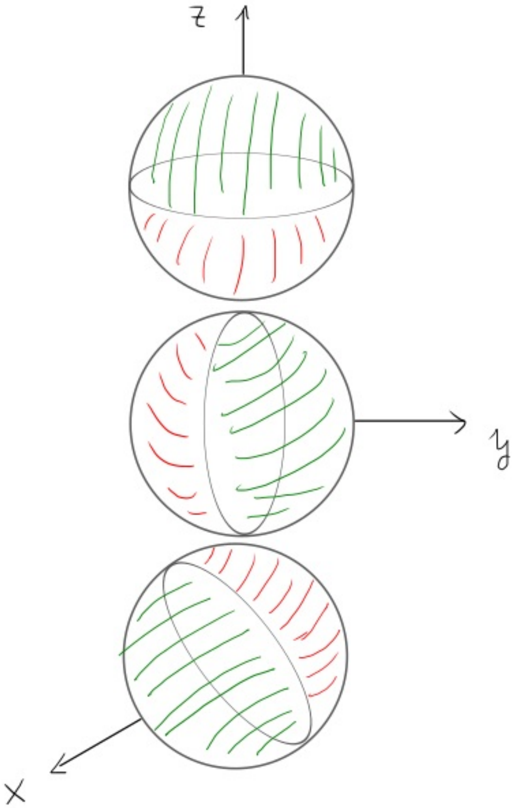
$$\chi_{y_-} : (x, -\sqrt{1-x^2-z^2}, z) \mapsto (x, z)$$

$$\chi_{x_+} : (+\sqrt{1-y^2-z^2}, y, z) \mapsto (y, z)$$

$$\chi_{x_-} : (-\sqrt{1-y^2-z^2}, y, z) \mapsto (y, z)$$

projection on (y,z) plane

S^2: The sphere Define 6 charts-hemispheres



$$\chi_{z_+}: (x, y, +\sqrt{1-x^2-y^2}) \longmapsto (x, y)$$

$$\chi_{z_-}: (x, y, -\sqrt{1-x^2-y^2}) \longmapsto (x, y)$$

$$\chi_{y_+}: (x, +\sqrt{1-x^2-z^2}, z) \longmapsto (x, z)$$

$$\chi_{y_-}: (x, -\sqrt{1-x^2-z^2}, z) \longmapsto (x, z)$$

$$\chi_{x_+}: (+\sqrt{1-y^2-z^2}, y, z) \longmapsto (y, z)$$

$$\chi_{x_-}: (-\sqrt{1-y^2-z^2}, y, z) \longmapsto (y, z)$$

A transition map:

$$\chi_{y^-} \circ \chi_{x^+}^{-1}:$$

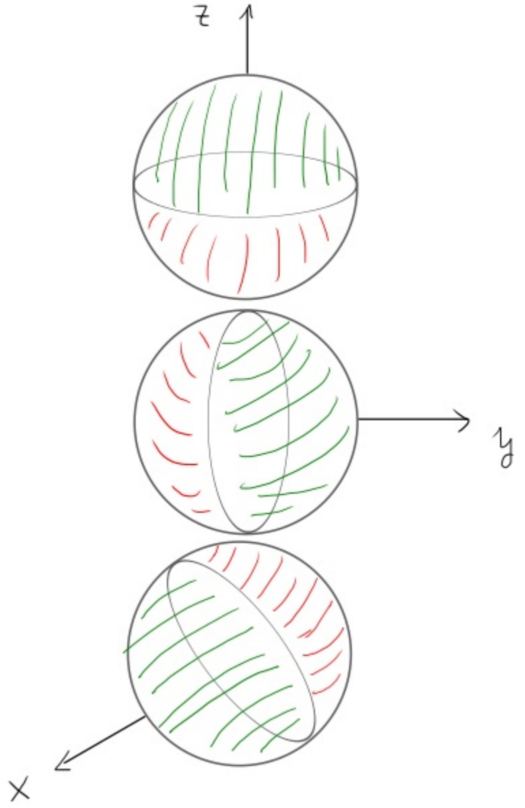
$$(y, z) \longmapsto (x, z)$$

with

$$x = \sqrt{1-y^2-z^2}$$

$$z = z$$

S^2: The sphere Define 6 charts-hemispheres



$\chi_{z_+}: (x, y, +\sqrt{1-x^2-y^2}) \mapsto (x, y)$

$\chi_{z_-}: (x, y, -\sqrt{1-x^2-y^2}) \mapsto (x, y)$

$\chi_{y_+}: (x, +\sqrt{1-x^2-z^2}, z) \mapsto (x, z)$

$\chi_{y_-}: (x, -\sqrt{1-x^2-z^2}, z) \mapsto (x, z)$

$\chi_{x_+}: (+\sqrt{1-y^2-z^2}, y, z) \mapsto (y, z)$

$\chi_{x_-}: (-\sqrt{1-y^2-z^2}, y, z) \mapsto (y, z)$

A transition map:

$\chi_{y^-} \circ \chi_{x^+}^{-1}$:

$(y, z) \mapsto (x, z)$

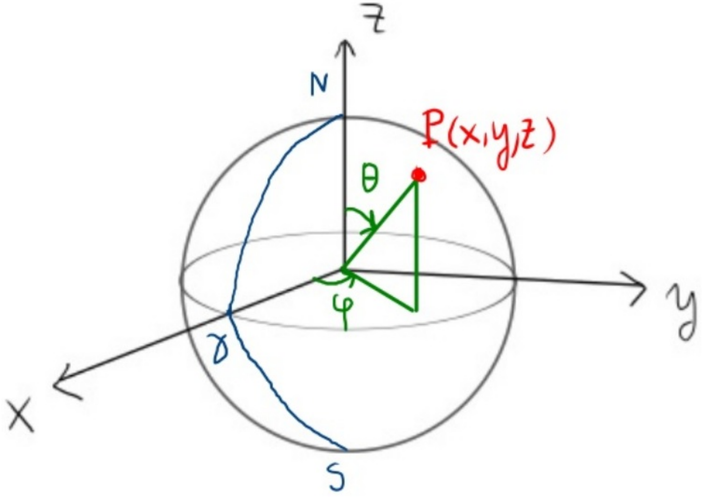
with

$x = \sqrt{1-y^2-z^2}$

$z = z$ Differentiable!

We need all 6 charts to cover S^2: we can do better!

S^2: The sphere



Another chart!

(U_θ, χ_θ): $U_\theta = S^2 \setminus \{\gamma\}$

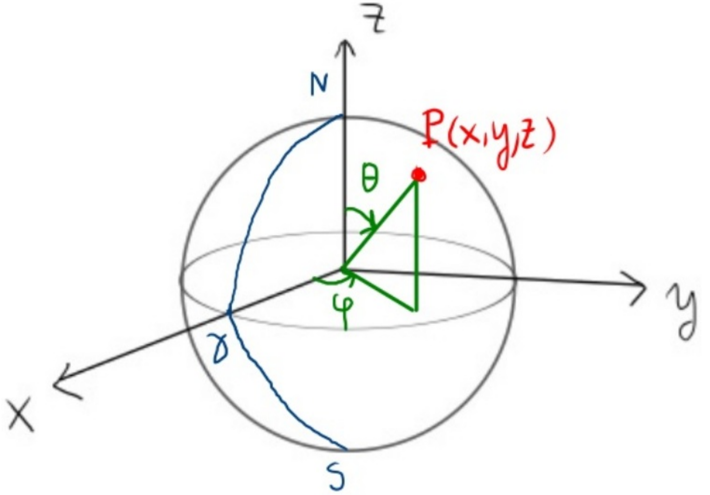
$\chi_\theta : (x,y,z) \longmapsto (\theta, \varphi)$ $\quad 0<\theta<\pi$, $\quad 0<\varphi<2\pi$

$x = \sin\theta \cos\varphi$

$y = \sin\theta \sin\varphi$

$z = \cos\theta$

S^2: The sphere



(U_θ, χ_θ): $U_\theta = S^2 \setminus \{\gamma\}$

$\chi_\theta : (x, y, z) \mapsto (\theta, \varphi)$ $\quad 0<\theta<\pi$, $0<\varphi<2\pi$

$x = \sin\theta \cos\varphi$

$y = \sin\theta \sin\varphi$

$z = \cos\theta$

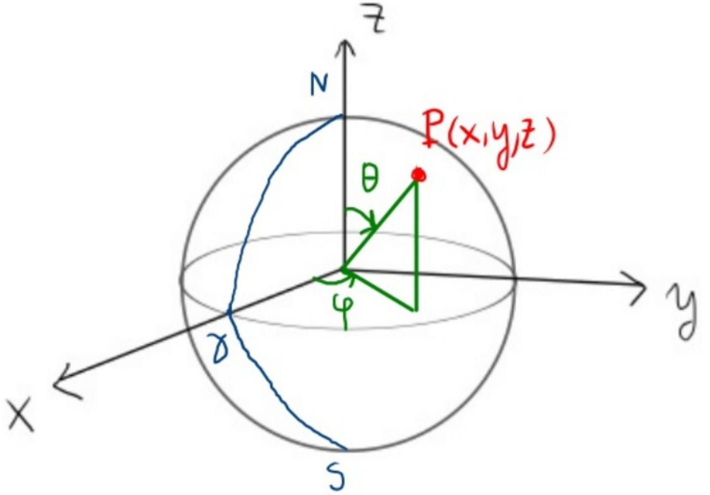
Transition functions

$\chi_{y+} \circ \chi_\theta^{-1} : (\theta, \varphi) \mapsto (x, z)$

$x = \sin\theta \cos\varphi \qquad 0<\theta<\pi$

$z = \cos\theta \qquad 0<\varphi<\pi$

S^2: The sphere



(U_θ, χ_θ): $U_\theta = S^2 \setminus \{\gamma\}$

$\chi_\theta : (x,y,z) \mapsto (\theta, \varphi)$ $\quad 0<\theta<\pi$, $0<\varphi<2\pi$

$x = \sin\theta\cos\varphi$

$y = \sin\theta\sin\varphi$

$z = \cos\theta$

Transition functions

$\chi_{y+} \circ \chi_\theta^{-1} : (\theta, \varphi) \mapsto (x, z)$

$x = \sin\theta\cos\varphi \qquad 0<\theta<\pi$

$z = \cos\theta \qquad 0<\varphi<\pi$

$\chi_\theta \circ \chi_{y+}^{-1} : (x, z) \mapsto (\theta, \varphi)$

$\theta = \tan^{-1}\sqrt{\frac{1}{z^2} - 1} \qquad \tan\theta = \frac{\sqrt{x^2+y^2}}{z}$

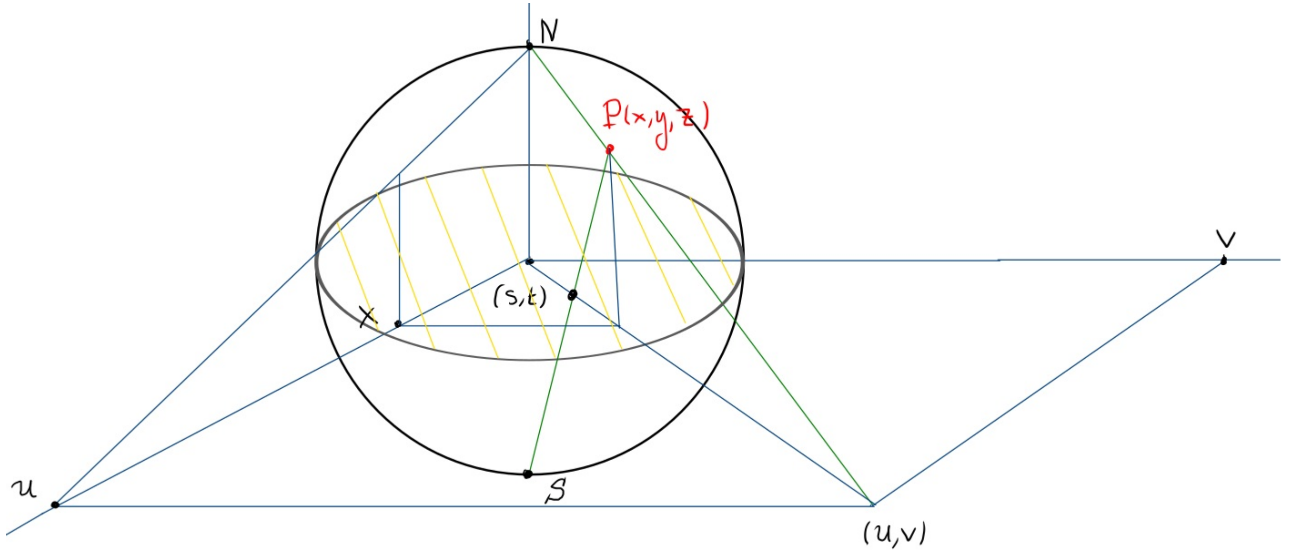
$\varphi = \tan^{-1}\frac{\sqrt{1-x^2-z^2}}{x} \qquad \tan\varphi = \frac{y}{x}$

S^2: The sphere

(U_N, χ_N): $U_N = S^2 \setminus \{N\}$

$\chi_N : (x, y, z) \mapsto (u, v)$ $\quad -\infty < u < +\infty$, $\; -\infty < v < +\infty$

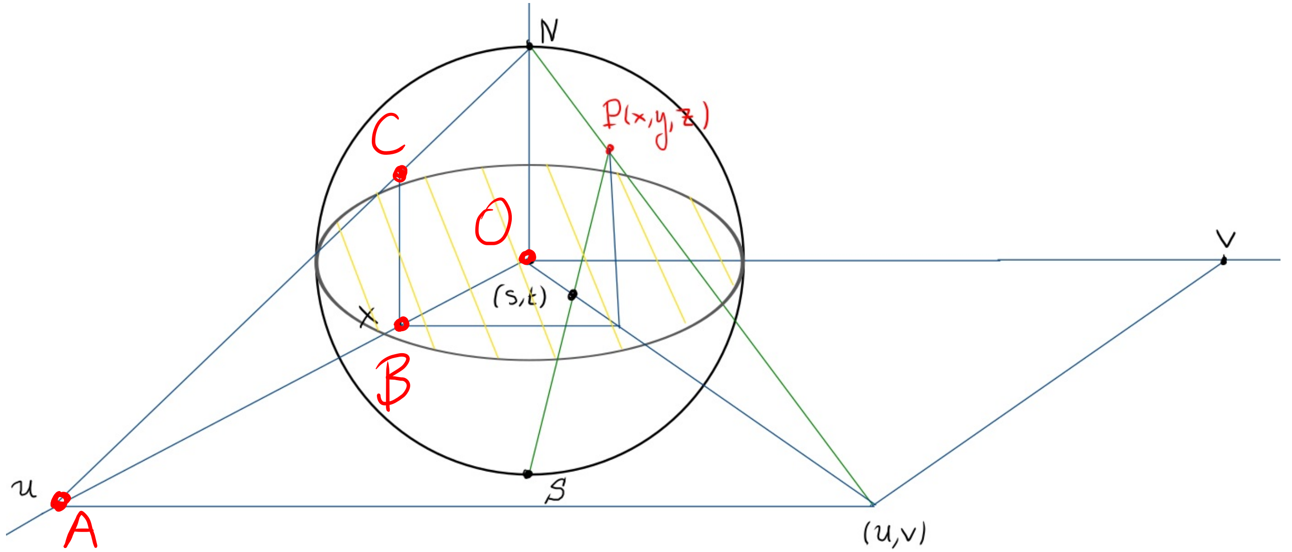
$$u = \frac{x}{1-z} \qquad v = \frac{y}{1-z}$$



S^2: The sphere

(U_N, χ_N): $U_N = S^2 \setminus \{N\}$

$\chi_N : (x, y, z) \mapsto (u, v)$ $\quad -\infty < u < +\infty$, $\; -\infty < v < +\infty$



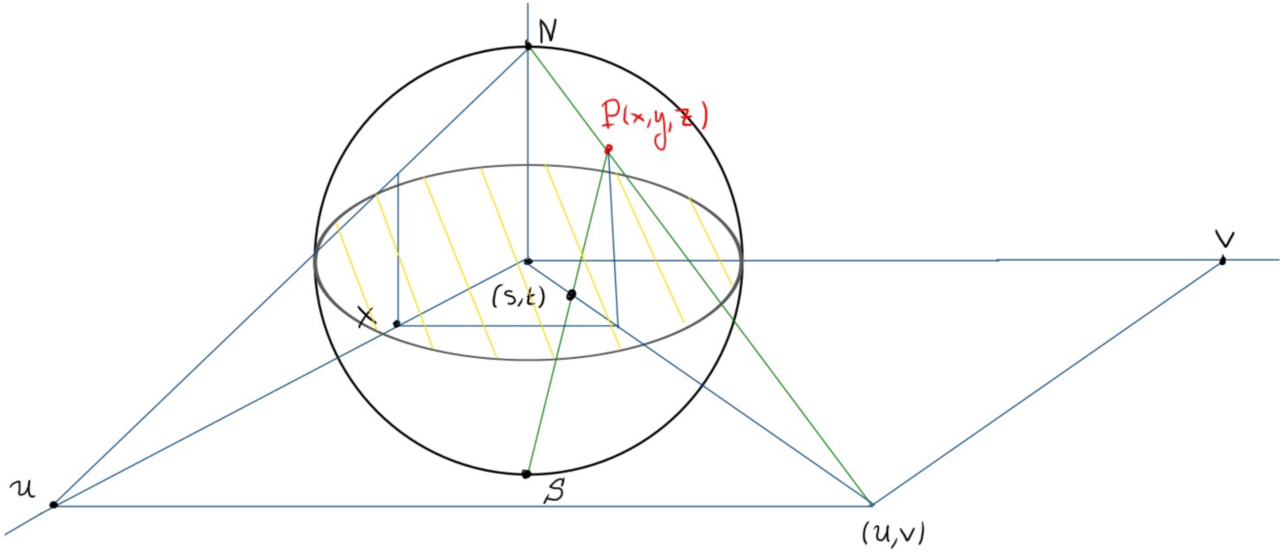
$$u = \frac{x}{1-z} \qquad v = \frac{y}{1-z}$$

$$\frac{OA}{OB} = \frac{ON}{CB} \Rightarrow \frac{u}{u-x} = \frac{1}{z} \Rightarrow u = \frac{x}{1-z}$$

S^2: The sphere

$(U_N, \chi_N) : U_N = S^2 \setminus \{N\}$

$\chi_N : (x,y,z) \mapsto (u,v)$ $\quad -\infty < u < +\infty$, $-\infty < v < +\infty$



$$u = \frac{x}{1-z} \qquad v = \frac{y}{1-z}$$

$$\chi_N \circ \chi_\theta^{-1} : (\theta, \varphi) \mapsto (u,v)$$

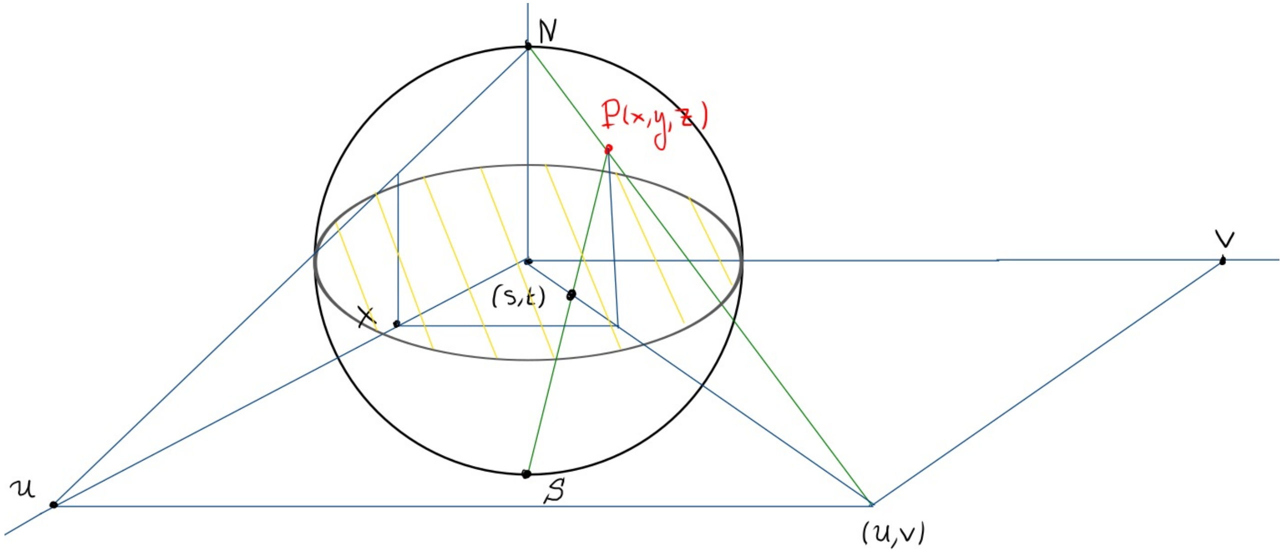
$$u = \frac{\sin\theta \cos\varphi}{1-\cos\theta}$$

$$v = \frac{\sin\theta \sin\varphi}{1-\cos\theta}$$

S^2: The sphere

$(U_S, X_S): U = S^2 \setminus \{S\}$

$X: (x,y,z) \mapsto (s,t)$ $\quad -\infty < s < +\infty$, $\; -\infty < t < +\infty$



$$s = \frac{x}{1+z} \qquad t = \frac{y}{1+z}$$

$$X_S \circ X_\theta^{-1} : (\theta, \varphi) \mapsto (s,t)$$

$$s = \frac{\sin\theta \cos\varphi}{1+\cos\theta}$$

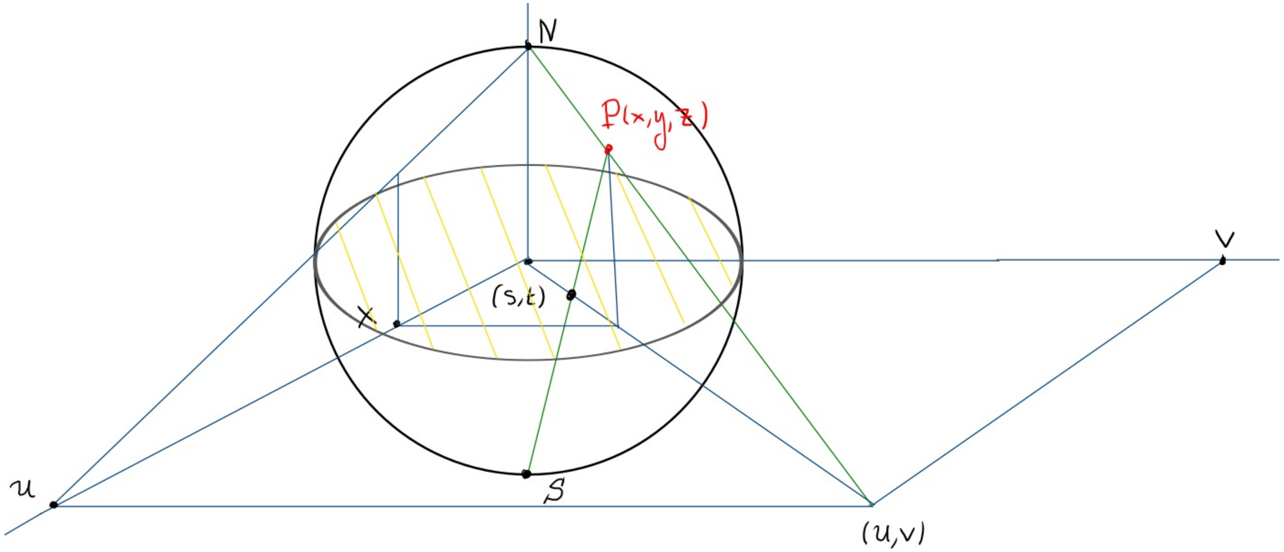
$$t = \frac{\sin\theta \sin\varphi}{1+\cos\theta}$$

S^2: The sphere

$(U_S, \chi_S): U = S^2 \setminus \{S\}$

$\chi : (x,y,z) \mapsto (s,t)$ $\quad -\infty < s < +\infty$, $-\infty < t < +\infty$

$s = \frac{x}{1+z} \qquad t = \frac{y}{1+z}$

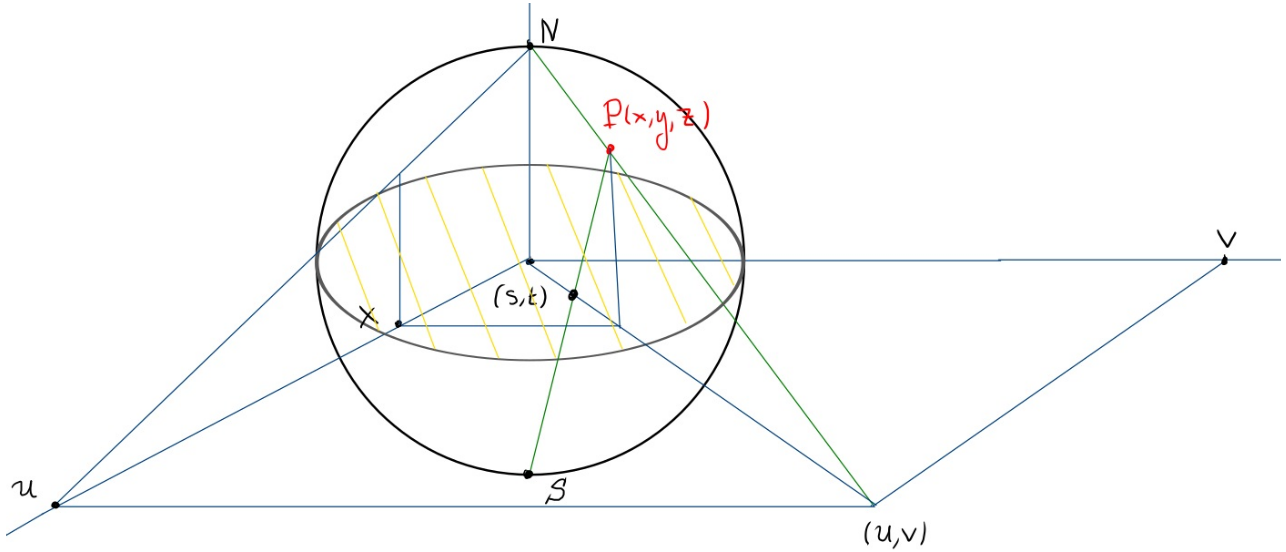


$\chi_S \circ \chi_N^{-1} : (u,v) \mapsto (s,t)$

$s = \frac{u}{u^2+v^2}$

$t = \frac{v}{u^2+v^2}$

$$\left.\begin{aligned} us+vt &= \frac{x^2}{1-z^2}+\frac{y^2}{1-z^2}=\frac{x^2+y^2}{x^2+y^2}=1 \\ ut &= vs = \frac{xy}{1-z^2} \end{aligned}\right\} \Rightarrow$$



$$\left.\begin{aligned} us+tv &= 1 \\ vs-ut &= 0 \end{aligned}\right\} \Rightarrow \quad s=\frac{u}{u^2+v^2} \qquad t=\frac{v}{u^2+v^2}$$

$$\chi_S \circ \chi_N^{-1} : (u,v) \mapsto (s,t)$$

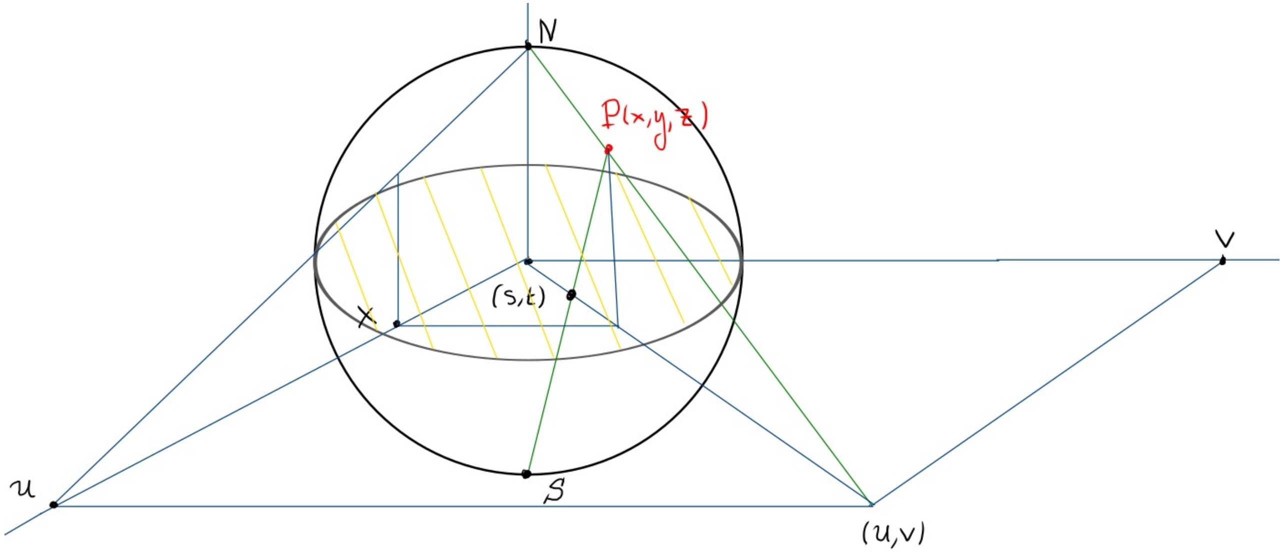
$$s = \frac{u}{u^2+v^2}$$

$$t = \frac{v}{u^2+v^2}$$

S^2: The sphere

$(U_S, \chi_S): U = S^2 \setminus \{S\}$

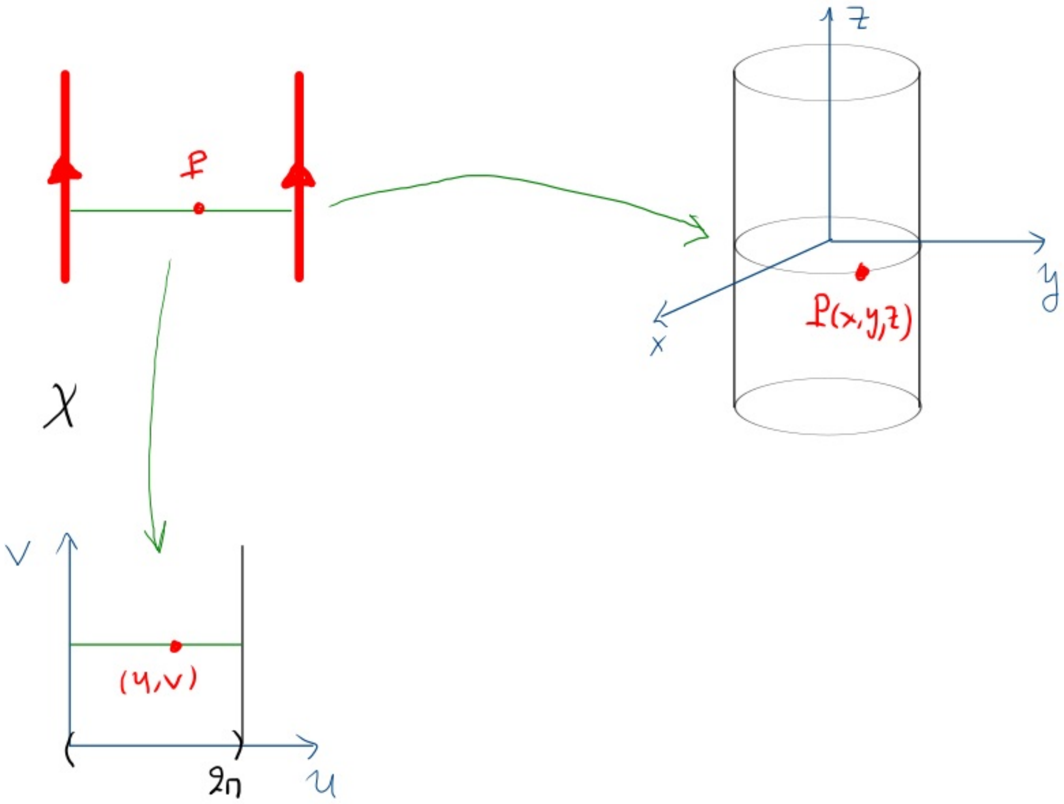
$\chi: (x,y,z) \mapsto (s,t)$ $\quad -\infty < s < +\infty$, $-\infty < t < +\infty$



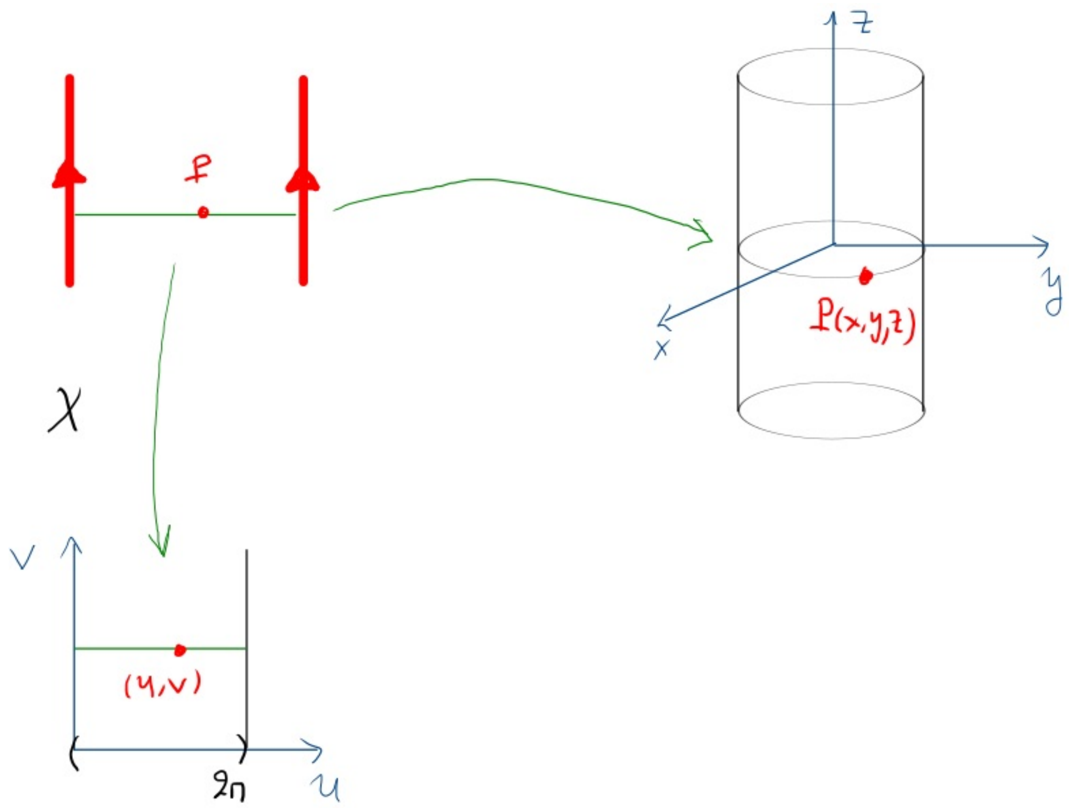
- all transition maps are differentiable
- $\{(U_N, \chi_N), (U_S, \chi_S)\}$ an atlas of S^2

  minimal: no atlas with one chart ($S^2 \not\cong \mathbb{R}^2$)

$S^1 \times \mathbb{R}$: The cylinder



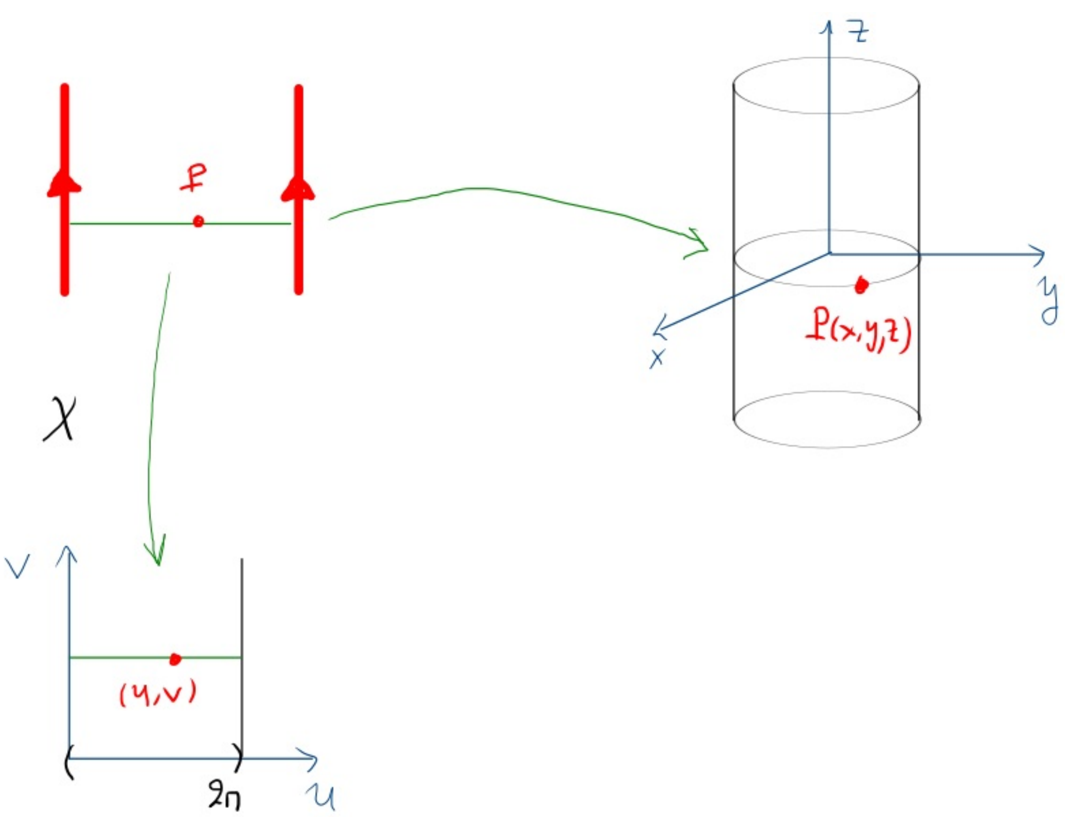
$S^1 \times \mathbb{R}$: The cylinder



Embedding:

$$x = R\cos u \qquad 0 < u < 2\pi$$
$$y = R\sin u \qquad -\infty < v < +\infty$$
$$z = v$$

$S^1 \times \mathbb{R}$: The cylinder



Embedding:

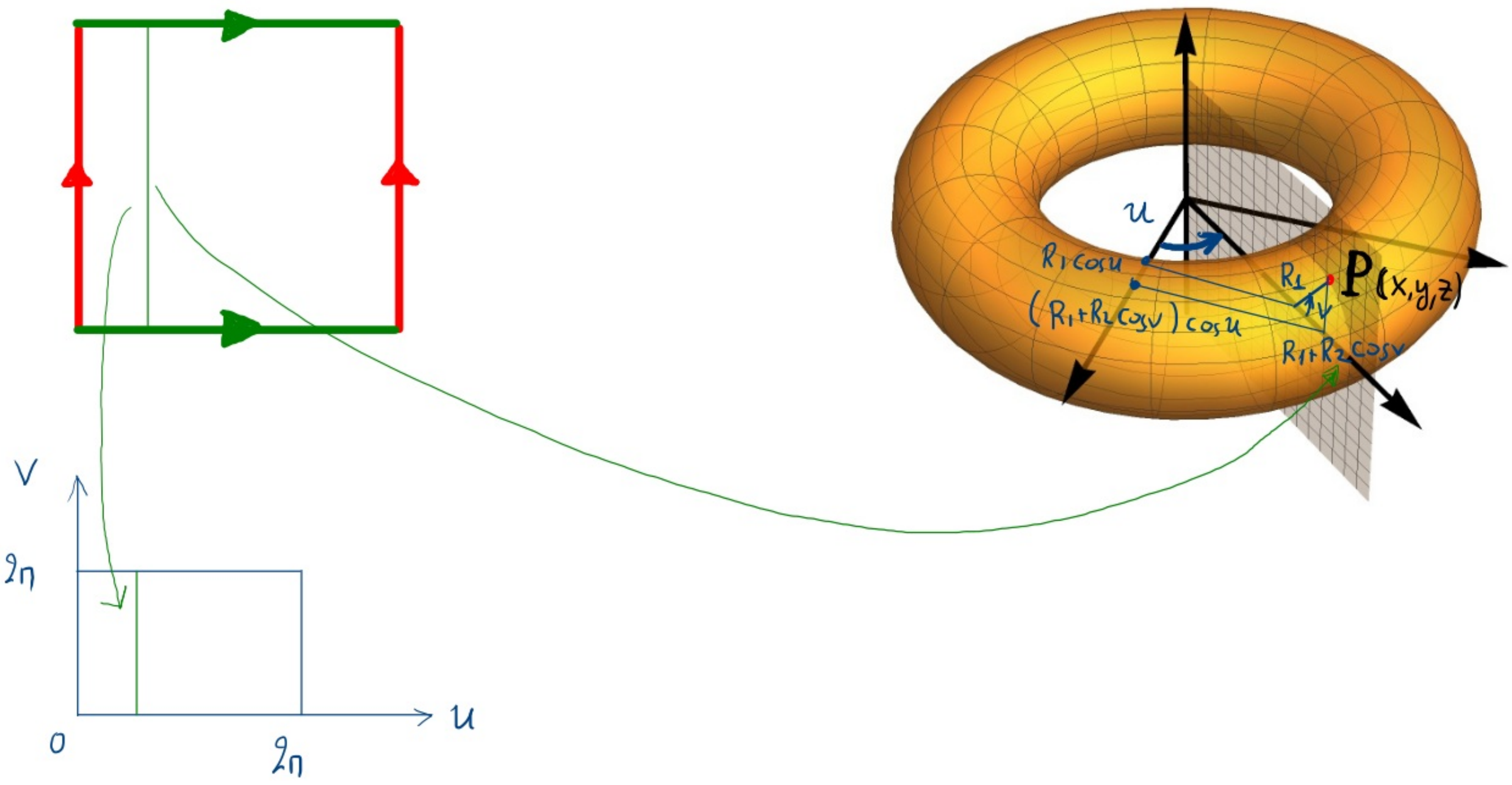
$$x = R\cos u \qquad 0 < u < 2\pi$$
$$y = R\sin u \qquad -\infty < v < +\infty$$
$$z = v$$

Chart:

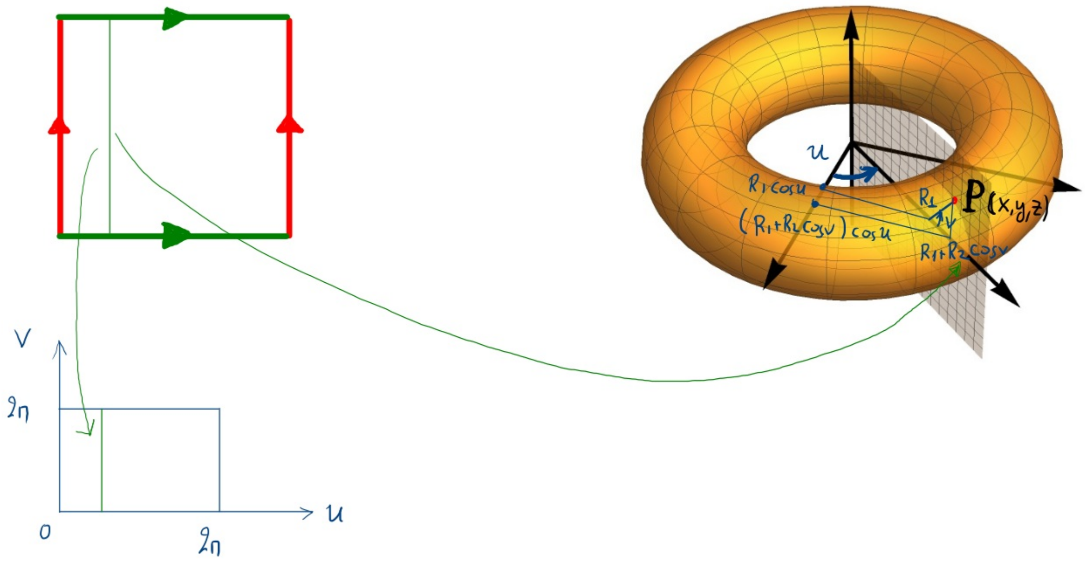
(U, χ) : $U = S^1 \times \mathbb{R} \setminus \{x = R \text{ line}\}$

$\chi : (x, y, z) \longmapsto (u, v)$

$T^2 = S^1 \times S^1$: The torus



$T^2 = S^1 \times S^1$: The torus



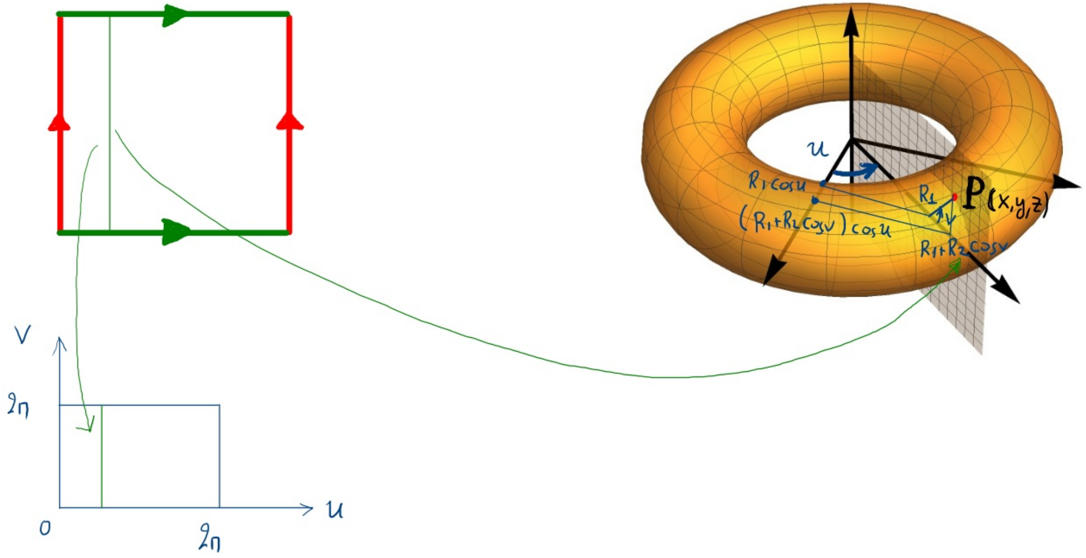
Embedding:

$$x = (R_1 + R_2 \cos v)\cos u \qquad 0 < u < 2\pi$$

$$y = (R_1 + R_2 \cos v)\sin u \qquad 0 < v < 2\pi$$

$$z = R_2 \sin v$$

$T^2 = S^1 \times S^1$: The torus



Embedding:

$$x = (R_1+R_2\cos v)\cos u \qquad 0<u<2\pi$$

$$y = (R_1+R_2\cos v)\sin u \qquad 0<v<2\pi$$

$$z = R_2 \sin v$$

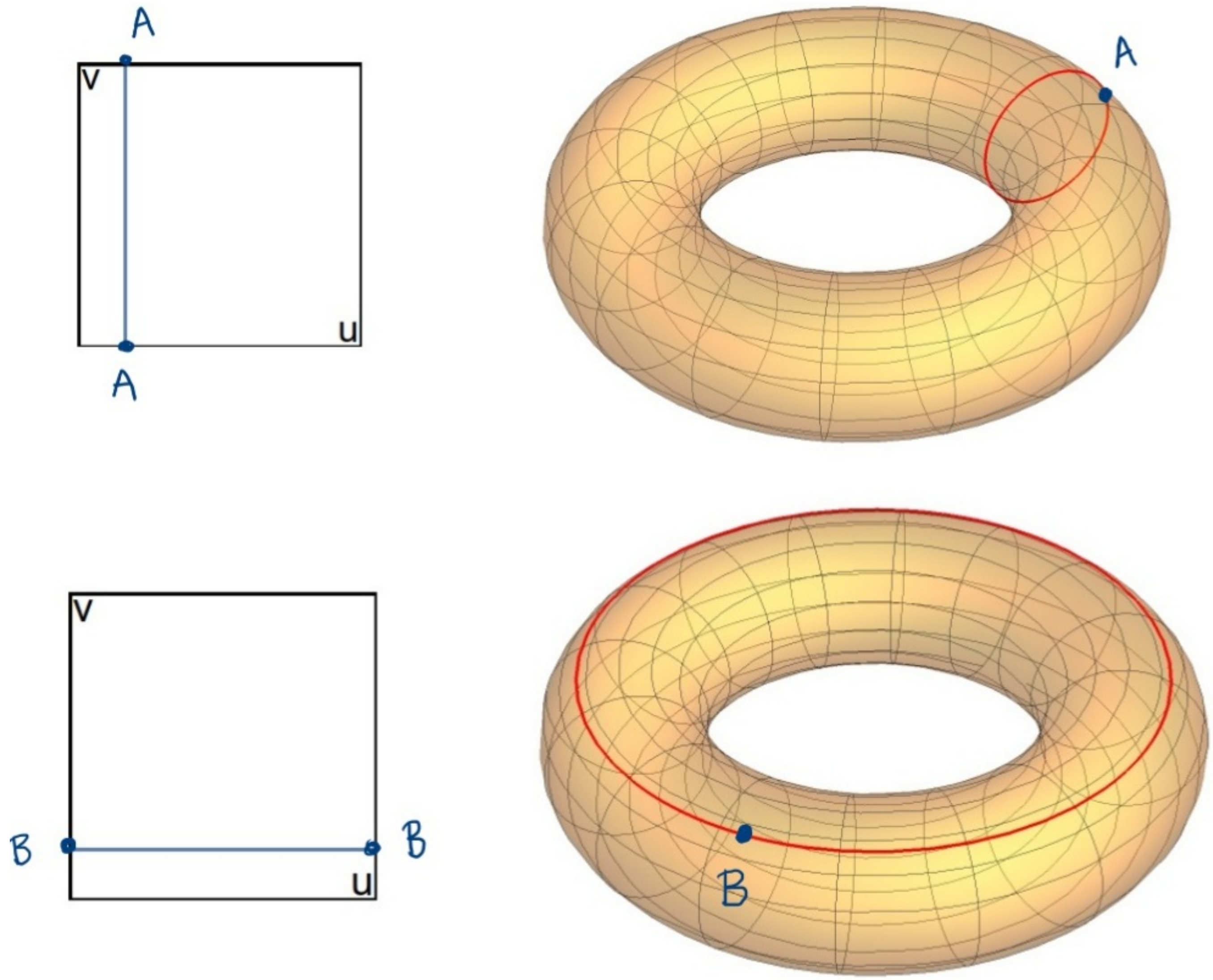
$$X : (x,y,z) \mapsto (u,v)$$

Chart: (U, X)

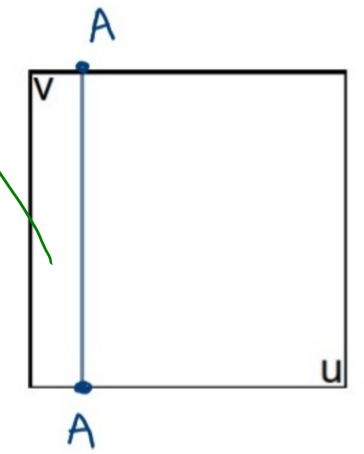
$U = T^2 \setminus (C_1 \cup C_2)$

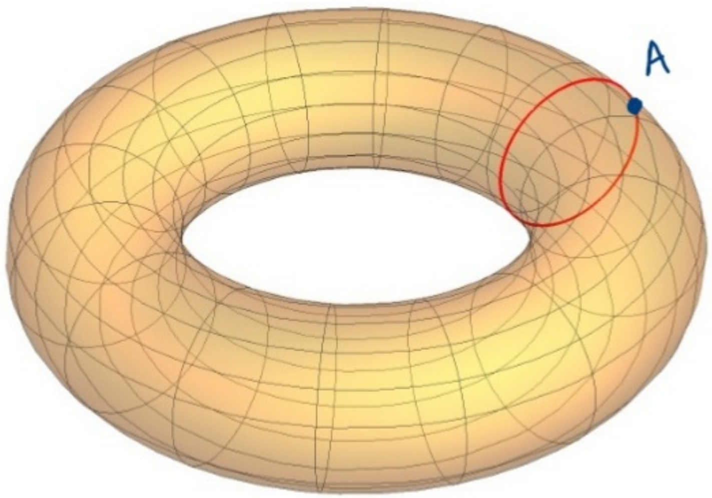
$C_1 = \{$ the red circle $\}$

$C_2 = \{$ the green circle $\}$

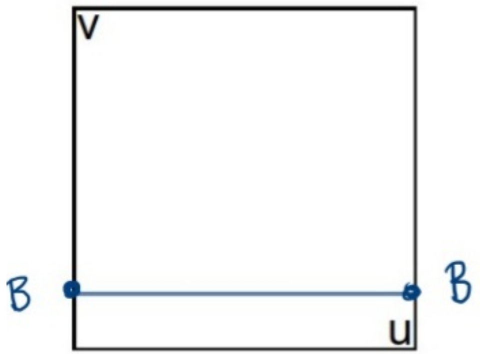

A
v
u
A
A
v
B
B
u
B

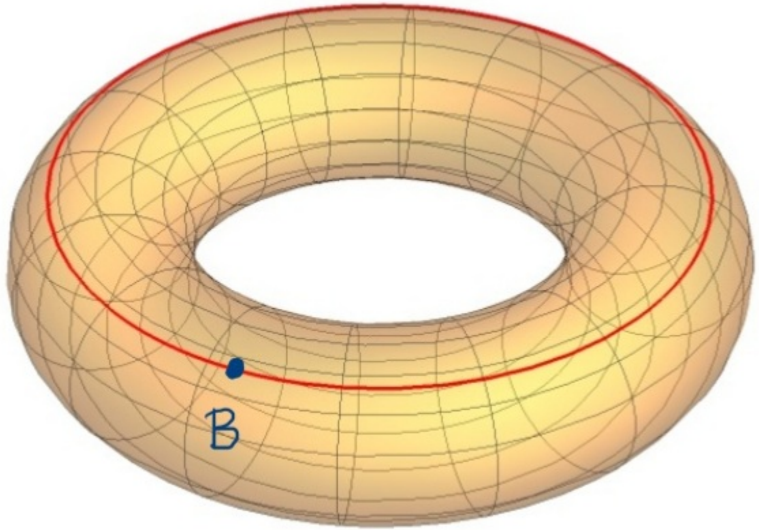
Better description!

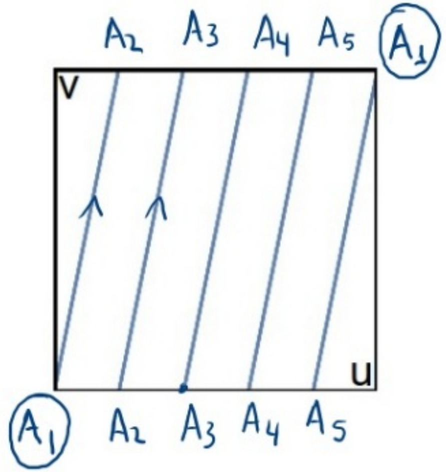
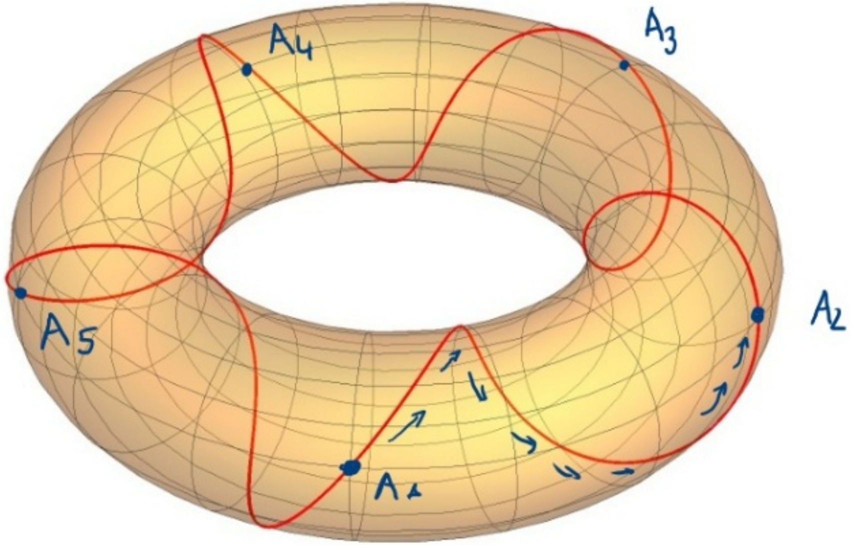
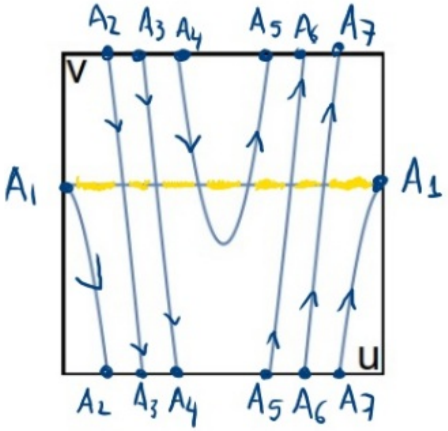
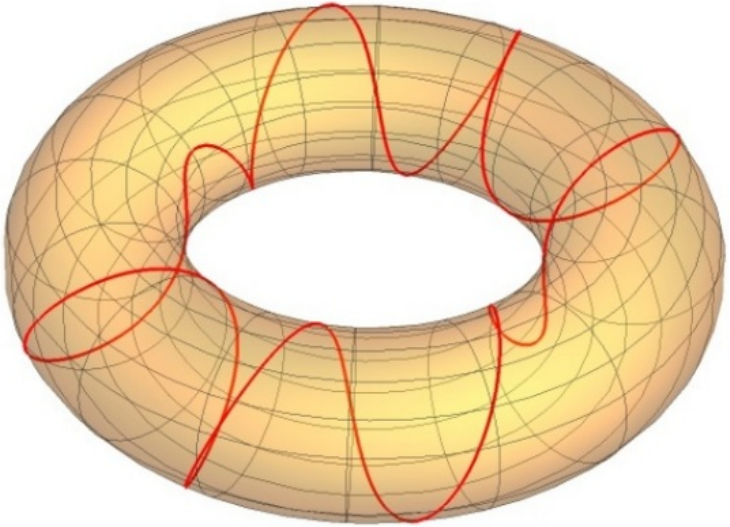




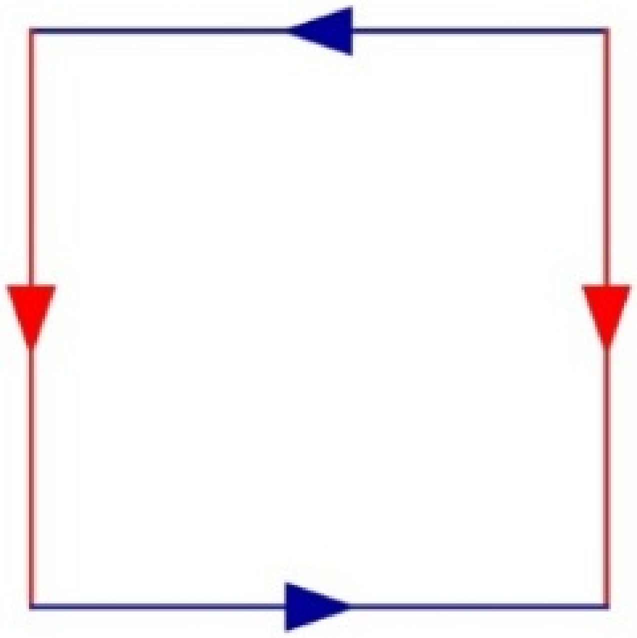
Can't visualize T^3 as embedding in 4d



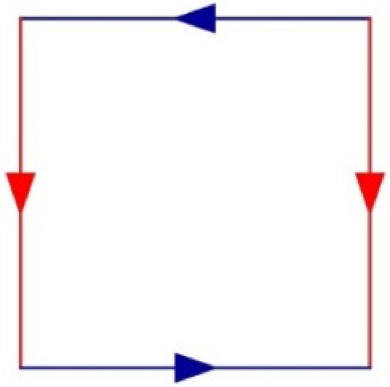



A_2 A_3 A_4 A_5 A_1
v
u
A_1 A_2 A_3 A_4 A_5

A_4
A_3
A_5
A_2
A_1

A_2 A_3 A_4 A_5 A_6 A_7
v
A_1
A_1
u
A_2 A_3 A_4 A_5 A_6 A_7


Klein Bottle



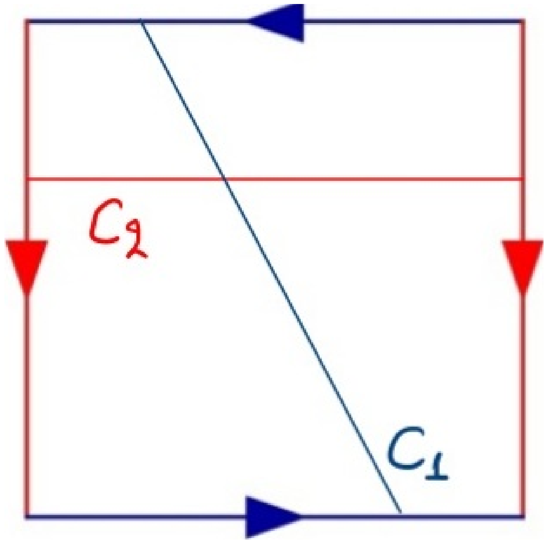
Klein Bottle



- Non-orientable
- Not embeddable in $\mathbb{R}^3$

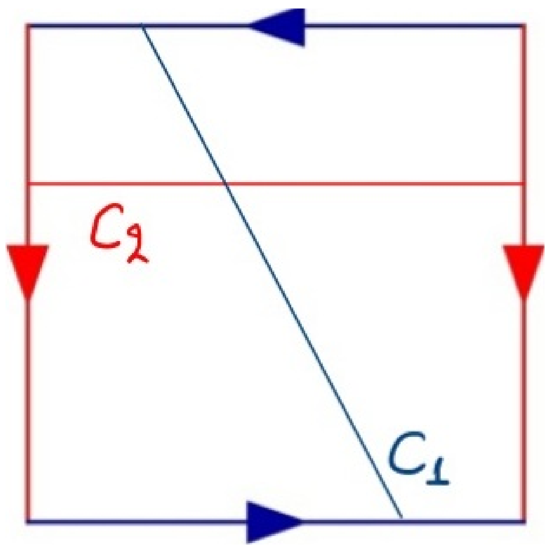
(ok in $\mathbb{R}^4$ due to Whitney's thm)

Klein Bottle

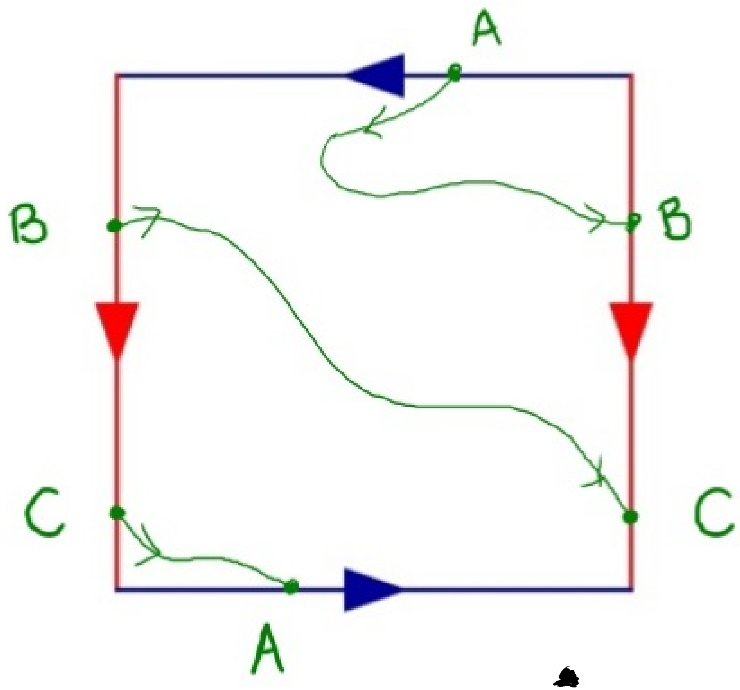


C_1 and C_2 are circles

Klein Bottle



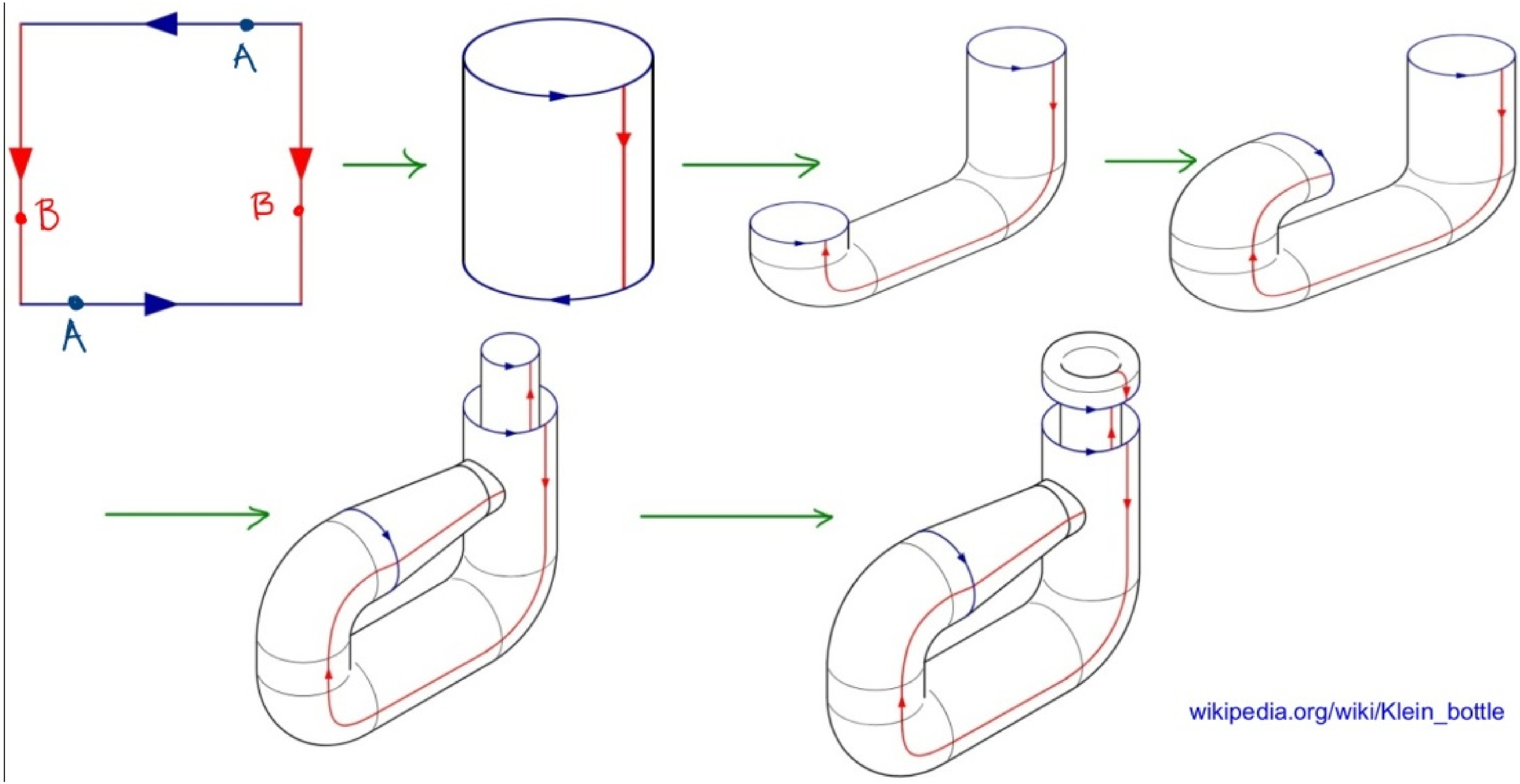
C_1 and C_2 are circles



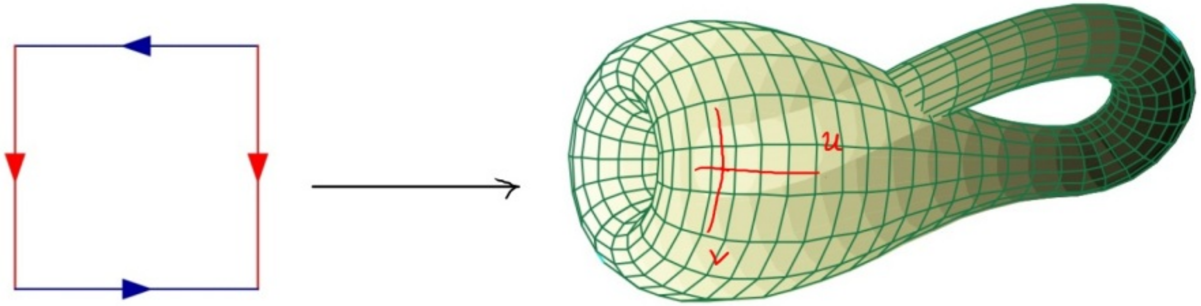
a closed loop

- notice the direction of velocities at A and C

Klein Bottle



Klein Bottle



$$x(u,v) = -\frac{2}{15}\cos u\left(3\cos v - 30\sin u + 90\cos^4 u\sin u - 60\cos^6 u\sin u + 5\cos u\cos v\sin u\right)$$

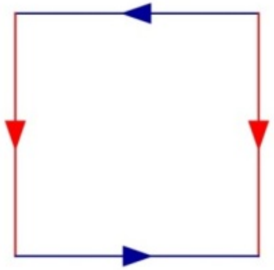
$$y(u,v) = -\frac{1}{15}\sin u\left(3\cos v - 3\cos^2 u\cos v - 48\cos^4 u\cos v + 48\cos^6 u\cos v - 60\sin u + 5\cos u\cos v\sin u - 5\cos^3 u\cos v\sin u - 80\cos^5 u\cos v\sin u + 80\cos^7 u\cos v\sin u\right)$$

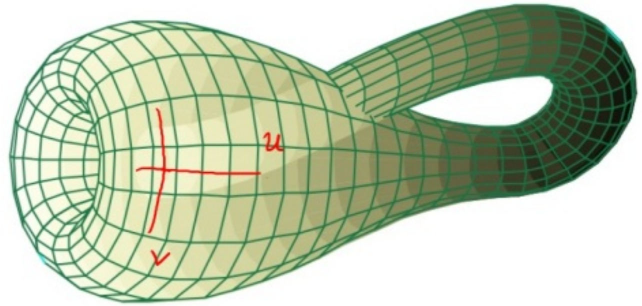
$$z(u,v) = \frac{2}{15}\left(3 + 5\cos u\sin u\right)\sin v$$

$0 < u < \pi$ $\qquad$ $0 < v < 2\pi$

wikipedia.org/wiki/Klein_bottle

Klein Bottle





4-D non-intersecting [edit] Embedding in $\mathbb{R}^4$

A non-intersecting 4-D parametrization can be modeled after that of the flat torus:

$$x = R\left(\cos\frac{\theta}{2}\cos v - \sin\frac{\theta}{2}\sin 2v\right)$$
$$y = R\left(\sin\frac{\theta}{2}\cos v + \cos\frac{\theta}{2}\sin 2v\right)$$
$$z = P\cos\theta\,(1 + \epsilon\sin v)$$
$$w = P\sin\theta\,(1 + \epsilon\sin v)$$

wikipedia.org/wiki/Klein_bottle