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Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών

«Φυσική και Τεχνολογικές Εφαρμογές»

Traversable Wormholes and the Simpson-Visser Model

Προσπελάσιμες Σκουληκότρυπες και το Μοντέλο των Sipmson και Visser

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

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Σκοπός αυτής της διπλωματικής εργασίας είναι η επέκταση του μοντέλου των Simpson και Visser, που αποσκοπεί στην απαλοιφή των απειρισμών της Schwarzschild μετρικής, εισάγοντας επιπλέον φορτίο και κοσμολογική σταθερά. Αρχικά, η εργασία ξεκινά με μία εισαγωγή στην οποία αποπειράται να γίνει κατανοητό πως η Schwarzschild μελανή οπή δεν αποτελεί προσπελάσιμη σκουληκότρυπα (traversable wormhole), ακόμη και αν αυτή διαθέτει μία χωρική γεωμετρία, που θυμίζει σκουληκότρυπα. Στη συνέχεια, θέτονται τα βασικά κριτήρια που πρέπει να πληρεί η μετρική ώστε να περιγράφει μία traversable wormhole. Κατ' ουσίαν, το κομμάτι αυτό της διπλωματικής αποτελεί μία ανασκόπηση (review) επάνω στις προσπελάσιμες σκουληκότρυπες κατά Morris και Thorne, χρησιμοποιόντας το φορμαλισμό του Bronnikov. Οι σκουληκότρυπες χαρακτηρίζονται ώς στατικοί και σφαιρικά συμμετρικοί χωρόχρονοι χωρίς κέντρο και ορίζοντες γεγονότων. Δηλαδή, ως μία γεωμετρία στην οποία η ακτίνα των σφαιρών φτάνει σε μία ελάχιστη τιμή μεγαλύτερη του μηδενός, που σηματοδοτεί τον "λαιμό" της σκουληκότρυπας και επιπλέον, χωρίς null Killing horizons, που είναι ο ορισμός του ορίζοντα γεγονότων για αυτή την κλάση μετρικών. Για να το πετύχουμε αυτό, βλέπουμε πως ο τανυστής ενέργειας-ορμής που περιγράφει την κατανομή της ύλης που απαιτείται για την κατασκευή της σκουληκότρυπας, στα πλάισια της Γενικής Θεωρίας της Σχετικότητας παραβιάζει την Null Energy Condition (NEC) και έτσι η καθιερωμένη στην βιβλιογραφία ως "εξωτική" (exotic) μορφή ύλης είναι αναπόφεκτη. Προς το τέλος αυτής της ανασκόπησης, επανερχόμαστε στον φορμαλισμό των Morris και Thorne και παρουσιάζουμε την πιο απλή μορφή μίας προσπελάσιμης σκουληκότρυπας, καθώς και το Penrose διάγραμμα αυτού του είδους των χωρόχρονων, έτσι ώστε να απεικονίσουμε την αιτιακή τους δομή. Μία αιτιακή δομή που δεν διαφέρει σε τίποτα από αυτή του επίπεδου Minkowski χωρόχρονου, εκτός από την ερμηνεία της. Στο τρίτο κεφάλαιο, παρουσιάζεται η προαναφερθείσα τεχνική των Simpson και Visser, εισάγοντας στην μετρική της μελανής οπής την παράμετρο η, για τις τιμές της οποίας, η περιγραφή της μετρικής ξεκινά από μία μελανή οπή μέχρι και μία προσπελάσιμη σκουληκότρυπα. Όσον αφορά την τελευταία, αυτή η παραμέτρος καταλήγει να αναπαριστά την ακτίνα του λαιμού της. Παρουσιάζουμε την τεχνική αυτή στην γενική της μορφή και έτσι είμαστε σε θέση να την επεκτείνουμε πέραν της Schwarzschild μετρικής. Όπως προείπαμε, εισάγουμε φορτίο, το οποίο σημαίνει πως εφαρμόζουμε την τεχνική στην Reissner–Nordström μετρική και επιπλέον, εισάγουμε την κοσμολογική σταθερά το οποίο σημαίνει πως εφαρμόζουμε την τεχνική στην Schwarzschild και Reissner–Nordström dS/AdS. Στο τέταρτο κεφάλαιο, παρουσιάζουμε έναν τρόπο για να διακρίνουμε την αρχική μελανή οπή και την συνεπαγόμενη σκουληκότρυπα με παρατηρήσιμα μεγέθη. Συγκεκριμένα, μελετάμε την θέση των κυκλικών τροχιών γύρω από τον λαιμό της σκουληκότρυπας και βλέπουμε πως καθώς αυξάνουμε την τιμή της εισαγόμενης παραμέτρου η, η ISCO και η φωτονική σφαίρα (photon sphere) έρχονται πιο κοντά στον λαιμό, μέχρι την τελική τους εξαφάνιση. Στο τελευταίο κεφάλαιο, πέρα από τα τελικά συμπεράσματα αυτής της μελέτης, γίνονται και μερικά σχόλια επάνω σε ανοιχτά ζητήματα γύρω από αυτούς τους χωρόχρονους.

Συγκεκριμένα, τα αποτελέσματα του κεφαλαίου 3 συνοψίζονται στα εξής:

1. Στην περίπτωση που εισάγουμε την κοσμολογική σταθερά στις εξίσωσεις πεδίου, οι δύο χωροχρονικές περιοχές που συνδέει η σκουληκότρυπα είναι ασυμπτωτικά επίπεδες/dS/AdS, για μηδενική, θετική και αρνητική κοσμολογική σταθερά, αντίστoιχα.

2. Για κάθε σκουληκότρυπα η NEC παραβιάζεται για όλο το εύρος του χωρόχρονου που περιγράφεται από την εκάστοτε μετρική.

3. Εξαίρεση αποτελεί η περίπτωση της θετικής κοσμολογικής σταθεράς, για την οποία εμφανίζεται ο κοσμολογικός ορίζοντας μακρυά από τον λαιμό. Αποτελεί εξαίρεση, διότι επάνω σε αυτήν την υπερεπιφάνεια η καθαρά χρονική συνιστώσα της μετρικής μηδενίζεται και έτσι η NEC δεν παραβιάζεται.

Όσον αφορά το κεφάλαιο 4, έχουμε τα εξής:

1. Για τις γνωστές μελανές οπές που η προτεινόμενη μετρική περιγράφει, λαμβάνουμε τις γνωστές θέσεις της φωτονικής σφαίρας και της ISCO.

2. Για τις προτεινόμενες σκουληκότρυπες, είναι δυνατό να μην ορίζεται ούτε φωτονική σφαίρα, ούτε ISCO.

3. Για τις προτεινόμενες σκουληκότρυπες, υπάρχει η περίπτωση να ορίζεται η ISCO, ενώ η φωτονική σφαίρα όχι.

4. Για τις προτεινόμενες σκουληκότρυπες, υπάρχει η δυνατότητα να ορίζεται και η φωτονική σφαίρα και η ISCO.

5. Όλες οι παραπάνω περιπτώσεις λαμβάνονται για διαφορετικές τιμές της παραμέτρου η, που χαρακτηρίζει το μοντέλο των Simpson και Visser, σε σχέση με τις υπόλοιπες παραμέτρους του συστήματος. Παρ' όλα αυτά, καμία ποιοτικά διαφορετική συμπεριφορά δεν εμφανίζεται με την εισαγωγή φορτίου και κοσμολογικής σταθεράς. Οι μόνες διαφορές είναι ποσοτικές.

Κλείνοντας την εργασία, βλέπουμε πως η προκείπτουσα μετρική από το μοντέλο των Simpson και Visser δεν μπορεί να προκύψει από την δράση της Γενικής Θεωρίας της Σχετικότητας, ακόμη και με την εισαγωγή κάποιου βαθμωτού πεδίου, ελάχιστα συνδεδεμένου (minimally-coupled) με τον μετρικό τανυστή. Αυτό σημαίνει, πως για να βρούμε την θεωρία πίσω από την μετρική αυτή θα πρέπει να ξεφύγουμε απο το πλαίσιο της Γενικής Θεωρίας της Σχετικότητας και να περάσουμε σε τροποποιημένες θεωρίες βαρύτητας. Για τον λόγο αυτό, σκιαγραφούμε μία πολύ ενδιαφέρουσα προοπτική για τις σκουληκότρυπες σε τροποιημένες θεωρίες βαρύτητας. Μία προοπτική που μας δίνει την δυνατότητα να κατασκευάσουμε μοντέλα σκουληκότρυπων για τα οποία δεν απαιτείται η εισαγωγή εξωτικής ύλης, αφού η ύπαρξη του λαιμού στην γεωμετρία δεν απαιτεί την παραβίαση της NEC από τον τανυστή ενέργειας-ορμής του υλικού πεδίου. Η παραβίαση της NEC, έρχεται από γεωμετρικούς όρους που εισάγονται στην θεωρία λόγω της τροποποίησης της, οι οποίοι ερμηνεύονται ως κάποιου είδους βαρυτικό ρευστό. Τέλος, βλέπουμε πως η μετρική των Simpson και Visser αποτελεί ένα εύφορο πεδίο μελετής για την περιγραφή αλλαγών φάσεων μεταξύ ομαλών (regular) μελανών οπών και σκουληκότρυπων.

Abstract

The goal of this thesis is to extend the Simpson-Visser technique for regularising the Schwarzschild metric by the introduction of a cosmological constant and charge. Before that, this thesis starts with an introduction that clarifies that the Schwarzschild black hole, even if it possesses a wormhole-like geometry, is not a traversable wormhole; something that is forbidden by *the* principle of General Relativity. After that, we pose the basic criteria for a metric to describe a traversable wormhole in principle. This is a review for traversable wormholes in the sense of Morris and Thorne, in which the Bronnikov formalism is enforced. Wormholes are characterized as static and spherical symmetric spacetimes without centre and horizons; that is, a geometry possessing a minimum coordinate sphere radius different than zero, which is the definition of the throat of the wormhole and no null Killing horizons, which is the definition of a horizon in this class of spacetimes. In order to succeed that, a NEC violating Energy-Momentum tensor is unavoidable, so "exotic matter" is appropriate for the structure of the throat. After, recovering the original Morris and Thorne formalism and present the simplest example of a wormhole, we extract the Penrose diagram of such a spacetime in order to illustrate its causal structure. A causal structure which is like that of Minkowski, but with a different interpretation. What we are going to see in chapter 3 is the technique of Simpson and Visser in order to regularize the Schwarzschild metric by the introduction of some parameter η. Specifically, on the values of this parameter depends the kind of the geometry that the metric describes, starting from the original Schwarzschild black hole to a traversable wormhole. For the wormhole, it turns out that this parameter corresponds to the throat radius. The intermediate "states" are those of a regular black hole and a one-way traversable wormhole. We present this technique in its general state, which allows us to extend the technique of Simpson and Visser to more spacetimes rather than the Schwarzschild one. Namely, we extend this procedure by introducing a cosmological constant and charge (Reissner–Nordström). In chapter 4, we present an observational distinction between the initial black hole and the corresponding wormhole. With the circular orbits being our tool, we see that as we grow the parameter η, the ISCO and the photon sphere become smaller and smaller until their final disappearance. In the final chapter, some comments for future work on this spacetimes are presented.

Acknowledgements

I do not know if this thesis is good, but this "chapter" is not for sure. People, that I live with, know me in so many ways of my character and I have lived with them in so many different conditions, that an acknowledgement about this thesis is strictly out of my goal.

I want to thank you because you keep me still in life. I hope that our mutual acknowledgement is proved daily.

But through this master and this thesis I met two more people. My colleague, Nikos Chatzifotis, is the second person that I've met so far with whom I'd like to do physics together. I thank you for the discussions (all of these hours) about this thesis and in general about physics. I hope that our future conditions are be based in good initial ones. I hope the same for Lefteris Papantonopoulos, who is my supervisor for this thesis. I want to thank him for this.

Contents

Chapter 1

Introduction

Einstein-Rosen Bridge or Schwarzschild Wormhole

The study of wormholes in physics finds its origins in 1935, when Einstein together with Rosen published an article in which they try to describe particles (neutral and charged) under the prism of field theory. This was a part in the attempt for a «unified foundation on which the theoretical treatment of all phenomena could be based»[1]. Contrary to the perception of physicists who interpreted particles as singularities of the fields, Einstein and Rosen argue that «a singularity brings so much arbitrariness into the theory that it actually nullifies its laws». So, «Every field theory [...] must therefore adhere to the fundamental principle that singularities of the field are to be excluded».

Today, it is clear along the physicist community the distinction between the curvature and the coordinate singularities. Curvature singularities are intrinsic to the geometry, while coordinate singularities can be eliminated by coordinate transformations. However, at the years that the paper was written this distinction was not clear and many physicist assumed that the coordinate singularity of the Schwarzschild metric, i.e. the event horizon, *was* the singularity [2]. Having these in mind, in this paper we see an attempt to eliminate the coordinate singularity of the Schwarzschild and the Reissner–Nordström metric, by a coordinate transformation which seems to lead to a bridge in spacetime. That is, a connection between two asymptotically flat spacetimes. Let's see the main idea behind this, in the case of the Schwarzschild metric[1], because in here is stated the basic wormhole concept in its "immature state". The Schwarzschild metric is the following:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2 d\Omega^2 \tag{1.1}$$

Of course, the coordinate singularity is for $r = R_s$. If you take the transformation: $v^2 = r - R_s$, then the metric becomes:

$$ds^2 = -\frac{v^2}{v^2 + R_s} dt^2 + 4(v^2 + R_s) dv^2 + (v^2 + R_s)^2 d\Omega^2 \tag{1.2}$$

[1]Einstein and Rosen are making a similar transformation to the Reissner–Nordström metric and they find similar results. They conclude to a similar kind of bridge, with which they argue that charged particles can be described. What we are considering about, although, are wormholes and the reason why this "charged bridge" is not a wormhole is exactly the same. So, we left it out for thriftiness.

Indeed Einstein and Rosen accomplished their goal. The above metric is free of singularities; the singularity of $r = R_s$ have been replaced with $v = 0$, for which no infinity is present. According to the relation between v and r, we see that the above metric is able to cover only the region $r > R_s$. Hence, the singularity (curvature) at $r = 0$ disappeared, too. In addition, with v defined in $(-\infty, +\infty)$, at each limit of v to $\pm\infty$, the metric describes an asymptotically flat spacetime. For this reason, the region near $v = 0$ was named, by Einstein and Rosen, as a "bridge", which connects two asymptotically flat spacetimes. In addition, by taking $t = const$ and $u = const$ the above metric describes spherical surfaces of radius $u^2 + R_s$. The surface with $u = 0$ corresponds to the minimum radius of R_s; that is, the bridge is the narrowest part of the geometry.

In the above paragraph we see two crucial points. The first one is that the metric is free of singularities, while the second one is that the metric describes two asymptotically flat regions of spacetime, which are connected by the narrowest part of the geometry. The wormhole physics can be considered as a trial to avoid the curvature singularity at the center of the spacetime[2], $r = 0$, and this is the point that connects wormholes with the above consideration of Einstein and Rosen about the description of particles. It is true that singularities in a field theory have posed many problems through out the years. Beyond wormholes, the exclusion of possible singularities in gravity was tackled with the concept of regular black holes, in which we try to either regularize the center ($r = 0$) of the black hole or to avoid the center by building a spacetime without one. For the latter, the idea is similar to the definition of the bridge that we stated above. In order to avoid the center, we can constraint the geometry of spacetime to have a minimum radius larger than zero, $r \geq r_0 > 0$. The study of wormholes and regular black holes, although, show that this is not something that can be done without producing other issues. We see that in order to avoid a singular center of the spacetime, "exotic matter" has to be introduced (this will be clear in the subsequent chapters); that is, matter distributions that do not correspond to any classical form. But more about this later.

However, the Einstein-Rosen bridge does not describe a traversable wormhole. We can argue for this in many ways. Visser in his textbook about the Lorentzian Wormholes [2], argues that the bridge described by (1.2) is just a coordinate artifact. That this metric describes just a black hole where a "bad" coordinate system is enforced, which doubly covers the exterior of the event horizon. A few lines, above we imposed a way in order to avoid the center; that is, constrain the narrowest part of the geometry to be a sphere of non-vanishing radius. Someone could tell that the transformation that Einstein and Rosen took (namely, $v^2 = r - R_s$) does exact this work and constraints r to be larger than R_s. But this is not true. The Schwarzschild metric is a vacuum solution to the Einstein equations defined for $r \in (0, +\infty)$. For $r > R_s$ the metric is static, while for $r < R_s$ is not. A transformation like that of Einstein and Rosen does not constraint the geometry of the spacetime. It just ignores a part of the geometry. Specifically ignores the non-static part of the Schwarzschild metric, which is a part of the underlined geometry and cannot be excluded by just a coordinate transformation. This is why the Einstein-Rosen bridge is just a coordinate artifact and if you try to cross the $v = 0$ hypersurface you will not go to an other flat spacetime (as you do in wormholes), but instead, you will fall into the singularity.

The other argument, which is actually the most common one, is stated in view of the

[2]For a definition of the center of the spacetime, see Chapter 2

Kruskal diagram of the Schwarzschild metric and does not concern the metric (1.2) explicitly. It is an argument that the underlined spacetime of the Schwarzschild metric actually contains two asymptotically flat regions, albeit disconnected. The Kruskal diagram of the Schwarzschild metric presents the maximal extension of the underlined spacetime, in which we see four regions(see figure (1.1)). The two diagonal lines represent the future and past event horizons,which are lines of 45^0 degrees. Regions I and II are the expected regions of the exterior and the interior of the black hole, respectively . The regions III and

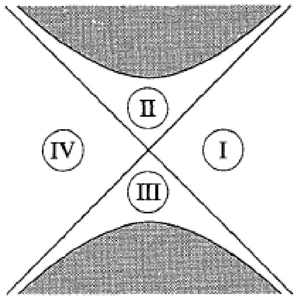

Figure 1.1: *The regions of the Kruskal diagram.*(This graph has been taken from [3])

IV are in someway unexpected. Region III can be thought of as the time reversal of the II; that is, a region from which things can emerge to region I, but nothing can get into it. This is the so called *white hole*. The shaded regions present the singularities in the center of the black and the white hole. Region IV is an other asymptotically flat spacetime region identical to I. Thence, it seems that in the Schwarzschild metric we find two asymptotically flat regions, which seems to be connected. But are they truly connected? If we take a slice of $t = const$, this hypersurface in the Kruskal diagram corresponds to a horizontal line passing through the point that the two diagonal lines are crossed, from region I to region IV. The spatial geometry of the $t = const$ hypersurface is of course given by (1.1) with $dt = 0$. If we make the embedding diagram of this spatial geometry, what we get is indeed a wormhole-like geometry, that we present in figure (2.1). However, if a particle could travel through this hypersurface, it would mean that this particle had exceeded the speed of light. But as we know this is **forbidden in principle**. In other words, the trajectories that connect the two asymptotically flat regions are *spacelike*. This is easily be shown by the Kruskal diagram, in which the null paths are lines of 45^0 degrees, like in Minkowski spacetime. Hence, as any timelike trajectory have to be inside the future lightcone, there is no way for a particle to have a worldline that crosses the left tilted horizon. For a photon, the best that can be done is to move parallel with that horizon; a path that leads it directly through the black hole. Thence, **it is *the* principle of general relativity, which implies that a black hole is not a traversable wormhole!**

The aim of this thesis

The reason that I chose to start the introduction with the Einstein-Rosen bridge is dual. The first one is that even today there is a lot of students that when they hear about wormholes, they have in mind the work of Einstein and Rosen. So, I find it important to make clear that the Einstein-Rosen bridge is not a traversable wormhole. The second is, that neglecting the fact that the Einstein-Rosen bridge is not a true traversable wormhole, the basic concept of a wormhole is stated there. The crucial boost to this topic of research was set my Morris and Thorne in 1988 [4].

It is not accidental that this thesis starts in someway with this paper. Morris and Thorne stated the criteria for a geometry in order to describe a wormhole, which has similar characteristics with the Einstein-Rosen bridge, but with the difference that those wormholes are truly traversable. So, in chapter 2 we introduce the basic criteria for a wormhole geometry to be traversable in principle. I do not use from the ground up the formalism of Morris and Thorne, but instead that of Bronnikov in [5]. The reason for that is that the notation of Bronnikov is quite more general, something that can help us to understand better the definitions, like as the centre of the spacetime or that of the throat of the wormhole. The advantage of this formalism is highlighted at the flaring out condition, in which we see how the latter is already guaranteed by the geometrical demands for the existence of the throat, something that follows from a discussion that in someway explains why the form of the metric given by Morris and Thorne is necessary.

In the following, we are concerned with the technique of Simpson and Visser introduced in [6] for the regularization of a black hole metric. This procedure produces new regular spacetimes, from a regular black hole to a traversable wormhole. In this thesis we focus on the construction of traversable wormholes. Specifically, we present the technique in its general state; that is, by starting from an arbitrary spherical symmetric black hole metric and then we state how we regularize this metric in order to construct a traversable wormhole. This generalization allows us to extend the technique of Simpson and Visser to more spacetimes rather than the Schwarzschild one. Moreover, by this general treatment we investigate the close relation between the no horizon condition and the flaring out condition. We extend this procedure by introducing a cosmological constant and charge (Reissner–Nordström). To these specific examples, we check the regularity of the spacetime by checking mainly the components of the Riemann tensor, we check the asymptotic behaviour far from the throat and we see how the NEC is violated in each case. For the asymptotic behaviour we see that in the case of a cosmological constant the two connected regions of spacetime are asymptotically dS or AdS. In the case of a positive cosmological constant (dS), we see that the NEC is not violated at the cosmological horizon.

In the last chapter, we see a way for distinguishing the initial black hole and the final traversable wormhole constructed by the Simpson and Visser technique by the location or even the existence of a photon sphere or ISCO. In order to do that, the effective potential for null and timelike geodesics is extracted for the general spherically symmetric and static metric and then is applied to the Schwarzschild and Reissner–Nordström cases, for which we applied the aforementioned technique.

Notations and Conventions

In this thesis the natural units $G = c = 1$ are enforced. Hence, the Einstein's field equations are:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.3}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

is the Einstein tensor and $T_{\mu\nu}$ the Stress-Energy tensor, while $g_{\mu\nu}$ is the metric tensor with signature: $(-,+,+,+)$

Chapter 2

A review of Morris and Thorne traversable wormholes

Morris's and Thorne's paper [4] is considered to be the renaissance of the wormhole study in general relativity. With this paper, the two physicists institute the basic criteria of a traversable wormhole, able for interstellar travel. The following study will not examine all of the 9 criteria, as the main focus shall be the examination of a wormhole geometry being traversable in principle, regardless of a human's ability to ford it or not. Thence, we will examine only the first four criteria, the so called "basic wormhole criteria". For each of these, a specific subsection will be allotted. Each section here forth will be named after one of these four criteria.

2.1 The metric should be static and spherically symmetric

This criterion is put forth for the sake of simplicity, as both Morris and Thorne underline. It is not necessary to take this criterion for granted as we begin our search. It is, however, wieldy. The static and spherically symmetric metrics are widely used, so their closer examination might be of great value in this early stage of our study. In general, spherical symmetric space times can be described by the following metric [5] :

$$ds^2 = -e^{2\gamma}dt^2 + e^{2\alpha}du^2 + e^{2\beta}d\Omega^2 \ \ , \ d\Omega^2 = d\theta^2 + sin^2(\theta)d\phi^2 \tag{2.1}$$

where α, β, γ are functions of the radial and time coordinates u and t, respectively. It is convenient to substitute $r = e^{\beta}$ and then r corresponds to the radius of the coordinate sphere $u = const$, $t = const$. If, in addition, the space time is static, there is always a coordinate system in which the functions α, β, γ are t independent. With the u coordinate still remaining unspecified, a particular relation between α, β, γ can fix the radial coordinate. Some examples are the following:

- **The tortoise coordinate**: $\alpha(u) = \gamma(u)$

- **The curvature coordinate**: $u = r$ and $\gamma = \gamma(r)$,$\alpha = \alpha(r)$

- **The quasiglobal coordinate** u, $\alpha(u) = -\gamma(u)$

A note: For the curvature coordinates we have actually taken $\beta(u) = ln(u)$, while the relation of γ and α remains unspecified. To the quasiglobal coordinate u, we constrained only the functions α and γ and not the β function. This means that the quasiglobal and the curvature coordinate can be combined, having $u = r$ and $\alpha = -\gamma$, simultaneously.

Killing vectors and symmetries

We think of the spacetime as a 4 dimensional manifold. The symmetries in General Relativity are characterized by the Killing vectors, that we can define on this manifold. The Killing vectors found in the spherical symmetry are those exact ones that also characterize the rotational symmetries of the S^2 (2-D sphere). For that, in a manifold with spherical symmetry, the following Killing vectors (R, S, T) are defined: [3]

$$\begin{aligned} R &= \partial_\phi \\ S &= cos\phi \ \partial_\theta - cot\theta \ sin\phi \ \partial_\phi \\ T &= -sin\phi \ \partial_\theta - cot\theta \ cos\phi \ \partial_\phi \end{aligned} \tag{2.2}$$

The forenamed form, though, is coordinate dependent, since the Killing vectors are being expressed in terms of the coordinate dependent basis vectors ∂_ϕ and ∂_θ. The coordinate independent relations that characterize the forenamed vectors are their exact commutation relations, which compose the structure of symmetry transformations and are the following:

$$\begin{aligned} [R, S] &= T \\ [S, T] &= R \\ [T, R] &= S \end{aligned} \tag{2.3}$$

In group theory, this is called as the Lie algebra of the symmetry generators. Killing vectors characterize symmetries of the spacetime and symmetries imply constants of motion, that is, conserved quantities. A vector K is a Killing vector, if it satisfies the Killing's equation:

$$\nabla_{(\mu} K_{\nu)} = 0 \tag{2.4}$$

[The conserved quantity] *The symmetry that the Killing vector K implies, produces a conserved quantity along a geodesic trajectory given by $K^\mu P_\mu$; that is, the projection of the tangent vector to the Killing vector.*

Proof. Let $x^\mu(\lambda)$ be a path, with a tangent vector $P^\mu = dx^\mu/d\lambda$. Now take the scalar $K^\mu P_\mu$ and see how it changes along a geodesic trajectory. To see this, we have to calculate the directional covariant derivative of $K^\mu P_\mu$ along the path $x^\mu(\lambda)$, as:

$$\begin{aligned} &\frac{D}{d\lambda}(K^\mu P_\mu) = P^\nu \nabla_\nu (K^\mu P_\mu) \\ \rightarrow &\frac{D}{d\lambda}(K^\mu P_\mu) = P^\nu P^\mu \nabla_\nu K_\mu + K_\mu P^\nu \nabla_\nu P^\mu \end{aligned} \tag{2.5}$$

The first term in right hand side contains a multiplication of a purely symmetric tensor $P^\nu P^\mu$ with a purely anti-symmetric tensor $\nabla_\nu K_\mu$, as the Killing's equation (2.4) implies. Hence, the first term vanishes. In the second term, the directional derivative of the tangent vector of the path, $P^\nu \nabla_\nu P^\mu$, appears. But a geodesic is a path that parallel transports its tangent vector; that is, the directional derivative of its tangent vector vanishes. So, the second term vanishes, too. Thus,

$$\frac{D}{d\lambda}(K^\mu P_\mu) = P^\nu \nabla_\nu (K^\mu P_\mu) = 0 \tag{2.6}$$

So, $K^\mu P_\mu$ is conserved. ■

Example. Take for example the Killing vector R. In Cartesian coordinates, this vector is expressed as follows:

$$R = \partial_\phi = -y\partial_x + x\partial_y \tag{2.7}$$

That is pretty straightforward and could be easily explained by just a coordinate transformation from polar to Cartesian coordinates. Hence, in component form:

$$R^\mu = -y\delta^\mu_x + x\delta^\mu_y \tag{2.8}$$

So, the conserved quantity is

$$R^\mu P_\mu = -yP_x + xP_y \tag{2.9}$$

This is the well-known z-component of the angular momentum of a particle. Therefore, the rotational symmetry around the z-axis implies conservation of the z-component of the angular momentum, as was expected. Similar results are produced for the other Killing vectors.

The timelike Killing vector

The fact that the metric is static, reflects the symmetry under time translation, which means that the Killing vector that generates this symmetry is:

$$K = \partial_t \tag{2.10}$$

This Killing vector in component form is written as:

$$K^\mu = \delta^\mu_t \tag{2.11}$$

It is a timelike vector as long as the γ function is finite, because:

$$g_{\mu\nu}K^\mu K^\nu = g_{\mu\nu}\delta^\mu_t \delta^\nu_t = g_{tt} = -e^{2\gamma} \tag{2.12}$$

Actually, this is not sufficient for the metric to be static. The existence of a time-like killing vector and consequently symmetric under time translations renders the metric stationary. In order for it to be static, one more condition is necessary. This condition refers to the timelike Killing vector and implies that this Killing vector is orthogonal to a class of hypersurfaces. This means that: [3]

$$K_{[\mu}\nabla_\nu K_{\sigma]} = 0 \tag{2.13}$$

Together with the Killing's equation (anti-symmetry of $\nabla_\mu K_\nu$) we have:

$$\boxed{K_\mu \nabla_\nu K_\sigma + K_\sigma \nabla_\mu K_\nu + K_\nu \nabla_\sigma K_\mu = 0} \tag{2.14}$$

So, if the timelike Killing vector satisfies the above equation, the metric is static. In the form of the metric that its components are t-independent, the class of the hypersurfaces that K is orthogonal are those defined by $t = const$. This feature is reflected to our metric by the fact that no-cross terms between the t- coordinate and any spatial coordinate are present. Thus, the only t term is the dt^2, which is also invariant under time reversal. Therefore, we can think of a static metric as a stationary one (a time-like Killing vector exists) and invariant under time reversal transformation. Keep in mind these equations. It will help us afterwards to define and exclude possible horizons of the metric.

Regularity and centre of the spacetime

As we know, the regularity of the spacetime is checked by the scalar invariants produced by the Riemann tensor. If the scalar invariants are finite for all of the space time points (events), then the space time is regular. Scalar invariants are produced by the contractions of the Riemann tensor or the Ricci tensor. One of them, is the Ricci scalar or scalar curvature, $R = R^{\mu}{}_{\mu}$. Of course, this scalar has to be finite all along the space time for a regular manifold, but it is not a useful candidate for checking the regularity. The most helpful candidate is the Kretschmann scalar, defined as:

$$\mathcal{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \tag{2.15}$$

For the static and spherically symmetric metric [5]

$$\mathcal{K} = 4K_1^2 + 8K_2^2 + 8K_3^2 + 4K_4^2 \tag{2.16}$$

where

$$\begin{aligned} K_1 &= e^{-\alpha-\gamma}(\gamma' e^{\gamma-\alpha})' = -R^{01}{}_{01} \\ K_2 &= e^{-2\alpha}\beta'\gamma' = -R^{02}{}_{02} = -R^{03}{}_{03} \\ K_3 &= e^{-\alpha-\beta}(\gamma' e^{\beta-\alpha})' = -R^{12}{}_{12} = -R^{13}{}_{13} \\ K_4 &= e^{-2\beta} + e^{-2\alpha}(\beta')^2 = -R^{23}{}_{23} \end{aligned} \tag{2.17}$$

More precisely, the geometry is regular if any non-zero component of the Riemann tensor is finite all along. This is a stronger statement than that of the scalars because, as we said, the scalar invariants are produced by the components of the Riemann tensor. Hence, if every component (non-zero) of the Riemann tensor is finite at some point, then every scalar invariant is finite too. The Kretschmann scalar contains all the non- vanishing components of the Riemann tensor squared and summed. Hence, if $\mathcal{K}$ is finite at some point, then every non-vanishing component of the Riemann tensor is finite too. This means, we could check for the regularity by computing only one scalar. That is, if $\mathcal{K}$ is finite for every point in space time, the space time is regular. Sometimes, although, it is more useful to calculate just the components of the Riemann tensor and check the regularity of each component. Then, you can check the regularity without calculation of any scalar.

The most common singular point is the center of the metric. With the term "center", we mean a point defined by

$$r = e^{\beta} = 0 \tag{2.18}$$

That is, the coordinate spheres ($u = const, t = const$) represented by points rather than 2D spheres. Of course, a center should be regular-nonsingular. The conditions under which a center is regular could be found in [5]. However, it is not necessary for a center to be present in our geometry. We can define a geometry without center by demanding that:

$$r = e^{\beta} \neq 0 \; \forall \; u \tag{2.19}$$

Then, every ($u = const, t = const$) hypersurface is a 2D sphere.
Wormholes are spacetimes without center; r takes a minimum value larger than zero. This minimum value is the throat of the wormhole, which we will examine in the following. Remember the discussion for the Einstein Rosen bridge. The above constraint is not just a coordinate artifact. It is a constraint of the geometry that is reflected to our coordinates. There is nothing beyond r_0, that we ignore. **The manifold itself has no center**. In addition, there are spacetimes without centers containing horizons, called "black bounce space times", which are actually black holes without the singular center and an expanding universe beyond them (black universes).

2.2 For a wormhole solution, there must be a throat connecting two spacetime regions

In a wormhole geometry two spacetime regions are separated and connected with a throat in between. A throat corresponds to the minimum value of the function $r = r(u)$. As we said, the wormhole geometry describes a spacetime with a metric without centre. Hence, this minimum value is larger than zero and, of course, is a regular minimum. The two different spacetime regions are determined from the sign of the u-coordinate. Specifically, for $u > 0$ we define the "one region",as for $u = 0$ we define the throat, $r_0 = r(u = 0)$, while for $u < 0$ we define "the other region".

If the wormhole connects two different spacetimes, usually referring to two universes, or two regions of the same spacetime (the same universe), it is a matter of global topology, something that does not affect the physics "near the throat". In this sense, even if we talk about two universes or two regions of the same spacetime, we are consistent for the purposes of this essay. What we are interested about, is the physics around the throat of the wormhole; hence, we are not concerned about issues of global topology.

However, as we go forth, we will come across the issue of the space time far from the throat. Meaning, we will wonder if it is asymptotically flat, dS or AdS. In the original paper of Morris and Thorne, we can see that the space time far away from the throat is flat, meaning Minkowski, either if we move towards one region (positive u) or towards the other (negative u). The statement "away from the throat", is mathematically expressed as the limit of u to $\pm\infty$. This statement is not in contradiction with the previous one, in which we constrained our study near the throat, neglecting the issues of global topology. The fact that we take the limit of the u coordinate to infinity , does not mean that we cover all the manifold-spacetime. This limit just means the end of the region of the manifold that we can cover with this specific coordinate system. The situation is almost alike with the case of the Schwarzschild black hole. In Schwarzschild coordinates, the coordinate time that is elapsed for a particle to cross the event horizon is infinite, which seems like the particle never reaches the event horizon. But, of course, this is not the case, as the proper time interval for the crossing is finite. That is one weakness of the Schwarzschild coordinate system. We have to change coordinate system, in order to cover all the spacetime manifold and then successfully describe this event. So, the fact that we are taking the limit to infinity does not mean that no event beyond that exists.

Moreover, the two connected space times are not necessarily asymptotically flat. They could also be asymptotically dS or AdS. Even more, the two spacetimes do not need to behave alike away from the throat in both directions. Let's remember our metric. It contains 3 functions, which define the specific form of the geometry. These functions are dependent from the coordinate u. If, for example, we take $\alpha(u) \neq \alpha(-u)$, then the behavior of the metric will be different to both directions. Conclusively, there is a chance that the throat is connecting an asymptotically flat spacetime and a dS space time or even an AdS space time. We restrict ourselves to the case of the α, β, γ functions to be even.

Let's now see the restrictions that must be imposed to our metric coefficients by the existence of the throat. As we previously said, the throat is the minimum of the function $r = r(u)$ which is taken to $u = 0$. This minimum is different from zero; a spacetime without a center. So the constraints of the $r(u)$ are:

$$\boxed{\begin{array}{l} r_{min} = r_0 \neq 0 \\ \dfrac{dr}{du}|_{u=0} = 0 \quad or \quad \beta'(0) = 0 \\ \dfrac{d^2r}{du^2}|_{u=0} > 0 \quad or \quad \beta''(0) > 0 \end{array}} \tag{2.20}$$

The embedded diagram

An other basic feature of the wormhole geometry is the spatial representation of the throat as a 2D Euclidean surface. These are the so called mathematics of embedding, with which we constraint two dimensions in order to represent the metric as surface in three dimensional space. Spherical symmetry allows us to constraint one angle coordinate to a constant value. The usual constraint is $\theta = \pi/2$, as it leaves the solid angle to take the simple form of $d\Omega^2 = d\phi^2$. Up to this constraint ds^2 represents a (2+1) dimensional space. Our purpose is to represent the spatial geometry of the throat. Hence, we "take a picture" of the metric at a time $t = const$, leaving in this way the metric to represent a 2-dimensional surface:

$$ds^2 = e^{2\alpha(u)}du^2 + r^2 d\phi^2 \tag{2.21}$$

In cylindrical coordinates the 3D Euclidean metric is:

$$ds_E^2 = dz^2 + dr_E^2 + r^2 d\phi_E^2 \tag{2.22}$$

So, a 2D surface defined by $z = z(r_E)$ takes the form: (having used the chain rule ($dz = \frac{dz}{dr_E} dr_E$):

$$ds_E^2 = \left[1 + \left(\frac{dz}{dr_E}\right)^2\right] dr_E^2 + r_E^2 \, d\phi_E^2 \tag{2.23}$$

Hence, in order to make the embedding, we have to express the metric (2.21) in terms of the r coordinate rather than u. This can be done only if we assume that the function of r in respect to u is reversible and $u = u(r)$ can be defined. In addition, we assume that the derivative of u in respect to r is also defined. Thus,

$$ds^2 = e^{2\alpha(r)} \left(\frac{du}{dr}\right)^2 dr^2 + r^2 d\phi^2 \tag{2.24}$$

What remains is to find the correspondence between (2.23) and (2.24). The matching of coordinates is $r \leftrightarrow r_E$ and $\phi \leftrightarrow \phi_E$, giving us:

$$\boxed{\frac{dz}{dr} = \pm\sqrt{e^{2\alpha(r)}\left(\frac{du}{dr}\right)^2 - 1}} \tag{2.25}$$

In the previous discussion we saw that at the throat, the first derivative of r in respect to u has to be zero. This condition implies that the derivative of u in respect to r reaches infinity: $du/dr|_{r=r_0} \to +\infty$. Hence, for a finite value of $\alpha(r)$ at the throat we see that (2.25) blows up to infinity at the throat; that is, the embedded surface is vertical at the throat. The only way with which (2.25) does not tend to infinity at the throat is $\alpha(r_0) \to -\infty$. But the perpendicularity of the diagram at the throat is a demand for a wormhole geometry; that is, $\alpha(r)$ is constrained to be finite at the throat.

For an asymptotically flat spacetime, far from the throat, (2.25) has to vanish. However, in the case that a cosmological constant is introduced, the limit of (2.25) at $r \to +\infty$ takes imaginary values. This reflects the statement that the validity of (2.25) is constrained near the throat [7]. Consequently, we can trust this equation only near the throat. The question of the asymptotic behavior is treated by the Ricci scalar and not by the embedding diagram, in general.

The shape of the embedded surface is presented in the following figure:

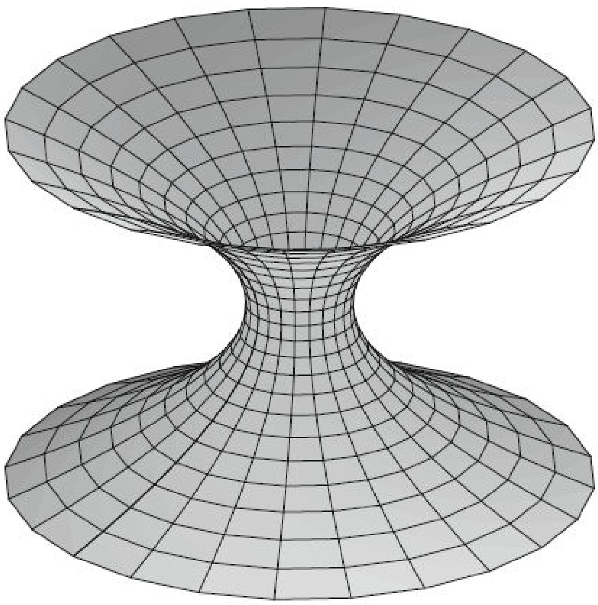

Figure 2.1: *The embedding diagram of a two-dimensional section along the equatorial plane (t = const, $\theta = \pi/2$) of a traversable wormhole. For a full visualization, of the surface sweep through a 2π rotation around the z-axis, as can be seen from the graphic above.*(This graphic has been taken from [8])

The flaring-out condition

Another demand of critical role about the embedding diagram is the flaring-out condition. This condition requires that the throat is open and imposes:

$$\boxed{\frac{d^2r}{dz^2}\Big|_{\text{throat}} > 0} \tag{2.26}$$

Inverting the equation (2.25) we get:

$$\frac{dr}{dz} = \pm\left(e^{2\alpha(r)}\left(\frac{du}{dr}\right)^2 - 1\right)^{-1/2} \tag{2.27}$$

which is a function of r rather than z. So, in order to calculate the second derivative in respect to z, we use the chain rule $d/dz = dr/dz\ d/dr$, which gives us:

$$\frac{d^2r}{dz^2} = -\frac{e^{2\alpha}\frac{du}{dr}}{\left[e^{2\alpha}\left(\frac{du}{dr}\right)^2 - 1\right]^2}\left[\frac{d\alpha}{dr}\frac{du}{dr} + \frac{d^2u}{dr^2}\right] \tag{2.28}$$

Suppose that we take $\alpha \equiv 0$ (this is not a random choice, it will be the case in the subsequent sections). Then, the above equation simplifies to the following one:

$$\frac{d^2r}{dz^2} = -\frac{1}{\left[\left(\frac{du}{dr}\right)^2 - 1\right]^2}\frac{du}{dr}\frac{d^2u}{dr^2} \tag{2.29}$$

So, what happens at the throat? We know that du/dr becomes infinite at $r = r_0$. Taking du/dr large, equation (2.29) looks like:

$$\frac{d^2r}{dz^2} \approx -\frac{1}{\left(\frac{du}{dr}\right)^3}\frac{d^2u}{dr^2} \tag{2.30}$$

The infinities in the above equation have to be eliminated, if we want to keep d^2z/dr^2 finite and positive (non-zero), as the flaring-out condition implies. In order to eliminate the infinities, there must be a specific relation between d^2u/dr^2 and du/dr; namely, they have to be proportional. **If the equations (2.20) constraint the possible relations between u and r, the latter requirement constraints this relation much more.**

Let's see how this work. Assume $h(r)$ to be a function of r. Then, if we take $du/dr = h^n(r)$, the second derivative becomes:

$$\frac{d^2u}{dr^2} = nh'(r)h^{n-1}(r) \tag{2.31}$$

where prime denotes d/dr. The above has to be proportional to $h^{3n}(r)$, which constraints n by $n - 1 = 3n$, giving $n = -1/2$. So, the du/dr must be of the following form:

$$\boxed{\frac{du}{dr} = \frac{1}{\sqrt{h(r)}}} \tag{2.32}$$

with $h(r)$ some unspecified function of r. Substitution to (2.29) gives:

$$\frac{d^2r}{dz^2} = \frac{h'(r)}{2(1-h(r))^2} \tag{2.33}$$

Moreover, if we calculate d^2r/du^2 (using again the chain rule) according to (2.32), we take:

$$\frac{d^2r}{du^2} = \frac{h'(r)}{2} \tag{2.34}$$

Hence,

$$\frac{d^2r}{dz^2} = \frac{1}{(1-h(r))^2}\frac{d^2r}{du^2} \tag{2.35}$$

But from the beginning of this section, we have constrained d^2r/du^2 to be positive at the throat. So, the latter equation implies that ***the flaring out condition is guaranteed by the constraints of the $r = r(u)$ function ab initio***.

Having now, the specific form for du/dr and consequently for dr/du, it is more useful to express the constraints (2.20) with respect to the function $h(r)$ rather than $\beta(r)$. Combining (2.20) and (2.32) we get:

$$\boxed{\begin{array}{l} h(r_0) = 0 \\ h'(r_0) > 0 \end{array}} \tag{2.36}$$

2.3 For a traversable wormhole, no horizon should be present

Beyond the curvature singularities, there are also the coordinate singularities. They are singularities which we can eliminate by just a coordinate transformation. Event horizons are of these kind of singularities of the metric and it is widely known, that they are closely related with the black holes; so, in order to distinguish a black hole from a wormhole geometry, we demand that no horizon has to be present near the throat.

Our metric (2.1) has signature $(-,+,+,+)$. The crucial point of the signature is to distinguish the (one) time from the (three) spatial coordinates. The time coordinate corresponds to the negative sign, while the three spatial coordinates correspond to the three positive signs. These signs refer to the sign of the spacetime interval ds^2 along the direction of each coordinate. If, for example, we take $dr = d\Omega = 0$, then we have $ds^2 = -e^{2\alpha}dt^2 < 0$. Meaning, if we move along a worldline in which only the t-coordinate changes (i.e. the worldline of a static point in space), then the spacetime interval is negative. We then say that the t-coordinate is timelike. If we do the same thing for the other coordinates, we will ascertain that the spacetime interval is positive; these coordinates are spacelike. Similarly, null coordinates are those with a zero spacetime interval. It is hard to think about of a world with two or more time directions, while the phenomena that we want to describe are taking place in space with three dimensions. Hence, we ascribe to time only one dimension and to space three dimensions by having in the metric signature *always* one negative and three positive signs; that is, our metric has a *Lorentzian signature*. However, there is no restriction for the character of each coordinate. It is possible for a spacelike coordinate to be transformed to a timelike coordinate and vice versa. **The Lorentzian signature, although, cannot be changed.** This means that a change of character of some coordinate comes with a change of character of an other coordinate. For example, if the t-coordinate changes to be spacelike, then some of the three spatial coordinates have to be changed and be timelike. Actually, this happens beyond an event horizon. In the case of the Schwarzschild metric, we have the following diagonal components:

$$\begin{aligned} g_{rr} &= \frac{1}{1-\frac{R_s}{r}} \\ g_{tt} &= -\left(1-\frac{R_s}{r}\right) \end{aligned} \tag{2.37}$$

The event horizon is the sphere of radius $r = R_s$. So , for $r > R_s$, the tt component is negative ($g_{tt} < 0$), while the rr component is positive ($g_{rr} > 0$); the t-coordinate is timelike, while the r coordinate is spacelike. On the other hand, for $r < R_s$ the tt component is positive, which means that the t coordinate becomes spacelike and the rr component is negative, which means that the r coordinate becomes timelike. Hence, beyond the horizon the coordinates have changed their character, with the Lorentzian signature preserved. This changing of characters makes the r coordinate temporal, from which follows that the $r = 0$ point stands in the future of an incoming observer; this is, an explanation of why everything is doomed to fall into the singularity $r = 0$. The other interesting fact is that the metric is no longer static. Indeed, beyond the event horizon the Killing vector ∂_t is spacelike rather than timelike, which means that for $r < R_s$, no timelike killing vector is defined. So, it is not stationary, too. With this in mind, ***the restriction that prevents any event horizon preserves the demand of the first criterion.***

Identification of an event horizon

How do we identify an event horizon in a spherical symmetric and static metric? The answer to this question comes from the Indian physicist C.V. Vishveshwara in 1968 [9]. As it is clear even from the title of the paper, the event horizon of the Schwarzschild metric has been taken as the starting point and, through it, Vishveshwara makes the generalization to an arbitrary static and spherically symmetric metric. In order to make this generalization, we have to identify the main characteristics of this surface and then to define same surfaces for the metric (2.1). These characteristics are the infinite redshift that is observed at $r = R_s$ and the fact that this surface is a null surface.

Null surfaces in a spherically symmetric and static metric can be defined with the aim of the timelike killing vector (2.10), by searching for surfaces for which $K^\mu K_\mu = 0$ (Killing Horizon). Consider the family of surfaces defined by the scalar $K^\mu K_\mu = const$. For these surfaces, the normal vector is:

$$n_\mu = \frac{1}{2}\nabla_\mu(K^\nu K_\nu) \rightarrow n_\mu = (\nabla_\mu K_\nu)K^\nu \tag{2.38}$$

The above, combined with (2.14) and (2.4), gives for the length of the vector:

$$\begin{aligned} n_\mu n^\mu &= (\nabla_\mu K_\nu)K^\nu(\nabla^\mu K_\sigma)K^\sigma \\ &= K^\nu K_\sigma(\nabla_\mu K_\nu)(\nabla^\mu K^\sigma) \\ &= -(K^\nu K_\nu)(\nabla_\sigma K_\mu)(\nabla^\mu K^\sigma) - K^\nu K_\mu(\nabla_\nu K_\sigma)(\nabla^\mu K^\sigma) \\ &= (K^\nu K_\nu)(\nabla_\sigma K_\mu)(\nabla^\sigma K^\mu) - K^\nu(\nabla_\sigma K_\nu)K_\mu(\nabla^\sigma K^\mu) \end{aligned} \tag{2.39}$$

The last term in the right hand side is equal to $n_\mu n^\mu$. Hence, we take:

$$n_\lambda n^\lambda = \frac{1}{2}(K^\sigma K_\sigma)(\nabla_\nu K_\mu)(\nabla^\nu K^\mu) \tag{2.40}$$

Thence, $n_\lambda n^\lambda$ is proportional to $(K_\mu K^\mu)$, which means that from $(K_\mu K^\mu) = 0$ follows that $n_\lambda n^\lambda = 0$, too. Hence, *hypersurfaces defined by* $(K_\mu K^\mu) = 0$ *are null hypersurfaces.*

Let's see now how $(K_\mu K^\mu) = 0$ affects the redshift of a static observer. A static source or observer is defined by the following (normalized) 4-velocity:

$$v^\alpha = \frac{K^\alpha}{\sqrt{-K^\beta K_\beta}} \tag{2.41}$$

Then, for a light ray with a 4-velocity P^α, the observer with the above velocity measures a frequency given by:[10]

$$\nu = -v^\alpha P_\alpha \to \nu = -\frac{K^\alpha P_\alpha}{\sqrt{-K^\beta K_\beta}} \tag{2.42}$$

Labeling with "s" the rest frame of the source and with "o" the rest frame of the observer, we get the following ratio of the two frequencies:

$$\frac{\nu_o}{\nu_s} = -\frac{(K^\alpha P_\alpha)_o}{(K^\alpha P_\alpha)_s}\frac{(-K^\alpha K_\alpha)_s^{1/2}}{(-K^\alpha K_\alpha)_o^{1/2}} \tag{2.43}$$

The observer's and the source's worldlines are connected by the null geodesic of the light ray. So, $(K^\alpha P_\alpha)_o$ and $(K^\alpha P_\alpha)_s$ are calculated on the same null geodesic. But in the previous section we proved that $K^\mu P_\mu$ is conserved along a geodesic. Hence, $(K^\alpha P_\alpha)_o = (K^\alpha P_\alpha)_s$, leaving us with:

$$\frac{\nu_o}{\nu_s} = -\frac{(-K^\alpha K_\alpha)_s^{1/2}}{(-K^\alpha K_\alpha)_o^{1/2}} \tag{2.44}$$

So, it is obvious that the ratio of the frequencies is inversely proportional to $(-K^\alpha K_\alpha)_o^{1/2}$. Consequently, for an observer approaching a null hypersurface defined by $K^\mu K_\mu = 0$, this ratio approaches infinity; that is, an infinite redshift is adopted.

In that way, Vishveshwara concludes that the two basic properties of the Schwarzschild's event horizon are satisfied by the null hypersurfaces, defined by $K^\mu K_\mu = 0$ (Killing horizon), for a general spherically symmetric and static metric. Thence, event horizons are identified by setting the tt component of the metric equal to zero (see eq. (2.12)):

$$K^\mu K_\mu = g_{tt} = 0 \tag{2.45}$$

Exclusion of event horizons and the traversability problem

If we want to avoid any event horizon in our geometry we have to keep g_{tt} strictly negative or that the γ function has to remain finite everywhere:

$$\boxed{g_{tt} < 0 \;\to\; e^{2\gamma} > 0} \tag{2.46}$$

However, it is not necessary for the above requirement to be satisfied far away from the throat. For example, if the spacetime far away from the throat is asymptotically de Sitter, a cosmological horizon is present and g_{tt} vanishes. The problem lies at the horizons near the throat. Not only for the distinction of wormholes from black holes, but due to the traversability problems that event horizons produce.

When we refer to the traversability of the wormhole, we mean that if a particle (with a timelike trajectory) or a photon (with a null trajectory) crosses the throat and then is transferred from the one region to the other, it has to be able to come back. Event horizons, though, constitute the boundary of the spacetime, beyond which the points of the spacetime are not able to be connected with the infinity via null or timelike trajectories. In other words, if a particle or a photon passes the event horizon from the region of the spacetime connected with infinity, it is impossible to come back to the exterior from which it came. As is commonly said, an event horizon is a one-way membrane. But instead, we want a two-way traversable wormhole. So, *event horizons near the throat have to be excluded.*

2.4 The wormhole metric has to satisfy Einstein's field equations (The Stress Energy Tensor)

Of course, our metric has to satisfy the field equations of General Relativity. This means that we move in the context of the General Relativity and its equations. From Birkhoff's theorem, it is known that a spherically symmetric and static metric is the general solution of the Einstein's equation in vacuum; that is, for a vanishing stress-energy tensor (For Birkhoff's theorem search any textbook for GR).However, it is possible to get a spherically symmetric and static metric with a non-vanishing matter distribution. One of these cases is that of the wormhole. Moreover, the non-vanishing stress-energy tensor describes "exotic matter", as it is called in the case of matter distributions with a stress-energy tensor which violates the Null Energy Condition (NEC).

Two separate paths could be followed in order to solve the Einstein equations. The first one begins by the definition of $T_{\mu\nu}$, meaning the clear determination of the matter field (e.g. dust, electromagnetic fields etc.). Through this step, the specification of the geometry is guaranteed by the field equations; namely the metric to which the specific $T_{\mu\nu}$ is matched. The second path could start in reverse. Commencing from a specific geometry, which we aim (theoretically) to construct, and via the field equations, we find the necessary matter distribution. We pose as an ansatz the desired metric components, $g_{\mu\nu}$, and via the field equations the corresponding $T_{\mu\nu}$ is derived. This procedure should provide too bizarre matter fields, depending on the metric we have ascribed. Matter fields in which, for example, a negative energy density should arise. These matter fields are called "exotic", due to the fact that no classical form of matter can have this property and arise as necessary for the wormhole construction.

The constraint to $T_{\mu\nu}$ that the Einstein's equations demand

Einstein's field equations is the relativistic generalization of the Newtonian theory of gravity. Relativistic means that the form of the equations is tensorial, while generalization of the Newtonian gravity means that under some conditions we reduce to the Poisson equation of the gravitational field. These conditions include the limit of low velocities (in respect to the speed of light) and the assumption of weak and static fields. However, in contrast to the equations of Newtonian gravity, in the context of the General Relativity we are facing with non-linear differential equations, which are much more difficult to be solved. This set of non-linear differential equations that are produced contains magnitudes concerning the geometry of the spacetime (constructed by the metric), as well as magnitudes concerning the matter involved (the Stress-Energy tensor). So, they describe how the spacetime responds to the existence of matter and how the matter moves in respect to the spacetime geometry.

Despite this, the only constraint for the Stress-Energy tensor that the field equations demand, is that it is conserved. Before we proceed, let's see how this works. The Stress-Energy tensor is proportional to the Einstein tensor (1.3), which is constructed by the Ricci tensor and the Ricci scalar. Ricci scalar and tensor are constructed by contracting the Riemann tensor. One very important property of the Riemann tensor, which is actually our starting point for the proof, is the Bianchi identity[3]:

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0 \tag{2.47}$$

We first take the contraction with $g^{\mu\lambda}$, getting:

$$\nabla^\mu R_{\rho\sigma\mu\nu} + \nabla_\sigma R^\mu{}_{\rho\mu\nu} + \nabla_\rho R_\sigma{}^\mu{}_{\mu\nu} = 0 \tag{2.48}$$

Substituting the Ricci tensor $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ and using the anti-symmetry of the Riemann tensor to have $R_\sigma{}^\mu{}_{\mu\nu} = -R^\mu{}_{\sigma\mu\nu} = -R_{\sigma\nu}$, we take:

$$\nabla^\mu R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\rho\nu} - \nabla_\rho R_{\sigma\nu} = 0 \tag{2.49}$$

To the latter equation, we take the contraction with $g^{\nu\sigma}$:

$$\nabla^\mu R_{\rho\mu} + \nabla^\nu R_{\nu\rho} - \nabla_\rho R = 0 \tag{2.50}$$

For the first term we used the symmetry of the Riemann tensor in interchange of the indices of the first pair together with an interchange between the indices of the last pair, in order to take the Ricci tensor, i.e. $R_\rho{}^\nu{}_{\mu\nu} = R^\nu{}_{\rho\nu\mu} = R_{\rho\mu}$. Before that, we made use of the metric compatibility ($\nabla_\rho g_{\mu\nu} = 0$), which allows us to lower and raise indices of the Riemann tensor inside the covariant derivative, i.e. $g^{\nu\sigma}\nabla^\mu R_{\rho\sigma\mu\nu} = \nabla^\mu R_\rho{}^\nu{}_{\mu\nu}$. For the last term we made use again of the metric compatibility in order to raise the first index of the Ricci tensor, from which the Ricci scalar appeared. Finally, in the second term we just raised the index of the covariant derivative. Using now the symmetry of the Ricci tensor under interchange of its indices, we see that the first two terms are identical. Then, it is straightforward to gain the following equation:

$$\nabla^\nu R_{\nu\rho} - \frac{1}{2}\nabla_\rho R = 0 \tag{2.51}$$

Now it is time to go back at the Einstein tensor. If we take its divergence and make use of the metric compatibility again, we take:

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla_\nu R \tag{2.52}$$

Thus, it is obvious that the Einstein tensor is divergenceless, i.e. covariant conserved. Hence, the Stress-Energy tensor is covariant conserved, too:

$$\nabla^\mu T_{\mu\nu} = 0 \tag{2.53}$$

The above set of four differential equations is the relativistic generalization of the energy ($\nu = 0$) and momentum ($\nu = 1, 2, 3$) conservation, in curved spacetime.

Energy conditions and NEC violation

However, as we said, nothing beyond that is constrained for $T_{\mu\nu}$. If this is the case, for any metric a conserved energy-momentum tensor is obtained, without any physical feature to be tested. But of course, this is not the case. Physical criteria have been formulated in the shape of *energy conditions*, that "realistic" or classical forms of matter satisfy. There are several energy conditions. Some of them are: the Null Energy Condition (NEC), the Weak Energy Condition (WEC), the Strong Energy Condition (SEC) e.t.c. All of them are conditions about scalars (in order to be coordinate independent) constructed by the Energy-Momentum tensor and timelike or null vectors. In wormhole physics, we are mainly concerned of the NEC, because its violation at the throat is a characteristic feature. In addition, NEC violation implies violation of the other conditions, too.[2]

The NEC postulates:

$$\text{"For any null vector } k^\mu: \; T_{\mu\nu}k^\mu k^\nu \geq 0\text{"} \tag{2.54}$$

For a general spherically symmetric and static metric, the most general Stress-Energy tensor is anisotropic ($T^u{}_u \neq T^\theta{}_\theta$) and has the following form[5]:

$$T^\mu{}_\nu = diag(-\rho, p_1, p_2, p_2) \tag{2.55}$$

The choice of the mixed components is not accidental. Due to the diagonal form of the metric, the mixed components of any tensor are in some kind coordinate independent, in the sense that they are unchangeable under coordinate transformations that preserve the diagonal form of the metric. Therefore, the mixed terms are the same as if we had expressed them on any orthonormal (not just orthogonal) basis. Moreover, they are the same with those components expressed on the orthonormal basis that construct the *local Lorentz frame*; that is, the orthonormal basis for which the metric takes the form of the Minkowski one. So, as this corresponds to an observer remaining at rest, we can make the following interpretation of the components: ρ is the *proper energy density*, while $p_{1,2}$ are the tension per unit area and the radial pressure, respectively. Our goal is to examine how NEC constraints the possible relations between these components.

Take the following null vector:

$$k_1^\mu = (e^{-\gamma}, e^{-\alpha}, 0, 0) \tag{2.56}$$

It is easy to prove that this vector is null:

$$g_{\mu\nu}k_1^\mu k_1^\nu = -e^{2\gamma}e^{-2\gamma} + e^{-2\gamma}e^{2\gamma} = 0 \tag{2.57}$$

Let's see now what $T^\mu_\nu k_{1\mu} k_1^\nu$ gives. With a diagonal T^μ_ν, only $\mu = \nu$ terms survive, implying:

$$T^\mu{}_\nu k_{1\mu} k_1^\nu = \rho + p_1 \tag{2.58}$$

Take now the null vector $k_2^\mu = (e^{-\gamma}, 0, e^{-\beta}, 0)$ and do the same. What we get is the second relation:

$$T_{\mu\nu}k_2^\mu k_2^\nu = \rho + p_2 \tag{2.59}$$

Hence, NEC implies that for the Stress-Energy components two conditions must be fulfilled, which we call NEC1 and NEC2, respectively:

$$\boxed{\rho + p_{1,2} \geq 0} \tag{2.60}$$

It's time to become more precise. For the metric (2.1) the Stress-Energy components are the following:

$$\begin{aligned} T^t_{\ t} &= -\rho = \frac{1}{8\pi}\left[-e^{-2\beta} + e^{-2\alpha}\left(-2\alpha'\beta' + 3(\beta')^2 + 2\beta''\right)\right] \\ T^r_{\ r} &= p_1 = \frac{1}{8\pi}\left[e^{-2\alpha}\left(-e^{2(\alpha-\beta)} + (\beta')^2 + 2\beta'\gamma'\right)\right] \\ T^\theta_{\ \theta} &= T^\phi_{\ \phi} = p_2 = \frac{1}{8\pi}\left[e^{-2\alpha}\left((\beta')^2 + \beta'\gamma' - \beta'\alpha' + \beta'' + (\gamma')^2 - \gamma'\alpha' + \gamma''\right)\right] \end{aligned} \tag{2.61}$$

where prime denotes d/du. Hence, NEC1 implies:

$$\rho + p_1 = \frac{e^{-2\alpha}}{4\pi}\left[\beta'\alpha' - (\beta')^2 + \beta'\gamma' - \beta''\right] \geq 0 \tag{2.62}$$

Remember the constraints of the β function, necessary for the throat existence. For $u = 0$ (throat) and (2.20), we get:

$$\rho + p_1 = -\frac{e^{-2\alpha}\beta''}{4\pi} \tag{2.63}$$

In the view of (2.20) the above is clearly negative. A condition that obviously implies violation of the NEC1. Thus, ***the existence of a throat to our geometry implies NEC violation, which consequently means that no classical form of matter is able to construct a wormhole. "Exotic" matter is unavoidable.*** *NOTE*: If the throat was null; that is, if the throat was an event horizon, NEC would not be violated. To see this, enforce the quasiglobal coordinated for which $\alpha = -\gamma$. For an event horizon $\gamma \to -\infty$. Thus, $\alpha \to +\infty$. So, if at the throat we have an event horizon $\rho + p_1 = 0$ and the NEC is not violated.

In the next section, we will see that this violation has as consequence that very fast observers measure a negative energy density at the throat. Moreover, for the radial pressure, it is straightforward to see that:

$$p_1 = -\frac{1}{8\pi r_0^2} \tag{2.64}$$

Hence, the radial pressure is necessarily negative at the throat.

2.5 Morris and Thorne metric

Let's proceed now to the specific metric that Morris and Thorne proposed in their paper [4], which is the standard form of a wormhole metric in literature. It is written down in terms of the Gaussian coordinates (see [5]), in which, we have $e^a \equiv 1$. Then, the coordinate u is labeled as l and is called as the *proper radial coordinate*, while the metric has the following form [2]:

$$\boxed{ds^2 = -e^{2\Phi(l)}dt^2 + dl^2 + r^2(l)d\Omega^2} \tag{2.65}$$

The function $\Phi(l)$ is the previously noted as γ- function. It is called the *redshift function*, as it determines the redshift between static observers and sources. This is clear from the section [2.3]. Hence, **$\Phi(l)$ has to be finite near the throat**.(for more details see section [2.3])

The shape of the wormhole is determined by the specification of the relation between r and l. In section [2.2], we saw that in the case of $\alpha \equiv 0$, this relation is determined from the derivative of l in respect to r and specifically to have the form of (2.32). In literature the function $h(r)$ is taken to be:

$$h(r) = 1 - \frac{b(r)}{r} \tag{2.66}$$

From the mathematics of embedding, we saw that the embedded diagram is specified from (2.25). Substituting to the latter equation, the $e^{\alpha} = 1$ ansatz and (2.32) in terms of $b(r)$, we get:

$$\frac{dz}{dr} = \pm \left(\frac{r}{b(r)} - 1 \right)^{-1/2} \tag{2.67}$$

Hence, exact specification of the $b(r)$ function, provides exact specification of the shape of the wormhhole. Due to that, $b(r)$ is called *the shape function.*

Using the chain rule $dl = \frac{dl}{dr}dr$, we can write the metric (2.65) in terms of the r-coordinate:

$$\boxed{ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2} \tag{2.68}$$

Notice that we wrote $\Phi = \Phi(r)$ rather than $\Phi = \Phi(l)$. This can be done because the function $r = r(l)$ is reversible, which means that $l = l(r)$ is defined, too. Hence, dependence on l can be replaced with dependence on r.

The throat is at $l = 0$, in which the nonzero minimum, r_0, of $r = r(l)$ is placed. Thus,

$$l(r) = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{1 - \frac{b(r')}{r'}}} \tag{2.69}$$

Moreover, at the throat $h(r)$ is constrained by (2.36), which implies for $b(r)$:

$$\boxed{\begin{aligned} &b(r_0) = r_0 \\ &b'(r_0) < 1 \end{aligned}} \tag{2.70}$$

These are the necessary constraints for $b(r)$, in order to describe the wormhole by this metric. Notice, that the condition (2.70) makes the g_{rr} component to blow up to infinity at the throat. This infinity does not denote any horizon (the identification of horizons concerns g_{tt}) or a curvature singularity. It is a coordinate singularity and nothing beyond that; meaning, our metric does not behave well near the throat using the r-radial coordinate. The problem is solved if we transform to the proper radial coordinate.

In addition, we have to mention that in order to the radial distance (2.69) be well defined, there must be this additional constraint:

$$b(r) \leq r \;, \forall r \tag{2.71}$$

where the equality holds for $r = r_0$. In the section [2.2] we proved that in the case of $\alpha = 0$ the flaring-out condition is guaranteed by the constraints of the $r = r(u)$ function. For this metric, as we said, we work in the Gaussian radial coordinate; that is, by definition

$e^\alpha \equiv 1 \rightarrow \alpha \equiv 0$. So, we claim that the flaring out condition is fulfilled by (2.70). Indeed, if we calculate d^2r/dz^2 from (2.67), using the chain rule again, we take:

$$\frac{d^2r}{dz^2} = \frac{b(r) - rb'(r)}{2b^2(r)} \tag{2.72}$$

It is straightforward that the above is positive at the throat, as it provides exactly the same constraint with (2.70).

In the previous chapter, we expressed the Stress-Energy tensor components in terms of the α, β, γ functions of the metric (2.1). For the metric of Morris and Thorne we have replaced γ with Φ, while the α function has been taken to zero. What is left over is the β function, which obviously can be derived from dr/dl. However, there is no need for this messy work. We are able to express the Stress-Energy components in respect of $r = e^\beta$ rather than l. Remember, though, that primes in (2.61) denote derivatives in respect to l (or u, it's the same). These derivatives have to be changed to derivatives of r, using the chain rule. What we get is:

$$\begin{aligned} \beta' &= \pm\frac{1}{r}\sqrt{1-\frac{b}{r}} \\ \beta'' &= \frac{3b - rb' - 2r}{2r^3} \\ \gamma' &= \pm\sqrt{1-\frac{b}{r}}\frac{d\Phi}{dr} \\ \gamma'' &= \left(1-\frac{b}{r}\right)\left[\frac{b-rb'}{2(r^2-rb)}\frac{d\Phi}{dr} + \frac{d^2\Phi}{dr^2}\right] \end{aligned} \tag{2.73}$$

Substituting these to (2.61), we get the following equations of structure:

$$\boxed{\begin{aligned} \rho &= \frac{b'}{8\pi r^2} \\ p_1 &= -\frac{1}{8\pi}\left[\frac{b}{r^3} - 2\left(1-\frac{b}{r}\right)\frac{1}{r}\Phi'\right] \\ p_2 &= \frac{1}{8\pi}\left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 + \frac{b-b'r}{2r^2(1-b/r)}\Phi' + \frac{1}{r}\Phi' - \frac{b'r-b}{2r^3(1-b/r)}\right] \end{aligned}} \tag{2.74}$$

where primes now denote derivatives in respect to r. Sometimes in literature (like in [8]) is is preferable to use the radial tension $\tau(r)$ rather than the radial pressure $p_1(r)$, where $\tau(r) = -p_1(r)$.

It is clear now what we meant at the beginning of this chapter about the path of solving the Einstein's equations for a wormhole. We argued that we first specify our desired geometry and then we find the matter field that is needed. Indeed, if we observe the equations above, it is obvious that specification of the functions $b(r)$ and $\Phi(r)$, which are related with the geometry of the wormhole, automatically specify the energy density and the pressures of the matter field.

With $b(r_0) = r_0$ at the throat, from the above expression of p_1, it is easy to verify that indeed the radial pressure is negative and equal to $-(8\pi r_0^2)^{-1}$. Look now at the energy density. Its sign depends on the sign of $b'(r)$. The only constraint that we have at the throat about $b'(r_0)$, is given by (2.70). So, as there is no restriction for the sign of $b'(r_0)$, there is no restriction for the sign of the energy density. It can be either negative or positive.

Negative energy density

However, this is true for our static observer at the throat. What about a traveler who cross the throat with a non-zero velocity $\vec{v}$?[1] In order to answer this question, we have to apply a Lorentz boost to the Stress-Energy tensor and see how it looks like for this boosted frame. For a boost to the radial direction we have the following matrix:

$$(\Lambda^{\mu}{}_{\nu}) = \begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.75}$$

where $\gamma = (1-v)^{1/2}$ and v is the traveler's velocity. We can calculate the inverse boost by $\Lambda_{\mu}{}^{\nu} = \eta_{\mu\tilde{\mu}}\eta^{\nu\tilde{\nu}}\Lambda^{\tilde{\mu}}{}_{\tilde{\nu}}$, with $\eta_{\mu\nu}$ be that of a Minkowski spacetime. What we get is:

$$(\Lambda_{\mu}{}^{\nu}) = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.76}$$

Pay attention here. The above Lorentz boost is defined for a flat spacetime, rather than curved. Mathematically, is defined in such a way that the Minkowskian metric is invariant under these transformations, i.e. $\Lambda_{\mu}{}^{\tilde{\mu}}\Lambda_{\nu}{}^{\tilde{\nu}}\eta_{\tilde{\mu}\tilde{\nu}} = \eta_{\mu\nu}$. So, these transformations are not defined for the metric (2.1). In order to apply this Lorentz boost, we have to change basis and go to the orthonormal basis in which the metric is equal to $\eta_{\mu\nu}$; that is, the local Lorentz frame. (A capability provided by the Equivalence Principle). However, as we said in section [2.4] the mixed components of any tensor are the same as if we were in this orthonormal basis. Thus, we can apply the Lorentz boost to the mixed components of the Stress-Energy tensor. If $T'^{\mu}{}_{\nu}$ are the boosted components, then:

$$T'^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\tilde{\mu}}\Lambda_{\nu}{}^{\tilde{\nu}}T^{\tilde{\mu}}{}_{\tilde{\nu}} \tag{2.77}$$

The energy density corresponds to the purely temporal component, T'^{0}_{0}. What we get is:

$$\begin{aligned} -\rho' &= T'^{0}{}_{0} = \Lambda^{0}{}_{\tilde{\mu}}\Lambda_{0}{}^{\tilde{\nu}}T^{\tilde{\mu}}{}_{\tilde{\nu}} \\ \rightarrow \rho' &= \gamma^2(\rho + v^2 p_1) \end{aligned} \tag{2.78}$$

Hence, for sufficiently high velocities ($v \rightarrow 1$) the energy density ρ' tends to $\gamma^2(\rho + p_1)$, which is negative, as the NEC1 violation implies (see eq.(2.58)). Thus, a negative energy density will be measured by these travellers.

[1]This is not a traversability condition for a human being. The purpose of this paragraph is to designate the consequences of NEC violation.

The simplest example of a Morris and Thorne wormhole

The simplest example of a traversable wormhole is given by Morris and Thorne in the box 2 of [4], with a metric:

$$ds^2 = -dt^2 + dl^2 + (r_0^2 + l^2)d\Omega^2 \tag{2.79}$$

Comparing with (2.68), it is obvious that we are in Gaussian coordinates, while for the redshift function, we have $\Phi = 0$. The function $r(l)$ is:

$$r(l) = \sqrt{r_0^2 + l^2} \tag{2.80}$$

or

$$\frac{dl}{dr} = \pm\frac{1}{\sqrt{1 - \frac{r_0^2}{r^2}}} \tag{2.81}$$

Hence, we speak for a shape function:

$$b(r) = \frac{r_0^2}{r} \tag{2.82}$$

The constraint of no horizons is obviously fulfilled, as the redshift function is constant and zero. What remains are the constraints of the shape function. The first constraint (2.70) it is satisfied, easily checked by just a substitution of $r = r_0$ to (2.82). The first derivative of the shape function is:

$$b'(r) = -\frac{r_0^2}{r^2} \tag{2.83}$$

Hence, for $r = r_0$, we get:

$$b'(r_0) = -1 \tag{2.84}$$

So, (2.70) is satisfied, while in view of the (2.74), we see that in this case the energy density is negative at the throat, even for the static observer standing there.

2.6 The causal structure of a traversable wormhole

In this section we proceed to the construction of the Penrose diagram for the metric (2.65). In general, the Penrose diagram of some metric is an attempt to draw in a finite piece of paper the entire spacetime, preserving the causal relations between the events (spacetime points). The causal relations are not determined, although, by the whole metric, but instead from the dt and dl part. We constrain then the angular part by taking $d\Omega = 0$, leaving:

$$ds^2 = -e^{2\Phi(l)}dt^2 + dl^2 \tag{2.85}$$

where $t, l \in (-\infty, +\infty)$. Under this constraint when we will draw the corresponding diagram each point on it it would represent a 2D sphere of radius $r(l)$, rather than a point.

Even from Special Relativity we know that the causal relation between events is determined by angles in the spacetime diagram. To be more precise, we know that if we attach a point in the spacetime diagram, the events that are in causal relation with it are those lying inside the past or future lightcone. For a Minkowski spacetime the null paths that shape the lightcones at every point are lines of 45^0 degrees. This feature of the Minkowski spacetime is very useful due to the fact that the causal relations between points is easily identified. So, if we are able to make the above metric look like the Minkowski metric the causal relations would be easily identified. The other main goal is to make the range of the coordinates finite in order to draw the entire spacetime in a finite portion of paper.

The above considerations will be accomplished by successive coordinate transformations, which we will present in steps. The key transformation, although, is not a coordinate transformation; is the so called *conformal transformation* and is the key transformation because it preserves angles. So, by this kind of transformation we will be able to "compactify" the coordinates, while preserving the casual relations, at the same time. For this reason we also refer for these diagrams as *conformal diagrams*.
[Conformal Transformation] *Given a spacetime M with metric $g_{\mu\nu}$, we may construct a new metric $\tilde{g}_{\mu\nu}$ by a conformal transformation:*

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{2.86}$$

where $\Omega(x)$ a smooth, non-vanishing function.

Let's start. Before we proceed to the steps, we define the connected spacetime regions as: region 1 for $l < 0$ and region 2 for $l > 0$.

- **Change the radial coordinate:** Make a coordinate transformation to the metric (2.85), by changing the l radial coordinate to the $l_\star$, related by:

$$\frac{dl_\star}{dl} = e^{-\Phi(l)} \tag{2.87}$$

  Then the metric becomes (this is the so called tortoise radial coordinate):

$$ds^2 = e^{2\Phi(l)} \left(-dt^2 + dl_\star^2\right) \tag{2.88}$$

  The derivative of $l_\star$ with respect to l is strictly positive as $\Phi(l)$ is a finite function of l. This means, that with an appropriate constant of integration we can set $l = 0$ matching with $l_\star = 0$. In this way, we identified the connected regions of spacetime and the throat in the new radial coordinate as follows:

$$\boxed{\begin{array}{l} l_\star = 0 : throat \\ l_\star < 0 : region1 \\ l_\star > 0 : region2 \end{array}}$$

- **Make the first conformal transformation:** Here, is the first time to make a conformal transformation. As we know $\Phi(l)$ is a finite function, which means that we can choose the Ω function of the conformal transformation to be equal with $e^{-\Phi(l)}$. Hence, our metric will be conformally mapped to the following one:

$$ds^2 \sim -dt^2 + dl_\star^2 \tag{2.89}$$

  where $\sim$ denotes that the metric in the right hand side is a conformal map of the initial metric. Our metric has already the form of (1+1) Minkowski spacetime. So, the next steps can be assumed as the steps for the construction of the Penrose diagram of the (1+1) Minkowski spacetime.

- **Transform to a pair of null coordinates:** We transform now to the null coordinates (v, w), defined by:

$$\begin{aligned} t &= \frac{1}{2}(v + w) \\ l_\star &= \frac{1}{2}(v - w) \end{aligned} \tag{2.90}$$

Then we get:

$$\boxed{\begin{array}{l} ds^2 \sim -dvdw \\ v = w : throat \\ v < w : region1 \\ v > w : region2 \\ v, w \in (-\infty, +\infty) \end{array}} \tag{2.91}$$

- **"Compactify" the coordinates:** The above two coordinates are still ranging from minus to plus infinity. Now it is time to compactify them, by making the following coordinate transformation:

$$\begin{aligned} v &= tan(p) \\ w &= tan(q) \end{aligned} \tag{2.92}$$

With these coordinates the $\pm\infty$ of the (v, w) coordinates is mapped to $\pm\pi/2$ of the (p, q) coordinates, having:

$$\boxed{\begin{array}{l} ds^2 \sim -\dfrac{dpdq}{cos^2(p)cos^2(q)} \\ p = q : throat \\ p < q : region1 \\ p > q : region2 \\ p, q \in (-\pi/2, +\pi/2) \end{array}} \tag{2.93}$$

- **One more conformal transformation:** As $cos(x)$ is a non-vanishing function in the interval $(-\pi/2, \pi/2)$, we can make again a conformal transformation, having $\Omega = cos^2(p)cos^2(q)$. This leaves us with the following metric:

$$ds^2 \sim -dpdq \tag{2.94}$$

- **Transform back to timelike and spacelike coordinates:** Let's make the final coordinate transformation, in order to make the metric to have the form of Minkowski spacetime. Define two coordinates (T, X) as:

$$\begin{aligned} T &= \frac{1}{2}(p + q) \\ X &= \frac{1}{2}(p - q) \end{aligned} \tag{2.95}$$

So, we get the following metric:

$$ds^2 \sim -dT^2 + dX^2 \tag{2.96}$$

which is like that of (1+1) Minkowski spacetime, with T having the role of the time coordinate, while X that of the spatial coordinate.
From (2.95) we can easily identify the regions and the boundaries as follows:

$$\boxed{\begin{aligned}&X = 0 : throat\\ &X < 0 : region1\\ &X > 0 : region2\\ &-\pi < X, T < +\pi\end{aligned}} \tag{2.97}$$

Putting T on the vertical axis and X on the horizontal axis, we get a diamond of equal sides that represents the whole spacetime as pictured in figure 2.2. This graph represents the causal structure of a traversable wormhole, which however is the same as that of a (1+1) Minkowski spacetime, but with a different interpretation. The throat which is the $X = 0$ vertical line is a timelike hypersurface which can be crossed by any timelike or null trajectory from each side. If we take two arbitrary points from each side and draw their future lightcones (see figure 2.2) we see that these lightcones intersect both regions. This means that the two regions are causal related; that is, signals can travel from region 1 to region 2 and vice versa. This is the situation for a traversable wormhole. If we go back to the discussion of why a black hole is not a traversable wormhole, now it is more clear why it's not. No future lightcone from a point at region IV intersects the region I and vice versa. Regions IV and I in figure (1.1) are not causally connected.

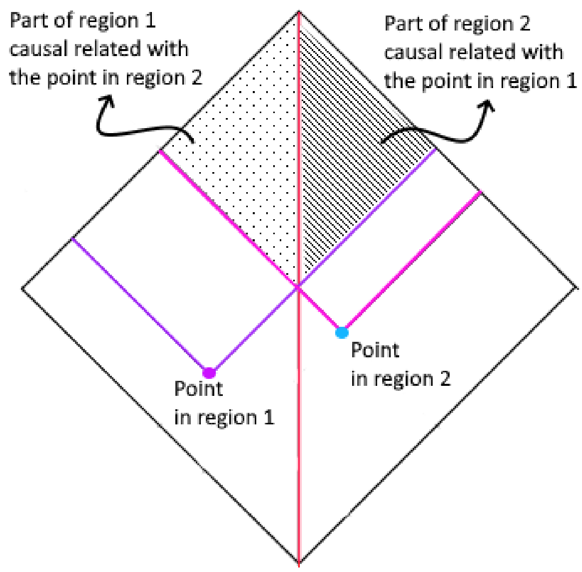

Figure 2.2: The causal structure of a traversable wormhole. Each point in the diagram correspond to a 2D sphere.

Chapter 3

The Simpson - Visser Traversable Wormholes

A few months ago, Simpson and Visser published a paper [6] in which they propose a candidate metric for a regular black hole. In the same vein of Einstein and Rosen, the regularization of the black holes is crucial in General Relativity, with the history of this research coming by Bardeen in 1968 until today[1]. When we refer to regularization, we refer to the special treatment of the curvature singularity at the centre of the black holes. As we previously mentioned, this can be done in two ways. We can either state some specific conditions for the centre of the spacetime in order to be regular or we can construct a spacetime without centre, i.e. to exclude the centre out of the spacetime. The second way is already known by the construction of the wormholes that we previously studied. The existence of the throat in the wormhole geometry leaves out the centre ($r = 0$) of the spacetime and this particular singular point is no longer part of the spacetime. In this way, we overcome the problems that the singular points introduce to the theory. However, by doing that, an other issue appears. The exclusion of the singular point by the introduction of a throat comes up with the necessity of exotic matter.

This is true not only for the wormholes but for the regular black holes, too. As is shown in the paper of Simpson and Visser the regularization of the metric implies NEC violation through all of the spacetime except for the possible horizons. What we are going to see in this chapter is the technique of Simpson and Visser in order to regularize the Schwarzschild metric by the introduction of some parameter η. Specifically, on the values of this parameter depends the kind of the geometry that the metric describes, starting from the original Schwarzschild black hole to a traversable wormhole. The intermediate "states" are those of a regular black hole and a one way traversable wormhole. Of course, as this thesis concerns wormholes we will emphasize in how we can construct traversable wormholes using this technique.

Firstly, we present the technique in its general state; that is, by starting from an arbitrary spherical symmetric black hole metric and then we state how we regularize this metric in order to construct a traversable wormhole. This generalization allows us to extend the technique of Simpson and Visser to more spacetimes rather than the Schwarzschild one. Namely, we extend this procedure by introducing a cosmological constant and charge

[1] For references see [6]

(Reissner–Nordström). To these specific examples, we check the regularity of the spacetime by checking mainly the components of the Riemann tensor, we check the asymptotic behaviour far from the throat and we see how the NEC is violated in each case. For the asymptotic behaviour we see that in the case of a cosmological constant the two connected regions of spacetime are asymptotically dS or AdS.

3.1 The technique of Simpson and Visser

We can think of the technique of Simpson and Visser as a procedure, which has a spherical symmetric black hole as an input and a traversable wormhole as an output. We start with a metric of the following form:

$$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2 d\Omega^2 \tag{3.1}$$

where the coordinates are running through the following intervals:

$$r \in (0, +\infty),\ t \in (-\infty, +\infty),\ \theta \in [0, \pi],\ \phi \in (0, 2\pi]$$

This particular choice of metric is not accidental. It is a form of a metric that includes the well known metrics of Schwarzschild, Reissner–Nordström with or without a cosmological constant. So, by describing the technique of Simpson and Visser over this metric we can extend this idea beyond the Schwarzschild metric. Keeping the characteristics of these black holes, we assume for the above metric to have an event horizon at r_h, meaning $F(r_h) = 0$, while, also, that it haves a singular center $r = 0$.

The crucial step to this technique is to make the function $F(r)$ not depending just to r, but instead to $\sqrt{r^2 + \eta^2}$. Be careful! This is not a coordinate transformation; it is just an ansatz substitution, by which we introduce a new parameter η. Having now the function $F\left(\sqrt{r^2 + \eta^2}\right)$ it is obvious that we can extend the range of the variable r to cover all of the real axis, having an even function and consequently a symmetric metric for $r < 0$ and $r > 0$.

Notation attention! Under this extension of the range of the initial variable r we change notation and from now on we speak of a variable $u \in (-\infty, +\infty)$ rather than r. This is made just for convenience with the previous discussion, leaving r to denote just the radius of the $(t = const, u = const)$ 2D coordinate spheres. Hence, we take the following metric as an output:

$$\boxed{ds^2 = -F\left(\sqrt{u^2 + \eta^2}\right) dt^2 + \frac{1}{F\left(\sqrt{u^2 + \eta^2}\right)} du^2 + r^2 d\Omega^2} \tag{3.2}$$

where the coordinates are running through the following intervals:

$$u \in (-\infty, +\infty),\ t \in (-\infty, +\infty),\ \theta \in [0, \pi],\ \phi \in (0, 2\pi]$$

Comparing with metric (2.1) of the previous chapter, we speak of a metric written down in the quasiglobal coordinate with:

$$\alpha(u) = -\gamma(u) = -\frac{1}{2} \ln F\left(\sqrt{u^2 + \eta^2}\right) \tag{3.3}$$

Moreover, we impose a relation between u and r of the form:

$$\boxed{r^2 = u^2 + \eta^2} \tag{3.4}$$

This relation, under the prism of the previous discussion about wormholes, can be understood as a constraint for r, that makes it to have a minimum at $u = 0$, $r_{min} = \eta$; that is, η corresponds to the throat of the wormhole, that we want to construct. Actually, this is exactly the same relation with the one we imposed in the simplest example of a Morris and Thorne wormhole with metric (2.79). However, the latter metric is more trivial than (3.2), with the main difference that in (2.79) we speak of a vanishing redshift function. In this case, we can speak of a non vanishing redshift function, given by $\gamma(u)$ in (3.3). Thence, we can see the metric (3.2) as a combination of the initial-input metric and the trivial wormhole example of Morris and Thorne.

In the previous chapter, we saw that for a wormhole to be traversable in principle two conditions are necessary to be satisfied. The first one is the necessary absence of any event horizon; a condition that we call as *no horizon condition*, while the second one is the so called *flaring out condition*, which implies that the throat of the wormhole must be open. Let's see how these conditions are implied for this metric.

The no horizon condition

With the aforementioned substitutions, we introduced the parameter η in the metric, with which the metric becomes regular all along the interval of u. Firstly, if $\eta = 0$ the spacetime has center, because r can reach the zero value. Secondly, it has a singular center at $|u| = r = 0$. In other words, taking $\eta \neq 0$, it is a way to avoid the geometrical singularity at the center of the spacetime by avoiding this center. Although, beyond that, we have to avoid any event horizon of the new metric, too. If the exclusion of the center of the spacetime is done by simple imposing $\eta \neq 0$, the exclusion of any event horizon is not accomplished for any non-vanishing value of η.

If r_h is the event horizon of (3.1), then $F(r_h) = 0$. Denoting with u_h any possible event horizon of the metric (3.2), we have that $F(\sqrt{u_h^2 + \eta^2}) = 0$, which means that:

$$u_h = \pm\sqrt{r_h^2 - \eta^2} \tag{3.5}$$

as by definition $F(r_h) = 0$. Hence, u_h cannot be defined if the throat radius, η, is bigger than r_h. Thus, we conclude that the no horizon condition constraints the parameter η as follows:

$$\boxed{\begin{gathered}\eta > r_h \\ \text{where } r_h \text{ is the event horizon of the initial metric.}\end{gathered}} \tag{3.6}$$

The flaring out condition

Previously, we had a discussion for the flaring out condition for which we concluded that this is satisfied from the appropriate relation between u and r ab initio. However, we prove that in the case of the Gaussian coordinates for which we have a vanishing α of the general metric (2.1). Here, this is not the case as we work in the quasiglobal coordinate with $\alpha(u)$ given by (3.3). So, we have to see what we get for this case.

The $u(r) = \pm\sqrt{r^2 - \eta^2}$ function is of course reversible, giving us $r = \sqrt{u^2 + \eta^2}$. Hence, in respect with r, for the α function we have $a(r) = -\frac{1}{2}\ln F(r)$. Substituting these to the general expression of (2.28), we take:

$$\frac{d^2r}{dz^2} = \frac{r\left(2\eta^2 F(r) + r(r^2 - \eta^2)F'(r)\right)}{2\left(r^2 - (r^2 - \eta^2)F(r)\right)^2} \tag{3.7}$$

The throat is located at $r = \eta$, giving us:

$$\frac{d^2r}{dz^2}|_{throat} = \frac{F(\eta)}{\eta} \tag{3.8}$$

By definition we have $\eta > 0$. Moreover, remember that the tt component of the metric (3.2) is $g_{tt} = -F(\sqrt{u^2 + \eta^2})$. Thence, **the flaring out condition imposes a constraint for g_{tt} at the throat**:

$$\boxed{(g_{tt})_{throat} < 0} \tag{3.9}$$

A constraint about g_{tt} is close related with horizons. So, it seems that the flaring out condition is closely related with the no horizon condition. Remember our discussion in chapter 2 about the event horizons. We said that they are the $g_{tt} = 0$ hypersurfaces for some $r = r_h$, which diverge the metric in two pieces. The static piece in which we have $g_{tt} < 0$ for $r > r_h$ and the non static piece for which we have $g_{tt} > 0$ for $r < r_h$. So, **the throat is constrained by the flaring out condition to be in the static piece strictly**. Moreover, consider two facts: (a) As we said in the previous chapter the exclusion of any event horizon preserves the metric to be strictly static; that is, $g_{tt} < 0$ everywhere. (b) The throat is the minimum radius r of the geometry, which means that the constraint (3.9) constraints g_{tt} to be negative everywhere.[2] So, **the no horizon condition and the flaring out condition are too closely related, with the one implying the other**.

The Stress-Energy Tensor and the NEC violation

Our goal for this paragraph is to see how the NEC is violated, according to the general metric (3.2).

For a Stress-Energy tensor in the form of (2.55), the calculations give us:

$$\begin{aligned}
\rho &= -\frac{(2\eta^2 + u^2)F(\sqrt{u^2+\eta^2}) + (u^2+\eta^2)(uF'(\sqrt{u^2+\eta^2}) - 1)}{8\pi(u^2+\eta^2)^2} \\
p_1 &= \frac{u^2 F(\sqrt{u^2+\eta^2}) + (\eta^2+u^2)(uF'(\sqrt{u^2+\eta^2}) - 1)}{8\pi(u^2+\eta^2)^2} \\
p_2 &= \frac{2\eta^2 F(\sqrt{u^2+\eta^2}) + (\eta^2+u^2)(2uF'(\sqrt{u^2+\eta^2}) + (\eta^2+u^2)F''(\sqrt{u^2+\eta^2}))}{16\pi(u^2+\eta^2)^2}
\end{aligned} \tag{3.10}$$

where prime denotes derivative in respect with u.

[2] We are not concerned in this discussion about cosmological horizons that appear in the case of a negative cosmological constant.

For NEC1, we get:

$$\begin{aligned}\rho + p_1 &= -\frac{\eta^2 F(\sqrt{u^2+\eta^2})}{4\pi(\eta^2+u^2)^2} \\ \rightarrow \boxed{\rho + p_1 = \frac{\eta^2}{4\pi(\eta^2+u^2)^2} g_{tt}}\end{aligned} \tag{3.11}$$

In the previous discussion we saw that g_{tt} has to be negative everywhere. So, it is obvious that NEC is violated not only at the throat, but instead, through out all of the spacetime.

However, in the case that we introduce a positive cosmological constant, a cosmological horizon will appear far from the throat. Then, this will make g_{tt} to reach the zero value which means that NEC1 is not anymore violated. So we conclude that **at the cosmological horizon the NEC1 is not violated.**

NEC beyond the cosmological horizon

Beyond the cosmological horizon where $g_{tt} > 0$, our metric is not static and the t coordinate is spacelike, while the u coordinate timelike. This swapping of the timelike/spacelike character between the t and u coordinate affects the interpretation of the Energy-Momentum components. Instead of (2.55), beyond the cosmological horizon we have:

$$T^{\mu}{}_{\nu} = diag(p_1, -\rho, p_2, p_2) \tag{3.12}$$

giving us:

$$\begin{aligned} p_1 &= \frac{(2\eta^2+u^2)F(\sqrt{u^2+\eta^2}) + (u^2+\eta^2)(uF'(\sqrt{u^2+\eta^2})-1)}{8\pi(u^2+\eta^2)^2} \\ \rho &= -\frac{u^2 F(\sqrt{u^2+\eta^2}) + (\eta^2+u^2)(uF'(\sqrt{u^2+\eta^2})-1)}{8\pi(u^2+\eta^2)^2} \\ p_2 &= \frac{2\eta^2 F(\sqrt{u^2+\eta^2}) + (\eta^2+u^2)(2uF'(\sqrt{u^2+\eta^2}) + (\eta^2+u^2)F''(\sqrt{u^2+\eta^2}))}{16\pi(u^2+\eta^2)^2}\end{aligned} \tag{3.13}$$

which yields:

$$\begin{aligned}\rho + p_1 &= \frac{\eta^2 F(\sqrt{u^2+\eta^2})}{4\pi(\eta^2+u^2)^2} \\ \rightarrow \boxed{\rho + p_1 = -\frac{\eta^2}{4\pi(\eta^2+u^2)^2} g_{tt}}\end{aligned} \tag{3.14}$$

for the region beyond the horizon where $g_{tt} > 0$. Hence, we can combine (3.11) and (3.14) in one expression which holds through out all of the spacetime:

$$\boxed{\rho + p_1 = -\frac{\eta^2}{4\pi(\eta^2+u^2)^2}\,|g_{tt}| \quad \forall u \in (-\infty, +\infty)} \tag{3.15}$$

Thus, **the NEC is violated through out all of the spacetime, except for any possible horizon, where $g_{tt} = 0$ and then $\rho + p_1 = 0$.**

3.2 From Schwarzschild black hole to traversable wormhole

In this case, we have as an input metric that of the Schwarzschild given in (1.1). Hence, the output metric is[3]:

$$\boxed{\begin{aligned}&ds^2 = -\left(1-\frac{2M}{\sqrt{u^2+\eta^2}}\right)dt^2 + \frac{du^2}{1-\frac{2M}{\sqrt{u^2+\eta^2}}} + r^2 d\Omega^2\\ &\text{where } r^2 = u^2+\eta^2 \text{ and } u,t \in (-\infty,+\infty)\end{aligned}} \tag{3.16}$$

Of course, the event horizon of (1.1) is located at $r_h = 2M$. So, the no horizon condition implies:

$$\boxed{\eta > 2M} \tag{3.17}$$

It is obvious that: $1 - 2M/\eta > 0\ ,\forall \eta > 2M$; that is, the flaring out condition is fulfilled.

Regularity

For the components of the Riemann tensor we have:

$$\begin{aligned}R^{tu}{}_{tu} &= -\frac{M(\eta^2-2u^2)}{\eta^2+u^2} & R^{t\theta}{}_{t\theta} = R^{t\phi}{}_{t\phi} &= -\frac{Mu^2}{(u^2+\eta^2)^{5/2}}\\ R^{\theta\phi}{}_{\theta\phi} &= \frac{2Mu^2+\eta^2\sqrt{\eta^2+u^2}}{(u^2+\eta^2)^{5/2}} & R^{u\theta}{}_{u\theta} = R^{u\phi}_{u\phi} &= \frac{2M\eta^2 - Mu^2 - \eta^2\sqrt{u^2+\eta^2}}{(u^2+\eta^2)^{5/2}}\end{aligned} \tag{3.18}$$

Giving:

$$\boxed{\mathcal{K} = \frac{4\left[8M\eta^2(u^2-\eta^2)\sqrt{u^2+\eta^2} + 3\eta^4(u^2+\eta^2) + 3M^2(4u^4-4u^2\eta^2+3\eta^4)\right]}{(u^2+\eta^2)^5}} \tag{3.19}$$

Everything above is regular. So, no curvature singularity is present.

Asymptotic behaviour

For the Ricci tensor only diagonal terms are non-vanishing:

$$\begin{aligned}R^t{}_t = -\frac{M\eta^2}{(u^2+\eta^2)^{5/2}} \quad R^u{}_u &= \eta^2\left[\frac{3M}{(u^2+\eta^2)^{5/2}} - \frac{2}{(u^2+\eta^2)^2}\right]\\ R^\theta{}_\theta = R^\phi{}_\phi &= \frac{2M\eta^2}{(u^2+\eta^2)^{5/2}}\end{aligned} \tag{3.20}$$

Giving for the Ricci scalar:

$$\boxed{R = \frac{2\eta^2(3M - \sqrt{\eta^2+u^2})}{(\eta^2+u^2)^{5/2}}} \tag{3.21}$$

[3] We substituted $R_s = 2M$, with $M > 0$

The Ricci scalar depends only on u^2. Thus, for $u \to \pm\infty$: $R \to 0$. So, the two connected regions of spacetime are asymptotically flat. Additionally, for an asymptotically flat spacetime it must be $r^2 \approx u^2$ for large values of u. This is obviously satisfied for $r^2 = u^2 + \eta^2$. This condition ensures a correct circumference to radius ratio of coordinate circles equal to 2π and is true to all of the following cases (asymptotically flat/dS/AdS), so it will not mentioned again.

Stress-Energy tensor and NEC violation

Like the Ricci tensor, the Stress-Energy tensor is diagonal, too.

$$-T^t{}_t = \rho = \eta^2 \frac{4M - \sqrt{u^2+\eta^2}}{8\pi(u^2+\eta^2)^{5/2}} \quad T^u{}_u = p_1 = -\frac{\eta^2}{8\pi(u^2+\eta^2)^2}$$
$$T^\theta{}_\theta = p_2 = T^\phi{}_\phi = \eta^2 \frac{-M + \sqrt{u^2+\eta^2}}{8\pi(u^2+\eta^2)^{5/2}} \tag{3.22}$$

According to (3.15):

$$\boxed{\rho + p_1 = -\frac{\eta^2}{4\pi(u^2+\eta^2)^2}\left|1 - \frac{2M}{\sqrt{u^2+\eta^2}}\right|} \tag{3.23}$$

So, NEC1 is clearly violated for $\eta > 2M$, as it is shown in figure 3.1. There is no horizon for $\eta > 2M$, so the above is strictly negative.

For NEC2, we take:

$$\rho + p_2 = \frac{3M\eta^2}{8\pi(u^2+\eta^2)^{5/2}} \tag{3.24}$$

Unlike NEC1, NEC2 is not violated for any $u \in (-\infty, +\infty)$.

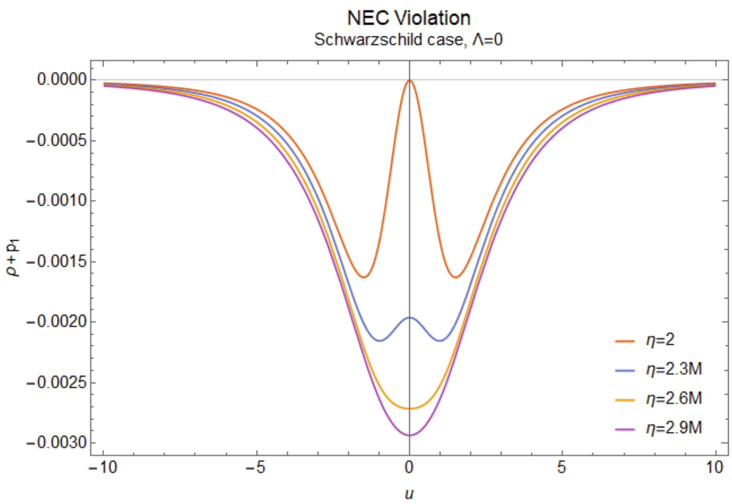

Figure 3.1: *In this graph we see how $\rho + p_1$ behaves with respect to the radial coordinate u, for $M = 1$ and some different values for η. For $\eta = 2M$ we see that the NEC is not violated at $u = 0$, as the latter is a null hypersurface. For $\eta > 2M$ we see the NEC violation to be maximized at the throat as we grow the throat radius η. Far from the throat $\rho + p_1$ goes to zero in all cases, which means that a minimal violations occurs.*

3.2.1 Introducing a negative cosmological constant

The Schwarzschild metric with a cosmological constant corresponds to a metric of the form of (3.1), with:

$$F(r) = 1 - \frac{2M}{r} + \frac{|\Lambda|}{3} r^2 \tag{3.25}$$

where we have a negative cosmological constant: $\Lambda = -|\Lambda| < 0$. Again, there is only one event horizon r_h given by (A.10). Hence, the no horizon condition implies:

$$\boxed{\begin{aligned} &\eta > \frac{2}{|\Lambda|} sinh\left[\frac{1}{3} sinh^{-1}\left(3M\sqrt{|\Lambda|}\right)\right] \\ &\text{or} \;\; \eta > 2M - \frac{8}{9} M^3 |\Lambda| \;\; \text{for} \;\; 9M^2|\Lambda| << 1 \end{aligned}} \tag{3.26}$$

The largest root of $F(r)$ is the r_h, while for large values r the function is obviously positive. Hence, $F(a) > 0 \; \forall a > r_h$ and the flaring out condition is fulfilled.

The wormhole metric is:

$$\boxed{\begin{aligned} &ds^2 = -\left(1 - \frac{2M}{\sqrt{u^2+\eta^2}} + \frac{|\Lambda|}{3}(u^2+\eta^2)\right) dt^2 + \frac{du^2}{1 - \frac{2M}{\sqrt{u^2+\eta^2}} + \frac{|\Lambda|}{3}(u^2+\eta^2)} + r^2 d\Omega^2 \\ &\text{where } r^2 = u^2 + \eta^2 \text{ and } u, t \in (-\infty, +\infty) \end{aligned}} \tag{3.27}$$

Regularity

The components of the Riemann tensor are similar with (3.28), but with an extra term, denoting the existence of the cosmological constant. Referring to the corresponding terms of a vanishing Λ with a subscript "0", we have the following:

$$\begin{aligned} &R^{tu}{}_{tu} = (R^{tu}{}_{tu})_0 - \frac{|\Lambda|}{3} \qquad R^{t\theta}{}_{t\theta} = R^{t\phi}{}_{t\phi} = (R^{t\theta}{}_{t\theta})_0 - \frac{|\Lambda| u^2}{3(u^2+\eta^2)} \\ &R^{\theta\phi}{}_{\theta\phi} = (R^{\theta\phi}{}_{\theta\phi})_0 - \frac{|\Lambda| u^2}{3(u^2+\eta^2)} \;\; , \;\; R^{u\theta}{}_{u\theta} = R^{u\phi}_{u\phi} = (R^{u\theta}{}_{u\theta})_0 - \frac{|\Lambda|}{3} \end{aligned} \tag{3.28}$$

The Kretschmann scalar is enormous, ugly and for these reasons useless in order to be written down. It contains three terms. The first one is the Kretschmann scalar in the case of zero cosmological constant, the second one is proportional to $|\Lambda|$, while the third one is proportional to $|\Lambda|^2$. But it is obvious even from the components of the Riemann tensor that spacetime is regular, as no component becomes infinite. Regularity is guaranteed.

Asymptotic behaviour

The components of the Ricci tensor are:

$$\begin{aligned} R^t_t = (R^t{}_t)_0 - \frac{3u^2+\eta^2}{3(u^2+\eta^2)}|\Lambda| \quad R^u{}_u = (R^u{}_u)_0 - |\Lambda| \\ R^\theta_\theta = R^\phi{}_\phi = (R^\theta{}_\theta)_0 - \frac{3u^2+\eta^2}{3(u^2+\eta^2)}|\Lambda| \end{aligned} \tag{3.29}$$

Giving a Ricci scalar:

$$\boxed{R = R_0 + \left(\frac{2\eta^2}{\eta^2 + u^2} - 4\right)|\Lambda|} \tag{3.30}$$

Taking the limit $u \to \pm\infty$, we take $R \to -4|\Lambda|$; that is, a constant negative scalar curvature, which corresponds to an AdS spacetime. Meaning, that in this case the wormhole connects two asymptotically AdS spacetimes.

Stress-Energy tensor and NEC violation

The components of the Stress-Energy tensor are:

$$\begin{aligned} -T^t{}_t = \rho = \rho_0 - \frac{3u^2 + 2\eta^2}{24\pi(u^2+\eta^2)^2}|\Lambda| \quad T^u{}_u = p_1 = (p_1)_0 + \frac{u^2}{8\pi(u^2+\eta^2)}|\Lambda| \\ T^\theta{}_\theta = p_2 = (p_2)_0 + \frac{3u^2 + 2\eta^2}{24\pi(u^2+\eta^2)}|\Lambda| \end{aligned} \tag{3.31}$$

According to (3.15):

$$\boxed{\rho + p_1 = -\frac{\eta^2}{4\pi(u^2+\eta^2)^2}\left|1 - \frac{2M}{\sqrt{u^2+\eta^2}} + \frac{|\Lambda|}{3}(u^2+\eta^2)\right|} \tag{3.32}$$

In the case of a negative cosmological constant, there in no horizon for η satisfying (3.26). If we plot the graph of $\rho + p_1$ for this case, this is qualitatively the same as 3.1, but shifted downwards.

For NEC2, we get:

$$\boxed{\rho + p_2 = (\rho + p_2)_0} \tag{3.33}$$

So, the introduction of the cosmological constant does not affect the NEC2.

3.2.2 Introducing a positive cosmological constant

Introduction of a positive cosmological constant affects the above work by a change of sign: $-|\Lambda| \to +\Lambda$. The wormhole metric is:

$$\boxed{\begin{aligned} &ds^2 = -\left(1 - \frac{2M}{\sqrt{u^2+\eta^2}} - \frac{\Lambda}{3}(u^2+\eta^2)\right)dt^2 + \frac{du^2}{1 - \frac{2M}{\sqrt{u^2+\eta^2}} - \frac{\Lambda}{3}(u^2+\eta^2)} + r^2 d\Omega^2 \\ &\text{where } r^2 = u^2 + \eta^2 \text{ and } u, t \in (-\infty, +\infty) \end{aligned}} \tag{3.34}$$

The crucial difference is the existence of the cosmological horizon, beyond the event horizon. If r_c is the cosmological horizon, for $r > r_c$ the corresponding $F(r)$ function is negative. Hence, according to (3.9) the throat radius η has to be in-between the event and the cosmological horizon given by (A.14). Thus:

$$\boxed{\frac{2}{\sqrt{\Lambda}} sin\left[\frac{1}{3} sin^{-1}\left(3M\sqrt{\Lambda}\right)\right] < \eta < \frac{2}{\sqrt{\Lambda}} sin\left[\frac{1}{3} sin^{-1}\left(3M\sqrt{\Lambda}\right) + \frac{2\pi}{3}\right]} \tag{3.35}$$

We are not going to present any tensor component for this case. As we said, just make the sign change to the cosmological constant and you will find them. I will mention only the important things.

For the Ricci scalar, we have that at the limit of $u \to \pm\infty$ tends to 4Λ; that is, a constant positive scalar curvature, which corresponds to a dS spacetime. Thence, the two connected regions of spacetime are asymptotically dS.

The other important feature comes to the NEC1. The aforementioned change of sign cosmological constant, gives us:

$$\boxed{\rho + p_1 = -\frac{\eta^2}{4\pi(u^2+\eta^2)^2}\left|1 - \frac{2M}{\sqrt{u^2+\eta^2}} - \frac{\Lambda}{3}(u^2+\eta^2)\right|} \tag{3.36}$$

For every η satisfying (3.35) a cosmological horizon is defined far from the throat at which the NEC1 is not violated. We plot the above behaviour in figure 3.2.

For NEC2, we see even from the previous results that is independent of Λ. So, nothing changes about NEC2 in this case. NEC2 is satisfied through all of the spacetime.

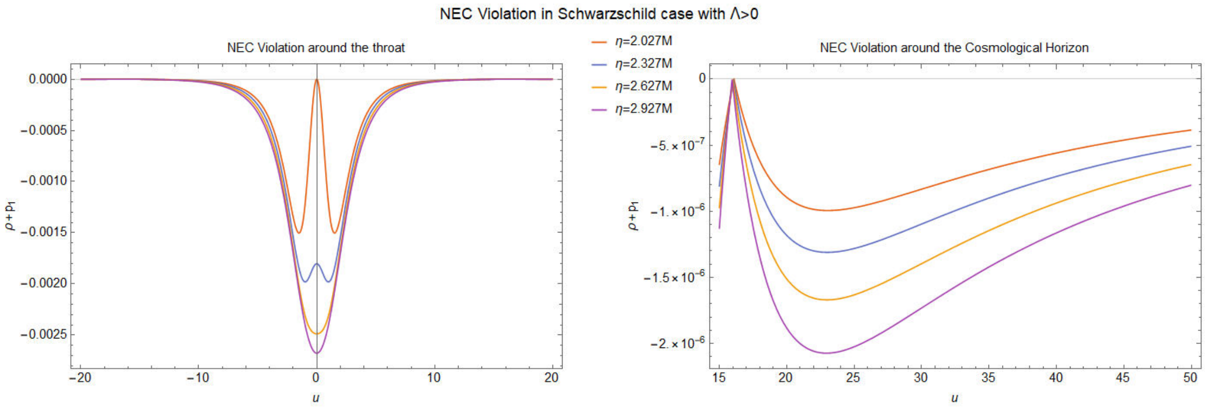

Figure 3.2: *This graph represents the NEC violation in the case that a positive cosmological constant is introduced, for $M = 1$ and $\Lambda = 0.01$. For these values the event and the cosmological horizon of the initial black hole are located at $r_h \approx 2.027M$ and $r_c \approx 16.217M$. In the first graph we see the behaviour of $\rho + p_1$ around the throat, which is like that of figure 3.1, albeit with different values of η, adapted according to the location of the initial event horizon. However, there is a cosmological horizon for each value of η. The behaviour of $\rho + p_1$ around the cosmological horizon is presented to the second graph, for $u > 0$ (The behaviour for $u < 0$ is symmetrical). At the cosmological horizon $\rho + p_1$ reaches the zero value, while beyond the cosmological horizon we see that remains negative, with a negative minimum value and then goes to zero for large values of the coordinate u.*

3.3 From Reissner–Nordström black hole to traversable wormhole

In this case, we have as an input metric that of the Reissner–Nordström given in (B.5). Hence, the output metric is:

$$\boxed{\begin{aligned} &ds^2 = -\left(1 - \frac{2M}{\sqrt{u^2+\eta^2}} + \frac{Q^2}{u^2+\eta^2}\right)dt^2 + \frac{du^2}{1 - \frac{2M}{\sqrt{u^2+\eta^2}} + \frac{Q^2}{u^2+\eta^2}} + r^2 d\Omega^2 \\ &\text{where } r^2 = u^2+\eta^2 \text{ and } u, t \in (-\infty, +\infty) \end{aligned}} \tag{3.37}$$

We consider the case of $M^2 > Q^2$ (see Appendix B). So, the event horizon of (B.5) is located at $r_h = M + \sqrt{M^2 - Q^2}$ and the no horizon condition implies:

$$\boxed{\eta > M + \sqrt{M^2 - Q^2}} \tag{3.38}$$

In order to the flaring-out condition be satisfied, g_{tt} has to be negative $\forall \eta > r_h$. In the Appendix B we see that r_h is the largest root of g_{tt} for (B.5), while for $r \to +\infty$ tends to -1, imposing that g_{tt} is purely negative after its largest root . Thence, $g_{tt}(\eta) < 0$, $\forall \eta > r_h$ and the flaring-out condition is fulfilled.

Regularity

For the components of the Riemann tensor, we have:

$$\begin{aligned} R^{tu}{}_{tu} &= \frac{M(2u^2-\eta^2)}{(u^2+\eta^2)^{5/2}} + \frac{\eta^2 - 3u^2}{(u^2+\eta^2)^3}Q^2 \\ R^{t\theta}{}_{t\theta} = R^{t\phi}{}_{t\phi} &= -\frac{Mu^2}{(u^2+\eta^2)^{5/2}} + \frac{u^2}{(u^2+\eta^2)^3}Q^2 \\ R^{\theta\phi}{}_{\theta\phi} &= \frac{\eta^2\sqrt{u^2+\eta^2} + 2Mu^2}{(u^2+\eta^2)^{5/2}} - \frac{u^2Q^2}{(u^2+\eta^2)^{5/2}} \\ R^{u\theta}{}_{u\theta} = R^{u\phi}_{u\phi} &= \frac{M(2\eta^2-u^2) - \eta^2\sqrt{u^2+\eta^2}}{(u^2+\eta^2)^{5/2}} + \frac{u^2-\eta^2}{(u^2+\eta^2)^3}Q^2 \end{aligned} \tag{3.39}$$

Without calculation of any scalar, it is obvious from the above simple and useful components that the spacetime does not contain any singular point for $\eta \neq 0$. Moreover, you can check by yourself that if you take $Q = 0$, then it will be reduced to (3.28), which is a statement of correctness for our calculations.

Asymptotic behaviour

The non-vanishing components of the Ricci tensor are the following ones:

$$\begin{aligned} R^t_t &= -\frac{M\eta^2}{(u^2+\eta^2)^{5/2}} + \frac{\eta^2-u^2}{(u^2+\eta^2)^3}Q^2 \\ R^u_u &= \frac{3M\eta^2}{(u^2+\eta^2)^{5/2}} - \frac{2\eta^2}{(u^2+\eta^2)^2} - \frac{Q^2}{(u^2+\eta^2)^2} \\ R^\theta_\theta = R^\phi_\phi &= \frac{2M\eta^2}{(u^+\eta^2)^{5/2}} + \frac{u^2-\eta^2}{(u^2+\eta^2)^3}Q^2 \end{aligned} \tag{3.40}$$

Giving for the Ricci scalar:

$$\boxed{R = \frac{2\eta^2(3M - \sqrt{\eta^2 + u^2})}{(\eta^2 + u^2)^{5/2}} - \frac{2\eta^2 Q^2}{(\eta^2 + u^2)^3}} \tag{3.41}$$

Taking the limit $u \to \pm\infty$, we see that R tends to zero; that is, two asymptotically flat spacetimes are connected.

Stress-Energy tensor and NEC violation

The components of the Stress-Energy tensor are:

$$\begin{aligned} -T^t_{\ t} &= \rho = \eta^2 \frac{4M - \sqrt{u^2 + \eta^2}}{8\pi(u^2 + \eta^2)^{5/2}} + \frac{Q^2(u^2 - 2\eta^2)}{8\pi(u^2 + \eta^2)^3} \\ T^u_{\ u} &= p_1 = -\frac{\eta^2}{8\pi(u^2 + \eta^2)^2} - \frac{Q^2 u^2}{8\pi(u^2 + \eta^2)^3} \\ T^\theta_{\ \theta} &= p_2 = T^\phi_{\ \phi} = \eta^2 \frac{-M + \sqrt{u^2 + \eta^2}}{8\pi(u^2 + \eta^2)^{5/2}} + \frac{Q^2 u^2}{8\pi(u^2 + \eta^2)^3} \end{aligned} \tag{3.42}$$

Thence, for NEC1 we take:

$$\boxed{\rho + p_1 = -\frac{\eta^2}{4\pi(u^2 + \eta^2)^2} \left| 1 - \frac{2M}{\sqrt{u^2 + \eta^2}} + \frac{Q^2}{u^2 + \eta^2} \right|} \tag{3.43}$$

No horizon is appeared in this case for every η satisfying (3.38). The behaviour of $\rho + p_1$ is plotted in figure 3.3, for the physical case $(0 < Q < M)$ and for the extreme case $(Q = M)$.

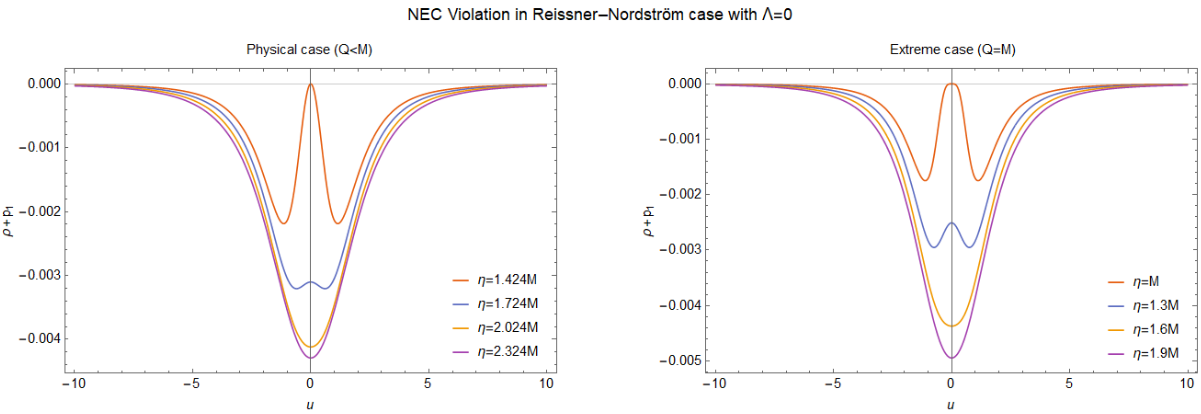

Figure 3.3: *In this graph we see the NEC violation in the case that we introduce charge. The first graph corresponds to the physical case, where we have taken* $Q^2 = 0.82M^2$*. For this value of* Q*, the event horizon of the initial (Reissner–Nordström) black hole is located at* $r_h \approx 1.424M$*. The second graph corresponds to the extreme case where* $Q = M$ *and there is only one horizon for the initial (extreme Reissner–Nordström) black hole at* $r_h = M$*. Qualitatively this graph is the same with that of figure 3.1. So, nothing important changes by the introduction of charge. The only difference is quantitative.*

3.3.1 Introducing a negative cosmological constant

As we see in the Appendix B the introduction of a negative cosmological constant affects the Reissner–Nordström black hole to have a metric of the form of (B.7) with $\Lambda = -|\Lambda|$. For this metric we have two horizons. The Cauchy and the event horizon related as $r_{cauchy}(M, Q, \Lambda) < r_h(M, Q, \Lambda)$. So, for the no horizon condition we just have:

$$\boxed{\eta > r_h(M, Q, \Lambda)} \tag{3.44}$$

The output-wormhole metric is the following:

$$\boxed{\begin{aligned} ds^2 = &-\left(1 - \frac{2M}{\sqrt{u^2+\eta^2}} + \frac{Q^2}{u^2+\eta^2} + \frac{|\Lambda|}{3}(u^2+\eta^2)\right) dt^2 \\ &+\left(1 - \frac{2M}{\sqrt{u^2+\eta^2}} + \frac{Q^2}{u^2+\eta^2} + \frac{|\Lambda|}{3}(u^2+\eta^2)\right)^{-1} du^2 + r^2 d\Omega^2 \\ &\text{where } r^2 = u^2+\eta^2 \text{ and } u, t \in (-\infty, +\infty) \end{aligned}} \tag{3.45}$$

The flaring out condition is satisfied cause the g_{tt} of (B.7) is negative for $r > r_h$.

Regularity, Asymptotic behaviour, Stress-Energy tensor and NEC violation

The contribution of the cosmological constant to the tensor components (Riemann and Stress-Energy tensor) is the same as in the case of Schwarzschild metric. The only difference is that to the terms with the subscript "0" correspond the terms with the non-vanishing Q given in the previous subsection. Hence, regularity is guaranteed, while for the limit of $u \to \pm\infty$ we take again $R \to -4|\Lambda|$; that is, two asymptotically AdS spacetimes. The NEC1 is violated again:

$$\boxed{\rho + p_1 = -\frac{\eta^2}{4\pi(u^2+\eta^2)^2}\left|1 - \frac{2M}{\sqrt{u^2+\eta^2}} + \frac{Q^2}{u^2+\eta^2} + \frac{|\Lambda|}{3}(u^2+\eta^2)\right|} \tag{3.46}$$

No cosmological horizon is introduced by the negative cosmological constant, so again NEC is violated for every $u \in (-\infty, +\infty)$. The graph of the above is qualitatively the same with figure 3.3 but shifted downwards.

3.3.2 Introducing a positive cosmological constant

In the case of a positive cosmological constant we have again three horizons (see Appendix B); namely, the Cauchy, the event and the cosmological horizon, $r_{cauchy} < r_h < r_c$. The g_{tt} of Reissner–Nordström metric with a positive cosmological constant (B.7) is negative between the event and the cosmological horizon. So, combination of the no horizon and the flaring out condition implies:

$$\boxed{r_h(M, Q, \Lambda) < \eta < r_c(M, Q, \Lambda)} \tag{3.47}$$

Except this, everything is pretty like the same as in the case of the negative cosmological constant, with the substitution of $|\Lambda| \to -\Lambda$. Of course, the two connected regions of the spacetime are asymptotically dS.($R \to 4\Lambda$ for $u \to \pm\infty$).

For the NEC violation we have the following:

$$\boxed{\rho + p_1 = -\frac{\eta^2}{4\pi(u^2+\eta^2)^2}\left|1 - \frac{2M}{\sqrt{u^2+\eta^2}} + \frac{Q^2}{u^2+\eta^2} - \frac{\Lambda}{3}(u^2+\eta^2)\right|} \tag{3.48}$$

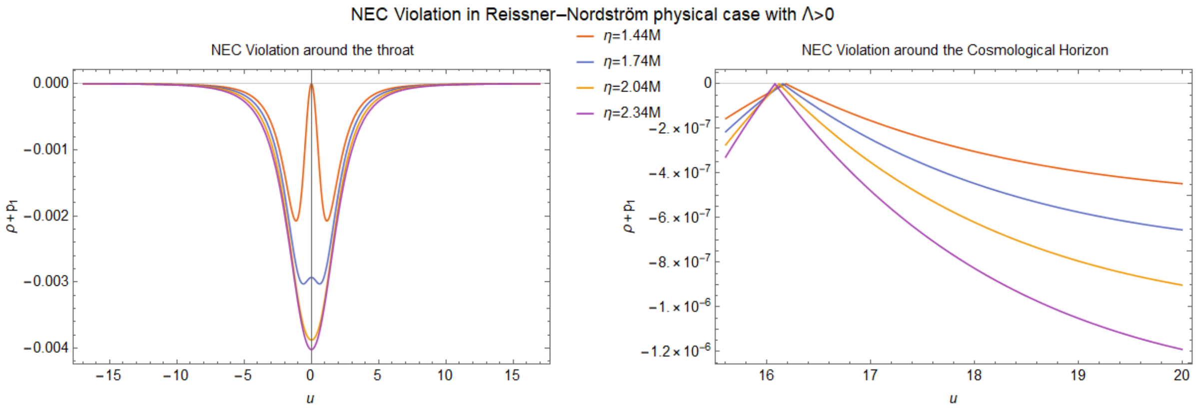

Figure 3.4: *In this graph we see the NEC violation in the case that we introduce charge and a positive cosmological constant. The graph corresponds to* $Q^2 = 0.82M^2$ *and* $\Lambda = 0.01$. *For this value of* Q *and* Λ, *the event horizon of the initial black hole is located at* $r_h \approx 1.44M$ *and the cosmological horizon at* $r_c \approx 16.24M$. *To the left, we see* $\rho - p_1$ *around the throat, while to the right near the cosmological horizon. At the cosmological horizon, we see that* $\rho + p_1$ *vanishes and after that goes to* 0^-.

3.4 Black-Bounce and One-way traversable wormhole

Until now, we studied the case of a traversable wormhole according to the Simpson-Visser regularization technique. However, this is not the only case.(This cases are not going to be analysed in detail because in this thesis we are concerned only about wormholes.) By this technique we are able to construct regular black holes; that is, a spacetime with an event horizon, where the singularity is a spacelike hypersurface. Such a spacetime in the case of the metric (3.16) is achieved for a throat radius in $0 < \eta < 2M$. Hence, is a regular black hole admitted by cutting the centre of the spacetime. This is why instead of a singularity we have a spacelike hypersurface which bounces into a separate copy of our universe. The Penrose diagram for such a spacetime is the following: The

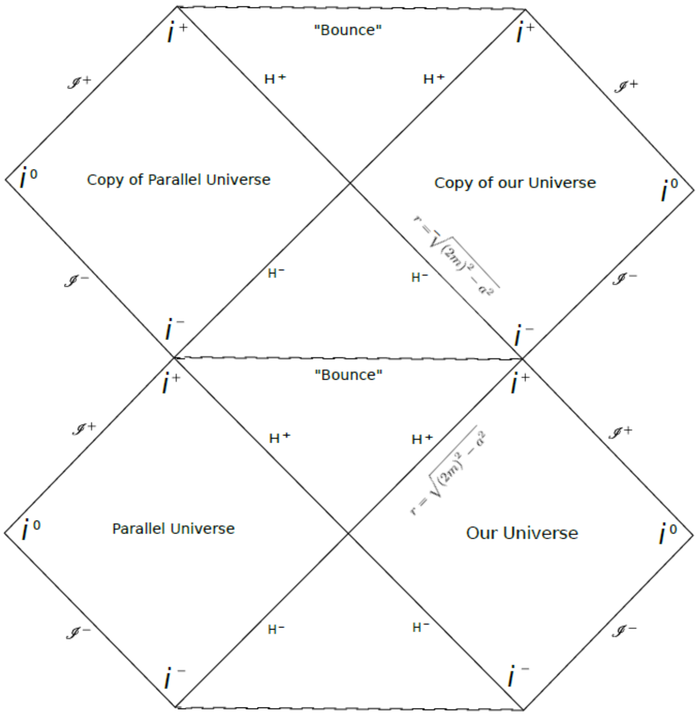

Figure 3.5: *Carter-Penrose diagram for the maximally extended spacetime when* $\eta \in (0, 2M)$*. In this example the time coordinate runs up the page, "bouncing" through the* $u = 0$ *hypersurface in each black hole region into a future copy of our own universe ad infinitum* [6]

other extreme case between the regular black hole and the traversable wormhole is that of the one way traversable wormhole. In this case, contrary to the Morris and Thorne criteria for a two way traversable wormhole the throat becomes null. This is achieved in the case of the metric (3.16) for $\eta = 2M$ and is characterized as one way because of the fact that if a particle or a photon passes the throat, due to its null nature, cannot come back. This is more clear to the Penrose diagram of this spacetime presented in figure (3.6). Morever, as the throat is null, NEC is not violated at the throat. (see (2.63) and discussion below)

We can summarize the possible spacetimes according to the sign of gtt at $u = 0$:

$$(g_{tt})_{u=0} = -\left(1 - \frac{2M}{\alpha}\right) \tag{3.49}$$

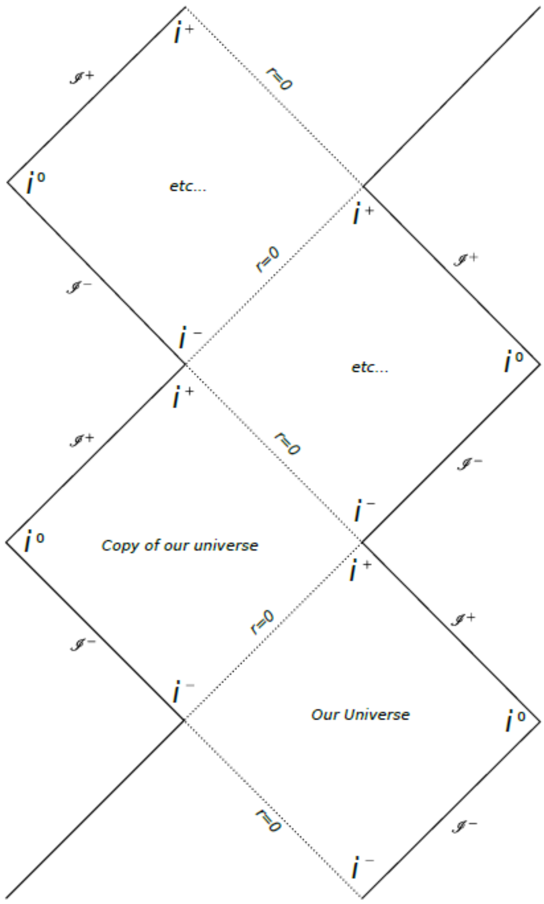

Figure 3.6: *Carter-Penrose diagram for the maximally extended spacetime when* $\eta = 2M$*. In this example we have a one-way wormhole geometry with a null throat.* [6]

as follows:

- for $a > 2M$: $(g_{tt})_{throat} < 0$. The $u = 0$ is a timelike hypersurface and the metric describes a two-way traversable wormhole

- for $a = 2M$: $(g_{tt})_{throat} = 0$. The $u = 0$ is a null hypersurface and the metric describes a one-way traversable wormhole

- for $a < 2M$: $(g_{tt})_{throat} > 0$. The $u = 0$ is a spacelike hypersurface and the metric describes a regular black hole without centre-black bounce
- for $\alpha = 0$ the $u = 0$ is a singularity and the metric describes a usual singular black hole

Chapter 4

Distinguishing the black hole from the wormhole

In the introduction, we met with the idea of Einstein and Rosen that a self-consistent field theory has to describe particles under the sense of a field. However, in this chapter we do not follow this idea and we describe particles as matter points that move along geodesics of the underlined geometry.

4.1 The effective potential in spherically symmetric and static metric

If $x^\mu(\lambda)$ is a geodesic and λ is an affine parameter, then we define the tangent vector as $P^\mu = dx^\mu/d\lambda$. The crucial equation is the one that distinguishes between the timelike and null geodesics. Particles with a non-vanishing mass are moving along the timelike geodesics, while massless particles (like the photon) move along the null geodesics. These geodesics are characterized by the norm of P^μ in the following way:

$$g_{\mu\nu}P^\mu P^\nu = \epsilon = \begin{cases} -1 & \text{timelike geodesics} \\ 0 & \text{null geodesics} \end{cases} \tag{4.1}$$

In the case of the timelike geodesics, the affine parameter can be taken to be the proper time (τ), while for the null geodesics remains just an arbitrary affine parameter. In the case of the metric (2.1), the above equation takes the following form:

$$-e^{2\gamma(u)}\dot{t}^2 + e^{2\alpha(u)}\dot{u}^2 + r^2\left(\dot{\theta}^2 + sin^2\theta\dot{\phi}^2\right) = \epsilon \tag{4.2}$$

where r is some function of u. Our goal is to simplify the above equation as much as we can, in view of the symmetries of the spacetime; that is, the spherical symmetry and the staticity of the spacetime (I follow [11]).

- **Spherical symmetry**: Spherical symmetry allows us to constrain the motion at a single plane, which we take to be the $\theta = \pi/2$ plane. Hence, the trajectories possible to this environment will be determined if we determine the radial (r) coordinate in respect of the angular (ϕ) coordinate. This is possible if we find an expression for the $dr/d\phi$ derivative and then to proceed to the calculation of the corresponding quadrature.

The other feature of the spherical symmetry is the existence of a rotational Killing vector; namely, the Killing vector: $R = \partial_\phi$. This Killing vector represents the rotational symmetry around the z-axis and as we saw in the example of (2.1) chapter, the corresponding conserved quantity can be interpreted as the angular momentum of the particle (per unit rest mass). So, we have the following constant of motion:

$$
\begin{aligned}
L = g_{\mu\nu} R^\mu P^\nu = g_{\mu\nu} \delta^\mu_\phi P^\nu = r^2 \dot{\phi} \\
\rightarrow \boxed{\dot{\phi} = \frac{L}{r^2}}
\end{aligned}
\tag{4.3}
$$

Thence, spherical symmetry allowed us to determine the derivatives of the angular coordinates in (4.2).

- **Staticity**: The fact that our metric is static is reflected by the existence of the timelike Killing vector (2.10). According to this Killing vector we define the following constant of motion:

$$
\begin{aligned}
E = -g_{\mu\nu} K^\mu P^\nu = -g_{\mu\nu} \delta^m u_t P^\nu = -g_{tt} \dot{t} \\
\rightarrow \boxed{\dot{t} = e^{-2\gamma(u)} E}
\end{aligned}
\tag{4.4}
$$

  where E can be interpreted as the particle's total energy (including the gravitational potential energy) per unit rest mass.

Substitution to (4.2), yields:

$$
e^{2(\alpha+\gamma)} \dot{u}^2 + e^{2\gamma} \left(-\epsilon + \frac{L^2}{r^2} \right) = E^2 \tag{4.5}
$$

Using the quasiglobal coordinates ($\alpha = -\gamma$), this simplifies to:

$$
\boxed{\dot{u}^2 + e^{2\gamma} \left(-\epsilon + \frac{L^2}{r^2} \right) = E^2} \tag{4.6}
$$

So, using the symmetries of the spacetime the problem has been reduced to an 1-dimensional problem. Specifically, the above equation mathematically is the same as a problem of classical mechanics, where $\dot{u}^2$ corresponds to the kinetic energy, E^2 to the particle's total energy, while the remaining term to the potential in which the particle moves. Thence, we define the *effective potential*:

$$
\boxed{V_\epsilon(u) = -g_{tt}(u) \left(-\epsilon + \frac{L^2}{r^2(u)} \right)} \tag{4.7}
$$

where we've substituted $e^{2\gamma} = -g_{tt}$. According to the effective potential we have the followings:

- Motion can take place only to regions where $V_{eff}(u) \leq E^2$

- Points where $E^2 = V_{eff}$ are called as turning points; that is, points where $\dot{u} = 0$ and are the bounds of the allowed region for motion

- Circular orbits of radius u_C: $V'_\epsilon(u_C) = 0$.

4.2 The circular orbits as a tool for distinction

In this section we are dealing with the metric (3.2) and specifically with an observational distinction between the initial black hole and the wormhole, that this metric can describe. Circular orbits are our tool. What are we going to see is that it is possible for a wormhole a circular orbit not to be defined, while for a black hole it does. Photon spheres and ISCOs (Innermost Stable Circular Orbit) are circular orbits. So, what do we actually prove is that it is possible to distinguish the wormhole of Simpson and Visser from the corresponding black hole depending on the location or even existence of a photon sphere or ISCO.

The effective potential according to the metric (3.2) becomes:

$$V_\epsilon(u) = F\left(\sqrt{u^2+\eta^2}\right)\left(-\epsilon + \frac{L^2}{u^2+\eta^2}\right) \tag{4.8}$$

Taking its first derivative in respect with u and setting it equal to zero, we get:

$$\begin{aligned} V'_\epsilon(u) &= \frac{dF(\sqrt{u^2+\eta^2})}{du}\left(-\epsilon + \frac{L^2}{u^2+\eta^2}\right) - F(\sqrt{u^2+\eta^2})\frac{L^2 u}{(u^2+\eta^2)^2} \\ \left[\text{with } \frac{d}{du} = \frac{u}{r}\frac{d}{dr}\right] &\rightarrow V'_\epsilon = \frac{u}{r}\left[\frac{dF(r)}{dr}\left(-\epsilon + \frac{L^2}{r^2}\right) - F(r)\frac{L^2}{r^3}\right] \\ [\text{if } V_\epsilon(r_C) = 0\,] &\rightarrow \boxed{\frac{u_C}{r_C}\left[\frac{dF(r_C)}{dr}\left(-\epsilon + \frac{L^2}{r_C^2}\right) - F(r_C)\frac{L^2}{r_C^3}\right] = 0} \end{aligned} \tag{4.9}$$

The above boxed equation determines the radius of the possible circular orbit. We firstly see that $u_C = 0$ satisfies this equation. Thus, it seems like that at the throat we have always a circular orbit. But actually this is not exactly the case. This circular orbit depends on the value of the wormhole throat radius, η. More on this in the following.

The important for us is the value r_C for which a circular orbit is appeared. This r_C corresponds to:

$$u_C = \pm\sqrt{r_C^2 - \eta^2} \tag{4.10}$$

For $\eta = 0$, as we know we reduce to the initial black hole metric. Thus, r_C is the circular orbit of the black hole description of the metric (3.2). In the same vein that we prevented any event horizon in order to accomplish the wormhole description of the metric, it is obvious that for some values of the wormhole throat the circular orbit is not allowed. Actually, out work is reduced to find the circular orbits of the initial black hole, with the passage to the wormhole being accomplished through the equation (4.10).

In all of our cases the circular orbits are of larger radius than that of the event horizon, $r_C > r_h$. So,

- for $\eta = 0$, r_C is the radius of the black hole's circular orbit.
- for $r_h < \eta < r_C$, $u_C = \sqrt{r_C^2 - \eta^2}$ is the circular orbit of the wormhole (in the region $u > 0$)
- for $\eta > r_C$, the circular orbit cannot be defined

Photon spheres correspond to the circular orbits of the null geodesics, while the ISCOs correspond to the stable orbits of the timelike geodesics. Thence, our work for an arbitrary circular orbit encloses both of them. Since, photon spheres and ISCOs do not coincide it is possible for some values of the throat ISCO to be defined, while the photon sphere to be not. But more on this possibilities in the subsequent sections in which we take specific examples. Moreover, from the above consideration it is obvious that as the throat becomes larger and larger the location of the circular orbits becomes smaller, until its non existence at all. This can be viewed that as we open the throat more and more it becomes larger from the event horizon and then no horizon is allowed. Then, by continuing this enlarging of the throat any circular orbit can be lost. Photon spheres and ISCOs are not an exception. So, these possible geometries come together with observational effects according to trajectories around them.

4.3 Distinguishing the Schwarzschild black hole from the traversable wormhole

According to (3.16), we have:

$$\boxed{\begin{aligned} V_\epsilon(u) &= \left(1-\frac{2M}{\sqrt{u^2+\eta^2}}\right)\left(-\epsilon+\frac{L^2}{u^2+\eta^2}\right) \\ V_\epsilon'(u) &= -\frac{2u}{(u^2+\eta^2)^{5/2}}\left[L^2\left(\sqrt{u^2+\eta^2}-3M\right)+M\epsilon(u^2+\eta^2)\right] \end{aligned}} \tag{4.11}$$

Photon Sphere

Taking $\epsilon = 0$, we reduce to:

$$V_0'(u) = -\frac{2uL^2}{(u^2+\eta^2)^{5/2}}\left(\sqrt{u^2+\eta^2}-3M\right) \tag{4.12}$$

Hence, $V_0'(u) = 0$ implies:

$$u_{photon} = \sqrt{(3M)^2-\eta^2} \tag{4.13}$$

So,

- for $\boxed{\eta = 0}$, we reduce to the photon sphere of the Schwarzschild black hole
- for $\boxed{2M < \eta \leq 3M}$, the photon sphere is allowed for the traversable wormhole
- for $\boxed{\eta > 3M}$, no photon sphere is allowed for the traversable wormhole

ISCO

Taking $\epsilon = -1$, we reduce to:

$$V_{-1}'(u) = \frac{2u}{(u^2+\eta^2)^{5/2}}\left[L^2\left(3M-\sqrt{u^2+\eta^2}\right)+M(u^2+\eta^2)\right] \tag{4.14}$$

Thus, $V'_{-1}(u) = 0$ implies for the angular momentum of the circular orbit located at u_C:

$$L_C^2 = \frac{Mr_C^2}{r_C - 3M} \tag{4.15}$$

ISCO corresponds to the radius u_C, or equivalently r_C, that minimizes the angular momentum. Taking the derivative of the above with respect to r_C:

$$\frac{\partial L_C^2}{\partial r_C} = \frac{Mr_C}{(r_C - 3M)^2}(r_C - 6M) \tag{4.16}$$

Hence, the minimum is for $r_C = 6M$, which means that ISCO is located at

$$u_{ISCO} = \sqrt{(6M)^2 - \eta^2}, \text{ with } L_C^2 = 12M^2 \tag{4.17}$$

Thus, the wormhole do not have an ISCO for $\eta > 6M$. Moreover, (4.15) constraints $r_C > 3M$, which implies $u_C^2 > u_{photon}^2$. This means, that if a photon sphere is allowed for the wormhole, then no timelike circular orbit can be defined beyond the photon sphere. But if the photon sphere is not allowed, which is for $\eta > 3M$, then there is no such a constraint for a timelike circular orbit. So, for the case of $u_C = 0$; **at the throat of the wormhole there can be a circular orbit only for $\eta > 3M$.**

Summarizing:

- for $\boxed{\eta = 0}$, we reduce to the ISCO of the Schwarzschild black hole
- for $\boxed{2M < \eta \leq 6M}$, ISCO is allowed for the traversable wormhole
- for $\boxed{\eta > 6M}$, no ISCO is allowed for the traversable wormhole

Comparing the values of η that allow the photon sphere and the ISCO we see that:

- for $\boxed{2M < \eta < 3M}$, ISCO and photon sphere are allowed
- for $\boxed{3M < \eta < 6M}$, ISCO is allowed, while the photon sphere is not

4.4 Distinguishing the Reissner–Nordström black hole from the traversable wormhole

According to (3.37), we have:

$$\boxed{\begin{aligned} &V_\epsilon(u) = \left(1 - \frac{2M}{\sqrt{u^2+\eta^2}} + \frac{Q^2}{u^2+\eta^2}\right)\left(-\epsilon + \frac{L^2}{u^2+\eta^2}\right) \\ &V'_\epsilon(u) = \frac{2u}{(u^2+\eta^2)^3}\left[-\epsilon M r^3 + (\epsilon Q^2 - L^2)r^2 + 3ML^2 r - 2L^2Q^2\right] \\ &\text{where} r^2 = u^2 + \eta^2 \end{aligned}} \tag{4.18}$$

Photon Sphere

Taking $\epsilon = 0$, we reduce to:

$$V_0'(u) = \frac{2uL^2}{(u^2+\eta^2)^3}\left[-r^2+3Mr-2Q^2\right] \tag{4.19}$$

Setting the above equal to zero we have to solve a second order polynomial equation. This means that two roots are produced:

$$r_{1,2} = \frac{1}{2}\left(3M \pm \sqrt{9M^2-8Q^2}\right) \tag{4.20}$$

Setting $Q = 0$ we see that the root with the minus sign goes to zero, while the root with plus sign goes to $3M$, which is the photon sphere of the Schwarzschild metric. Hence, the correct root is that with the plus sign, yielding for the location of the photon sphere ($u > 0$):

$$\boxed{u_{photon} = \sqrt{\frac{1}{4}\left(3M+\sqrt{9M^2-8Q^2}\right)^2 - \eta^2}} \tag{4.21}$$

Hence, the extreme limits are the followings:

- for $\boxed{\eta = 0}$, we reduce to the photon sphere of the Reissner–Nordström black hole
- for $\boxed{\frac{1}{2}\left(3M+\sqrt{9M^2-8Q^2}\right) > \eta > M + \sqrt{M^2-Q^2}}$, the photon sphere is allowed for the traversable wormhole
- for $\boxed{\eta > \frac{1}{2}\left(3M+\sqrt{9M^2-8Q^2}\right)}$, there is no photon sphere for the traversable wormhole

But are we sure that the location of the photon sphere of the Reissner–Nordström black hole is larger than its event horizon? In order to answer that question we take the function

$$h(M,Q) = \frac{1}{2}\left(3M+\sqrt{9M^2-8Q^2}\right) - M - M\sqrt{M^2-Q^2}$$

which is the difference between the photon sphere given by the plus sign of (4.20) and the Reissner–Nordström's event horizon. Setting $M = 1$ the graph of the above function is presented in figure 4.1.

So, it is obvious that this difference is always positive $\forall\ |Q| < M$ (physical case); meaning, that the photon sphere is always out of the event horizon, even in the extreme case where $Q = M$. Moreover, as it is seen from the above graph, for some value of Q/M close to 1, this difference reaches a minimum value.

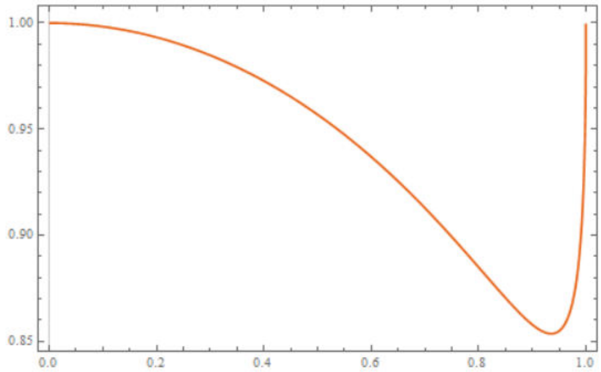

Figure 4.1: *The difference between the photon sphere and the event horizon of the Reissner–Nordström black hole, for $M = 1$, in respect with the charge Q,*

ISCO

Taking $\epsilon = -1$, we reduce to:

$$V_{-1}(u) = \frac{2u}{(u^2+\eta^2)^3}\left[Mr^3 - (Q^2+L^2)r^2 + 3ML^2r - 2L^2Q^2\right] \tag{4.22}$$

Setting the above equal to zero, we find for the angular momentum:

$$L_C^2 = \frac{Mr_C^3 - Q^2r_C^2}{r_C^2 - 3Mr_C + 2Q^2} \tag{4.23}$$

and

$$\frac{\partial L_C^2}{\partial r_C} = \frac{r_C\left(Mr_C^3 - 6M^2r_C^2 + 9MQ^2r_C - 4Q^4\right)}{\left(2Q^2 + r_C(r_C - 3M)\right)^2} \tag{4.24}$$

Thus, in order to find the r_C that minimizes the angular momentum we have to solve a cubic equation. The two out of three[1] roots are complex, while the other is real and equal to:

$$\begin{aligned} r_{ISCO} = 2M &+ \frac{4M^3 - 3MQ^2}{\left(8M^6 - 9M^4Q^2 + 2M^2Q^4 + \sqrt{5M^8Q^4 - 9M^6Q^6 + 4M^4Q^8}\right)^{1/3}} \\ &+ \frac{\left(8M^6 - 9M^4Q^2 + 2M^2Q^4 + \sqrt{5M^8Q^4 - 9M^6Q^6 + 4M^4Q^8}\right)^{1/3}}{M} \end{aligned} \tag{4.25}$$

Of course, the location of the ISCO is given by (4.10), by substituting the above.

Moreover, there is a similar constraint about the timelike circular orbits and the photon sphere. In (4.23) the denominator is the quartic equation that determines the location of the photon sphere. The situation is the same as before; if the photon can be defined according to the wormhole radius, no timelike circular orbit can be defined beyond the photon sphere. If the photon sphere cannot be defined, there is no such a constraint.

[1]This equation was solved with Mathematica

An example

For example, in the case of $Q^2 = 0.82M^2$: $r_{ISCO} \approx 4.48M$, while $r_h \approx 1.424M$ and $r_{photon} \approx 2.28M$.

So,

- for $\eta = 0$, we reduce to the photon sphere and the ISCO of the Reissner–Nordström black hole
- for $1.424M < \eta < 2.28M$, the photon sphere and the ISCO of the wormhole are both allowed

- for $2.28M < \eta < 4.48M$, the ISCO of the wormhole is allowed, while the photon sphere is not

- for $\eta > 4.48M$, ISCO and photon sphere for the wormhole are both not allowed

4.5 Effective Potentials and particles' behaviour

4.5.1 Schwarzschild case

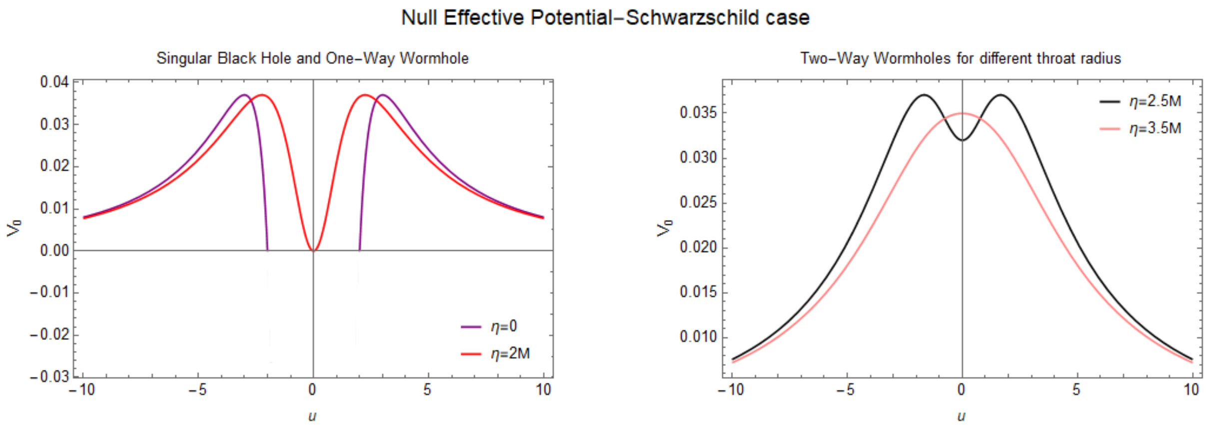

Figure 4.2: *This is the effective potential produced by the null geodesics for different values of the throat radius.*

To the left, we see the null potential in the case of the original Schwarzschild black hole ($\eta = 0$) and the one-way traversable wormhole ($\eta = 2M$). To the right, we see the null effective potential for the traversable wormhole, for $\eta = 2.5M$ and $\eta = 3.5M$. For $\eta = 2.5M$, which is between the Schwarzschild horizon and the Schwarzschild photon sphere, we see a stable circular orbit at the throat and two unstable orbits symmetrical in each region. The latter, are the photon spheres. For $\eta = 3.5M$, the throat radius is larger than the Schwarzschild photon sphere. Hence, no photon sphere is allowed and there is only one unstable circular orbit at the throat.

The effective potential of timelike geodesics

The effective potential for the singular black hole, the one way wormhole and two way traversable wormholes are plotted to figure 4.3, for an angular momentum $L^2 = 12M^2$.

In the case of the singular black hole, the one-way wormhole and the traversable wormhole of $\eta = 4M$, we see an ISCO to be allowed; that is, a value $u > 0$ for which the effective potential has a vanishing first order derivative. The exact location of each ISCO is determined by (4.10), from which is obvious that as the parameter η becomes larger, the ISCO comes closer to the $u = 0$ hypersurface. Every ISCO exhibits a marginal stability, $V'' = 0$, which means that they are not stable as their name define. For the traversable wormhole with $\eta = 7M$, no extrema is appeared for $u > 0$, which means that no ISCO is allowed, as was expected.

For the singular black hole, the graph is not valid beyond the point that vanishes the effective potential. This point corresponds to the event horizon of the black hole, beyond which the metric is non-static and our analysis about the geodesics is not valid. Remember, that we assumed a timelike Killing vector ∂_t in order to determine the effective potential. Beyond the event horizon ∂_t is not a Killing vector; this is why our analysis is not valid. This is true, also, for the $u = 0$ null hypersurface in the case of the one-way traversable wormhole. It is also true in the case of the null effective potential of figure 4.2.

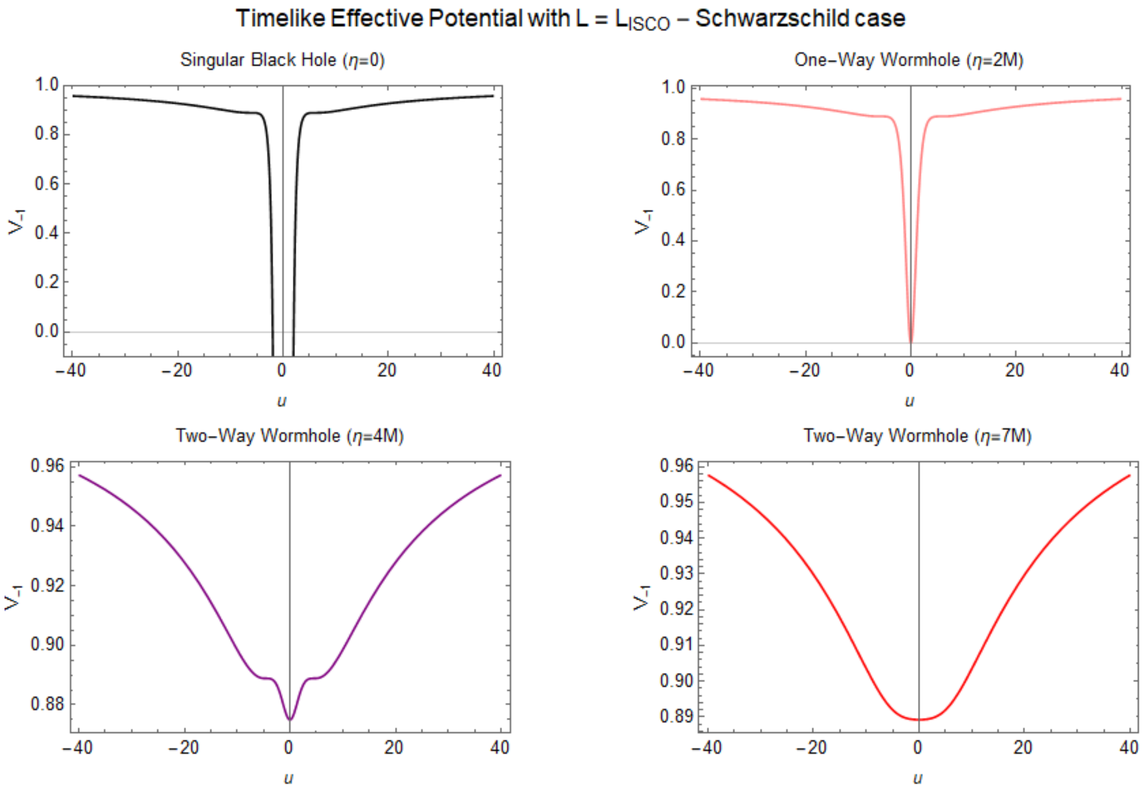

Figure 4.3: *Here we see the effective potential for timelike paths in the Schwarzschild case, for an angular momentum* $L^2 = 12M^2$*; that is, the angular momentum of the ISCO. All graphs above tend to* 1 *for* $u \to \pm\infty$

For the two-way traversable wormholes at $u = 0$ the effective potential is non-zero, while the effective potential has a minimum, which corresponds to a stable circular orbit.

What is happening for other values of the angular momentum L? In figure 4.4, we present the effective potential of each case for $L^2 = 10M^2$; that is, a smaller value of angular momentum than that of ISCO. What we see is that in all cases there is no extrema (stable or unstable) for $u > 0$. At $u = 0$ there is a stable circular orbit (of course not for the singular black hole and the one way wormhole). In figure 4.5 we present the effective potential of each case and for $L^2 = 14M^2$; that is, an angular momentum larger than that of the ISCO. Except for the two-way wormhole with $\eta = 7M$, all the other effective potentials have two extreme values for $u > 0$. One local maximum, which corresponds to an unstable orbit and one local minimum, which corresponds to a stable orbit. The unstable orbit is closer to the $u = 0$ hypersurface than the stable one. Thence, starting from large values of the angular momentum there are two circular orbits; one stable and one unstable. Lowering the value of the angular momentum these two local extreme values come closer to each other until a minimum value of the angular momentum in which they coincide. At this minimum of the angular momentum the ISCOs are located. Of course, this is not true for the two way traversable wormhole with $\eta = 7M$. In such a spacetime for $u > 0$ there is only one orbit, which is stable. At $u = 0$ there is not a stable orbit, albeit an unstable one. Hence, in this case if we lower the angular momentum, the stable orbit for $u > 0$ comes closer to the unstable orbit at the throat. At the minimum value of the angular momentum these two orbits coincide, forming a stable orbit at the throat and not an ISCO. The above consideration can be more clear with figure 4.6.

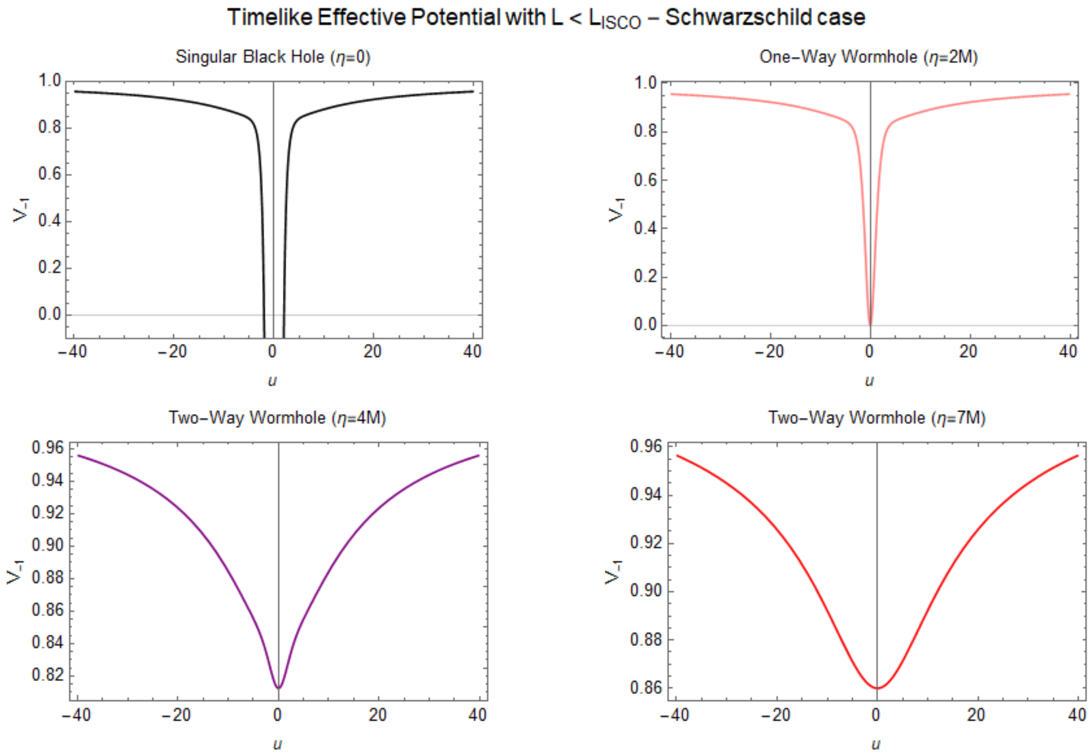

Figure 4.4: *Here we see the effective potential for timelike paths in the Schwarzschild case, for an angular momentum* $L^2 = 10M^2$.

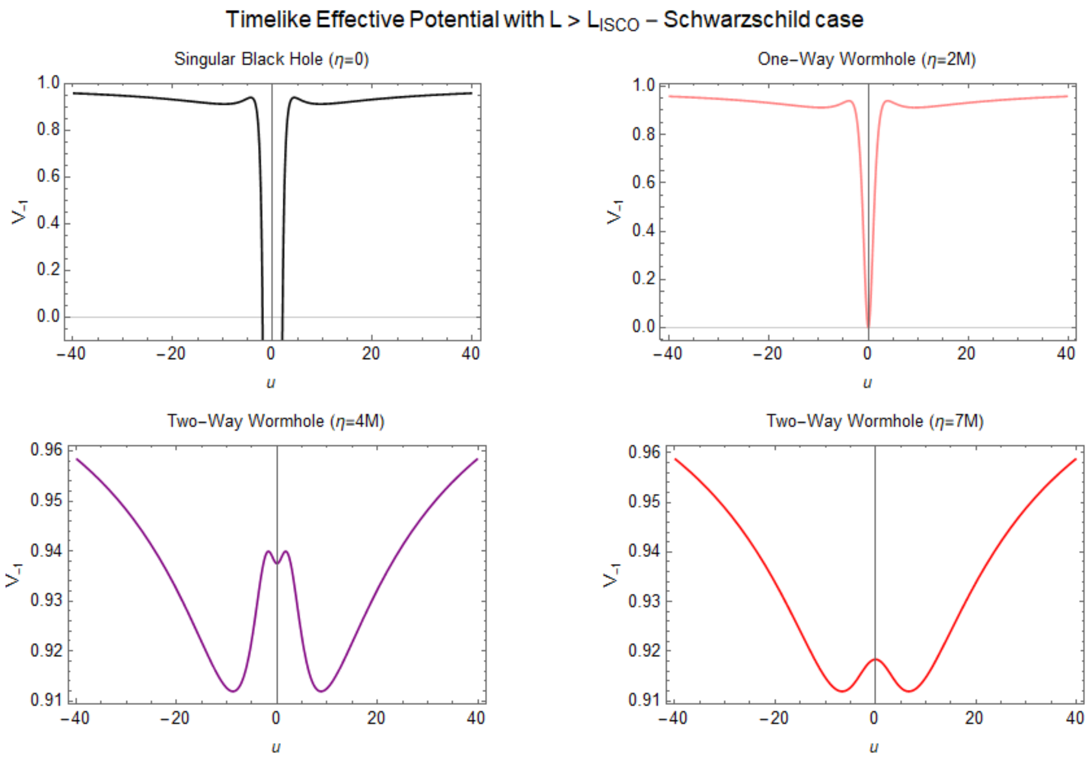

Figure 4.5: *Here we see the effective potential for timelike paths in the Schwarzschild case, for an angular momentum* $L^2 = 14M^2$.

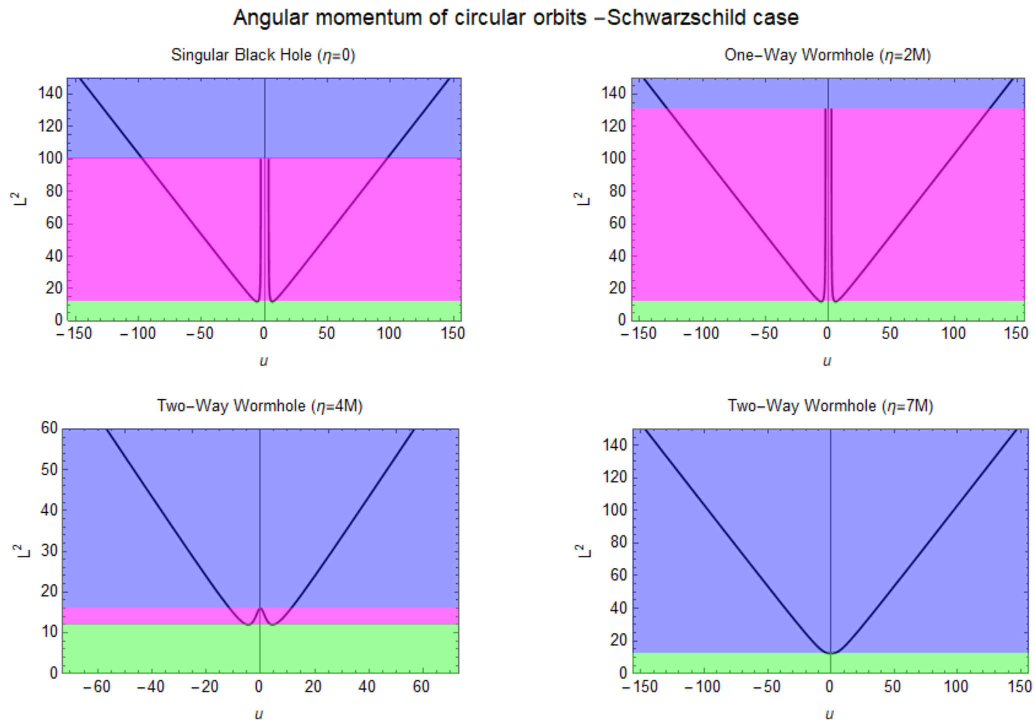

Figure 4.6: *In this graph we see the behaviour of the squared angular momentum* L^2 *in respect with the radial coordinate* u*, for different values of the* η *parameter.*

In the above graph, we see the squared angular momentum in respect with the radial coordinate u. For the colored regions of the above graph we have the following:

1. Blue regions correspond to values of the angular momentum that only one circular orbit is allowed.
2. Pink regions correspond to the angular momentum values that two circular orbits are allowed. One stable and one unstable, with the unstable closer to $u = 0$.
3. Green regions correspond to the angular momentum values that no circular orbit is allowed.

As we see, there is no pink region for the two-way traversable with a throat radius $\eta = 7M$. There is only a blue region which corresponds to only one circular orbit for $u > 0$, which is consistent with the previous discussion.

Remark 1

Question: Which is the upper limit for the wormhole radius η in order to a pink region exist? If it is not clear so far, let me clarify this.

Firstly, let me suppose a wormhole radius larger than the photon sphere. This will not affect our result, but it will make simpler the explanation, as in this way we avoid any constraint concerning the photon sphere and we focus on the timelike orbits. Actually, this is what we have done so far, as both of the two-way traversable wormholes that we have taken have $\eta > 3M$.

In figure 4.6, we see that the pink region is bounded from the L_{ISCO} and the angular

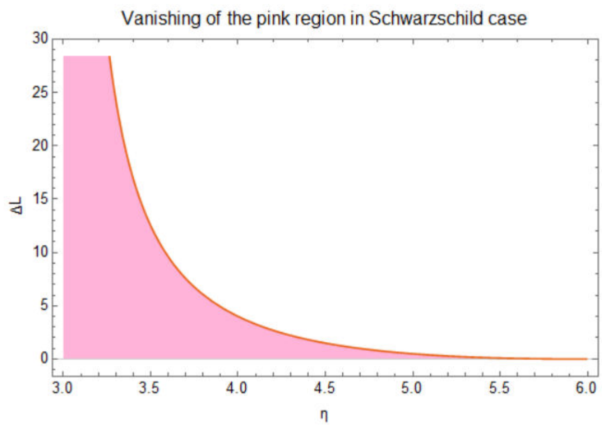

Figure 4.7: *In this graph is presented the function* $\Delta L = \Delta L(\eta)$*, which shows us how the pink region vanishes depending on the throat radius.*

momentum of the circular orbit at the throat, $L(0,\eta)$. So, we define:

$$\Delta L = L^2(0,\eta) - L^2_{ISCO} \tag{4.26}$$

which is a function of η.

For $3M < \eta < 6M$, ΔL is presented in figure 4.7. For $\eta \to 3M$ ΔL blows up to infinity, as was expected due to the denominator of (4.15). Albeit, for $\eta \to 6M$ goes to zero. This means, that for $\eta > 6M$ there is no pink region, and the wormhole can have only one circular orbit.

Remark 2

Figure 4.6 informs us that for a throat radius smaller than the ISCO, there is a limited interval of angular momentum values that two circular orbits exist. This interval is the aforementioned ΔL. In figure 4.5 we see the effective potential of the two way wormhole to have two local extreme values, corresponding to a stable and unstable circular orbit. But this effective potential is for $L^2 = 14M^2$. If we take a larger angular momentum there, which is out of the interval ΔL, then it must be only one extreme value. Indeed, this is case for an angular momentum $L^2 = 18M^2$, as it is shown in figure 4.8.

The angular momentum far away from the throat

The angular momentum of a circular orbit is given by (4.15). With $\sqrt{u_C^2 + \eta^2} \approx u_C$ far from the throat, we get for the angular momentum:

$$L_C^2 \approx M u_C \tag{4.27}$$

where u_c is the location of the circular orbit.

In Classical Mechanics, we know that the effective potential of a point particle of unit mass, moving around of a mass M which is located at $r = 0$, is the following:

$$U_{eff}(r) = \frac{L^2}{2r^2} - \frac{GM}{r^2} \tag{4.28}$$

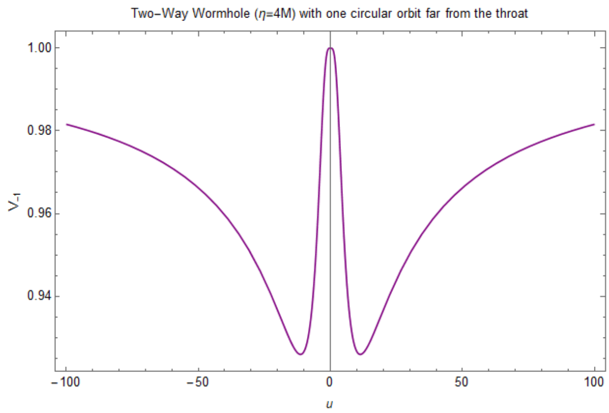

Figure 4.8: *This is the timelike effective potential for a two-way traversable wormhole, with a throat radius smaller than the ISCO, but with such an angular momentum that only one circular orbit exists. Namely, only the stable one, while the unstable has disappeared.*

where the second term denotes the gravitational potential, while the first term denotes the potential of the centrifugal force. Taking $U' = 0$, it is straightforward to show that:

$$L^2 = GMr \tag{4.29}$$

Thence, comparing the above with (4.27), we see that the angular momentum is in agreement with the Newtonian gravity far from the throat (Weak Field limit).

4.5.2 Reissner–Nordström case

In the previous analysis about the effective potentials and the circular orbits the crucial point is the same for the Reissner–Nordström case. There is a parameter η which we are able to vary and according to its values the spacetime describes a singular black hole, a one-way wormhole and a two-way wormhole. There is also the regular black hole but we do not consider this case for the effective potentials.

For the singular black hole ($\eta = 0$) there is always a photon sphere,r_{photon}, and an ISCO, r_{ISCO}, as it was shown in the previous sections.

Thus, for the **null effective potential**, the critical value of the parameter η is r_{photon}; namely, for the wormholes:

1. for $\eta < r_{photon}$ a photon sphere is allowed for the wormholes and a local maximum is appeared for the effective potential for $u > 0$, while at the throat there is a local minimum (stable orbit)

2. for $\eta > r_{photon}$, no photon sphere is allowed and the local maximum is disappeared, while the extreme value of the effective potential at the throat changes to a local maximum.

Indeed, this is the case for the Reissner–Nordström case, as it is shown in figure 4.9.

For the timelike effective potential, the critical value of the parameter η is r_{ISCO}; namely, for the wormholes:

1. for $\eta < r_{ISCO}$, there are two circular orbits, one stable and one unstable, for $u > 0$ and some specific values of the angular momentum

2. for $\eta > r_{ISCO}$, there is only one circular orbit (stable) for $u > 0$.

Indeed, this is the case as it shown in figure 4.10 , where we plot the angular momentum of a circular orbit in respect with the radial coordinate u.

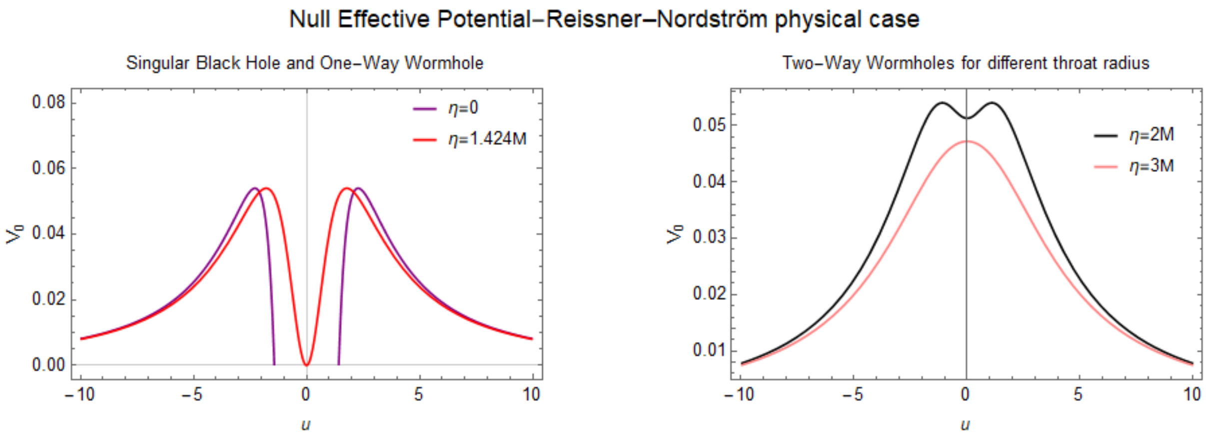

Figure 4.9: *This is the effective potential for the null geodesics for the Reissner–Nordström case, with $Q^2 = 0.82M^2$. For this charge the event horizon and the photon sphere of the Reissner–Nordström black hole are located at $r_h = 1.424M$ and $r_{photon} = 2.281M$.*

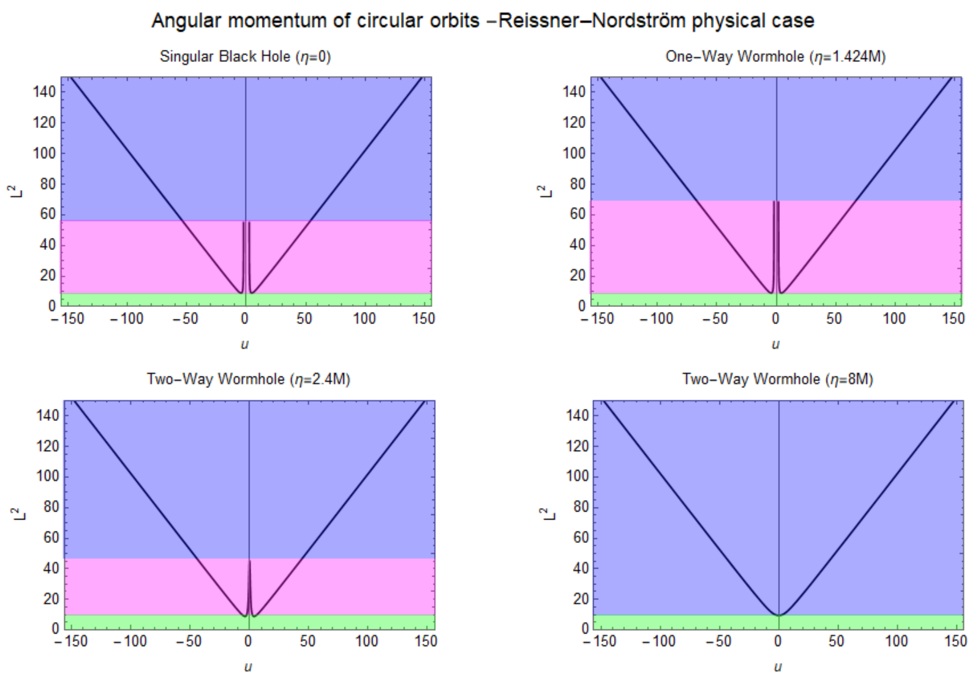

Figure 4.10: *This is the angular momentum for timelike circular orbits for different values of η. Concerns the Reissner–Nordström case, with $Q^2 = 0.82M^2$. For this charge the event horizon, the photon sphere and the ISCO of the Reissner–Nordström black hole are located at $r_h = 1.424M$, $r_{photon} = 2.281M$ and $r_{ISCO} = 4.489$, respectively. (For the extreme case $Q^2 = M^2$, no qualitatively change occurs)*

Finally, the vanishing of the pink region is presented in the following figure:

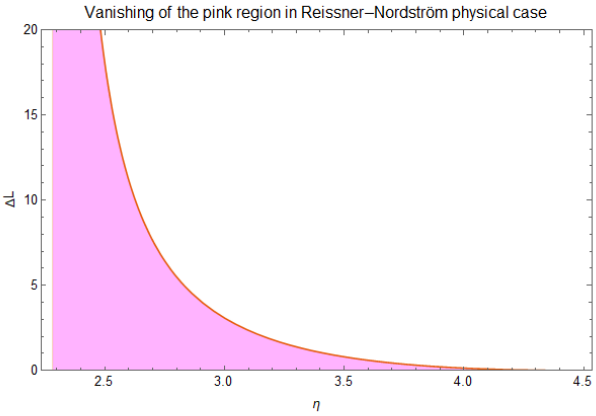

Figure 4.11: *In this graph is presented the function* $\Delta L = \Delta L(\eta)$*, which shows us how the pink region vanishes depending on the throat radius, for the Reissner–Nordström physical case.*

For $\eta \to 2.281M$, which is the photon sphere of the Reissner–Nordström black hole ΔL blows up to infinity, while for $\eta \to 4.489M$, which is the ISCO, reaches the zero value. This is exactly the same qualitatively behaviour with the Schwarzschild case.

Angular momentum far away from the throat

Taking the angular momentum (4.23) far from the throat ($u >> \eta$) we again reduce to: $L_C^2 \approx Mu_C$, which is valid with the Newtonian gravity and Classical Mechanics. The charge Q does not affect because we study uncharged particles; there is no Electrostatic interaction. The charge affects only geometrically the motion of uncharged particles, which is a relativistic effect.

Chapter 5

Conclusion

This thesis started with an attempt to clarify why the Schwarzschild metric and consequently the Einstein-Rosen bridge is not a traversable wormhole, even if possesses a wormhole-like geometry. To the paper of Einstein and Rosen, the idea of a wormhole stands in its immature age. To the paper of Morris and Thorne, finally takes its own way. In chapter 2, which is review of Morris and Thorne traversable wormholes, we saw the following ideas:

1. For a traversable wormhole, there must be no center in the geometry; definition of the throat
2. For a traversable wormhole, there must be no horizon near the throat; two-way traversability
3. In the context of General Relativity such a metric corresponds to an non-vanishing energy-momentum tensor, which violates the Null Energy Condition. Such a violation implies that for fast moving reference frames, a negative energy density might be appeared.
4. The causal structure of a traversable wormhole is like that of Minkowski spacetime, albeit with a different interpretation.

In chapter 3 we extended the Simpson-Visser model for constructing a traversable wormhole, beyond the Schwarzschild case. In a nutshell:

1. We introduced charge and a cosmological constant
2. In the case of a cosmological constant the connected regions are asymptotically flat,dS or AdS, depending on the cosmological constant.
3. For every wormhole the NEC is violated through out all of the spacetime
4. In the case of a positive cosmological constant NEC is not violated at the cosmological horizon

In chapter 4, we see how we are able to distinguish the wormholes and the black holes constructed in the previous chapter. Circular orbits was our tool. Specifically:

1. For the singular black hole a photon sphere and an ISCO are allowed.
2. For a traversable wormhole, photon sphere and ISCO might be not allowed

3. For a traversable wormhole, a photon sphere might be forbidden, while ISCO to be allowed
4. For a traversable wormhole, photon sphere and ISCO might be allowed.
5. The above considerations are qualitatively the same even if we introduce charge and a cosmological constant. The differences are only quantitative.

5.1 Beyond this thesis

The goal of this thesis was to extend the Simpson-Visser technique for regularizing the Schwarzschild metric by the introduction of a cosmological constant and charge. This idea came from a research that I made with my colleague Nikos about wormholes, when we suddenly found the paper of Simpson and Visser [6]. At that time, only a few works was done about these spacetimes, so it was an tempting topic for studying. However, as my thesis was about to be completed, two more papers were published. The first one [12], summarizes the features of some traversable wormholes, regular black holes and black-bounces, including those I studied in this thesis. The second one [13], concerns an extension of the Simpson-Visser technique to the Reissner–Nordström and Kerr black holes. The latter was a benefit for me in two ways. Firstly, it gave me a reference in order to check my results for the charge introduction. Secondly, it was encouraging because it was a clue that the initial idea of Nikos and I, was reasonable and that this kind of spacetimes is an active topic of research. So, let me make some comments in this direction.

Phantom fields as sources for wormholes

Through out of this thesis was made clear that the wormhole geometry demands a non-vanishing stress energy tensor. Specifically, a Stress-Energy tensor that violates the Null Energy Condition. But this is not enough. Our physical theories are field theories and field theories are constructed by actions that we vary in order to specify the corresponding equations of motion.

For static and spherically symmetric spacetimes, if we try to produce a wormhole geometry by a scalar field we unavoidably have to deal with phantom (or ghost) fields; that is, fields with negative kinetic energy. This is the NEC violation in terms of the scalar field. Let me explain this in brief, in the case of a General Relativity action with a minimally coupled scalar field, following [14].

The corresponding action is the following:

$$S = \frac{1}{2} \int \sqrt{-g}\, d^4x (R + 2\mathcal{L}_S)$$

where

$$\mathcal{L}_S = h(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$$

is the Lagrangian of the scalar field ϕ and $V(\phi)$ an arbitrary potential. If $h(\phi) > 0$ we have a canonical scalar field (positive kinetic energy), while for $h(\phi) < 0$ a phantom field. The fact that the other term in the action is just the Ricci scalar, means that we stay in the context of the original General Relativity.

Assuming $\phi = \phi(u)$ and a metric in quasiglobal coordinates:

$$ds^2 = A(x)dt^2 - \frac{dx^2}{A(x)} - r^2(x)d\Omega^2$$

where $r = r(x)$, the equations of motions reduce to the following set of differential equations:

$$\begin{aligned}
&2(Ar^2h\phi')' - Arh'\phi' = r^2\frac{dV}{d\phi} \\
&(A'r^2)' = -2r^2V \\
&\boxed{\frac{r''}{r} = -h(\phi)(\phi')^2} \\
&\boxed{A(r^2)'' - r^2A'' = 2} \\
&-1 + A'rr' + A(r')^2 = r^2(hA(\phi')^2 - V)
\end{aligned}$$

We know that the condition for the existence of the throat implies ($r >, r' = 0, r'' > 0$), where primes denote derivatives in respect with the radial coordinate x. So, the left hand side of the first boxed equation has to be positive (at least near the throat). So, the right hand side has to be positive, too, which cannot happen except for $h(\phi) < 0$; that is, a phantom field.

Can this action produce our wormhole? The answer is clearly negative. In our wormhole we have a specific relation between r and x, which is not valid with the desired $A(x)$ function, according to these equations. Look on the second boxed equation. It is the only equation that contains only terms of the metric. If we put in there $r = \sqrt{x^2 + \eta^2}$, the $A(x)$ function that we get is:

$$A(x) = r^2(x)\left(\frac{c}{\eta^2} + \frac{1}{x^2+\eta^2} + \frac{3m}{\eta^3}\left(\frac{\eta x}{\eta^2 + x^2} + Arctan(x/\eta)\right)\right) \tag{5.1}$$

which is far away from our desires (c,m are some integration contants). For the phantom field and the corresponding potential look [14].

Hence, the classical action of General Relativity minimally coupled with a scalar field is not able to produce our wormhole. So, we have to branch of the original gravity action or to introduce some other fields beyond the real scalar field, minimally or not coupled with the gravity action or both. This is an open question, which is of course crucial but too difficult with so many alternative theories as candidates...

Wormholes without exotic matter

The demand for a NEC violating energy-momentum tensor from the matter content is unavoidable only in the original Einsteinian theory of gravity. However, it is possible in modified theories of gravity to consider a matter energy momentum tensor that satisfies all of the energy conditions, while the metric describes a wormhole geometry. But a wormhole geometry is achieved by demanding a throat to the spacetime; a demand that constraints the $g_{\theta\theta}$ component of the metric. In chapter 1, we saw that the latter constraint is responsible for the violation of the Null Energy Condition. So, a question arises.

How is it possible to have an energy-momentum tensor ascribed to the matter field that does not violate the NEC, while preserves the wormhole geometry described by the metric, at the same time?

In a modified theory of gravity the field equations usually reduce to the following form[15]:

$$G_{\mu\nu} = \kappa^2 T^{eff}_{\mu\nu}$$

The difference of the above with (1.3) is that instead of the matter $T_{\mu\nu}$, in the r.h.s we have an effective energy-momentum tensor. The latter, contains the energy-momentum of the matter field, while also a term that is originated by the modification that we can impose on the gravity theory. In general, one has the following form:

$$T^{eff} = C\, T^{matter} + T^{curvature}$$

where C denotes some terms originated from the modification of the theory. We ascribed to the added term the superscript "curvature" because it contains curvature terms due to the modification of the theory, while this term can be interpreted as a gravitational fluid. The exact form of this component depends on the specific modification that we choose , but the general idea of how we can avoid exotic matter can be stated, even by the above general remark.

The Null Energy Condition according to the modified theory is stated in terms of the effective energy-momentum tensor, rather than the matter field itself. This is the crucial point that allows us to avoid the existence of exotic matter in order to hold the wormhole geometry. The Null Energy Condition is stated as follows:

$$\text{"For any null vector } k^\mu :\ T^{eff}_{\mu\nu} k^\mu k^\nu \geq 0\text{"}$$

As the effective energy-momentum tensor is divided into two terms, it is possible to have $T^{eff}_{\mu\nu} k^\mu k^\nu < 0$ and $T^{matter}_{\mu\nu} k^\mu k^\nu \geq 0$, simultaneously. The first of the latter inequalities is able to ensure a throat for the geometry described by our metric, while the second inequality ensures that the matter field does not describe exotic matter. Moreover, combining these inequalities and assuming that $C > 0$, we are left with a bound for the matter field threading the wormhole:

$$0 \leq T^{matter}_{\mu\nu} k^\mu k^\nu \leq -\frac{1}{C} T^{curvature}_{\mu\nu} k^\mu k^\nu$$

This class of wormholes in modified gravity theories are too interesting, due to the fact that allow wormhole construction without a exotic matter, in contrast to the case in original General Relativity. Of course, this is an active topic of research in a variety of modified gravity theories like as the $f(R)$ gravity, braneworlds e.t.c.

Black bounce to traversable wormhole transitions

Another, more realistic I think, topic for future work concerning these spacetimes are the transitions between the regular black holes and the traversable wormholes that this metric can describe. A first attempt to this direction has been done in [16]. In this paper a trial is presented in order to make the metric (3.16) non-static. As in this paper is stated, this is done "a la Vaidya", which means that they write the black bounce metric in Eddington–Finkelstein coordinates and then they allow the m parameter to depend on the null time coordinate:

$$ds^2 = -\left(1 - \frac{2m(w)}{\sqrt{u^2 + \eta^2}}\right) dw^2 - (\pm dw du) + (u^2 + \eta^2) d\Omega^2$$

where the plus/minus sign corresponds to the outgoing/ingoing null coordinate. For $\eta = 0$ the metric is reduced to the original Vaidya spacetime. (see [17])

For $\eta > 0$ the corresponding energy-momentum tensor has the following form:

$$T = T^{static} + T^{non-static}$$

where T^{static} is the energy-momentum for a constant m, while $T^{non-static}$ is the contribution due to the w dependence of m. For $T^{non-static}$ the only non-vanishing component is

$$T^{non-static}_{ww} = \mp \frac{2u}{(u^2+\eta^2)^{3/2}} \frac{dm}{dw}$$

where the minus/plus sign corresponds to the outgoing/ingoing null coordinate.[16]

Let's see how a transition from a black bounce to a traversable wormhole can be described by this model, in the case of the ingoing null coordinate. As for $m < \eta/2$ the metric describes a wormhole and for $m > \eta/2$ the black bounce, we want $dm/dw < 0$, with $w \to -\infty$: $m > \eta/2$ and $w \to +\infty$: $m < \eta/2$. In this case, the apparent horizon $u_{ah} = \sqrt{4m^2(w) - \eta^2}$ decreases; starting as spacelike (black bounce); for $m(w) = 2\eta$ becomes null (one-way wormhole) and then changes to timelike (wormhole); that is a transition from a regular black hole to a traversable wormhole. The corresponding Penrose diagram is given in figure 5.1. The fact that we use the null ingoing coordinate w means that energy flux is directed to the apparent horizon (in view of "our" universe which we suppose to be $u > 0$) of the black bounce. Hence, we can interpret it as accretion of energy. But what kind of energy is this that makes the apparent horizon decrease?

If we take the null vector

$$\xi^\mu = \left(1, \frac{1}{2}\left(1 - \frac{2m(w)}{\sqrt{u^2+\eta^2}}\right), 0, 0\right)$$

then we find:

$$T^{non-static}_{\mu\nu}\xi^\mu\xi^\nu = \frac{2u}{(u^2+\eta^2)^{3/2}} \frac{dm}{dw}$$

which is clearly negative for $dm/dw < 0$, implying NEC violation. Thus, the black bounce is converted to a traversable wormhole by accretion of negative/phantom energy.

In the case that we use the outgoing null coordinate we do not speak of accretion, but instead for emission. For the NEC in this case we have the null vector:

$$\xi^\mu = \left(1, -\frac{1}{2}\left(1 - \frac{2m(w)}{\sqrt{u^2+\eta^2}}\right), 0, 0\right)$$

and then for NEC we have:

$$T^{non-static}_{\mu\nu}\xi^\mu\xi^\nu = -\frac{2u}{(u^2+\eta^2)^{3/2}} \frac{dm}{dw}$$

which is positive for $dm/dw < 0$. Thence, we speak for a positive energy emission with the corresponding Penrose diagram given in figure 5.2

These models are too interesting, because they can describe regular black hole to wormhole transitions without a topological transition, as both spacetimes are without centres. However, this procedures cannot be formed by a collapsing stellar object, due to the fact that

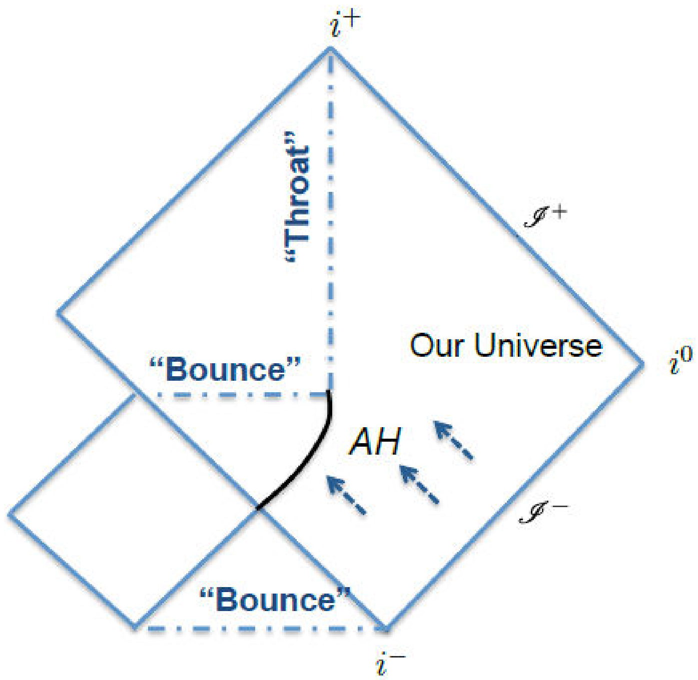

Figure 5.1: *Carter-Penrose diagram for a black bounce to wormhole transition due to the accretion of phantom energy. The arrows indicate the region where the phantom fluid is being accreted. There is a black-bounce in our universe, characterized by an apparent horizon, that converts into a wormhole. Therefore, there is no horizon in our universe. [16]*

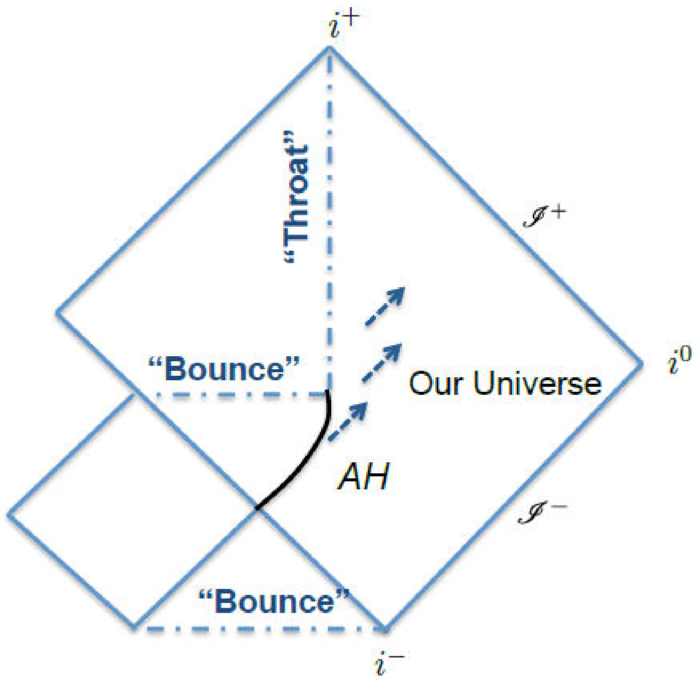

Figure 5.2: *Carter-Penrose diagram for a black-bounce to wormhole transition due to the emission of positive energy. This diagram is very similar to that shown in figure 5.1, however, now there is a (positive) flux being emitted by the black-bounce and wormhole. [16]*

in the limit of $m \to 0$ the metric describes a traversable wormhole rather than a Minkowski spacetime. In order to describe such procedures one must impose "time-dependence" to both m and η, which is a much more complicated concept. It is a research for future work. [12]

Appendix A

Horizons of the A(d)S Schwarzschild black hole

In the case that we add a cosmological constant (Λ), the Einstein's field equation is modified in the following way:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{A.1}$$

The addition of a cosmological constant, physically means that we assume an "always there" energy-momentum source, which is interpreted as the vacuum energy, given by an energy-momentum tensor proportional to the metric: $T^{(vacuum)} \sim g_{\mu\nu}$. In the case of the Schwarzschild solution the $T_{\mu\nu}$ in the right hand side is equal to zero; that is, no matter field is considered (vacuum solution). In addition, the Ricci scalar can be derived by a contraction to the field equations, equal to: $R = 4\Lambda$[1]. Hence, we get:

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{A.2}$$

Solving the above, we get the following metric:

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2} + r^2 d\Omega^2 \tag{A.3}$$

In order to find where the horizons of this metric are located, we have to take $g_{tt} = 0$, which gives the following cubic equation:

$$-\frac{\Lambda}{3} r^3 + r - 2M = 0 \tag{A.4}$$

The cubic equation

Assume the cubic equation:

$$\lambda x^3 - 3\lambda\delta^2 x + y = 0 \tag{A.5}$$

According to [18] there are three solutions, given by:

$$\begin{aligned} x_1 &= 2\delta \sin(\phi) \\ x_2 &= 2\delta \sin\left(\frac{2\pi}{3} + \phi\right) \\ x_3 &= 2\delta \sin\left(\frac{4\pi}{3} + \phi\right) \end{aligned} \tag{A.6}$$

[1]Where $g^{\mu\nu} g_{\mu\nu} = 4$, which is derived by $g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}{}_{\sigma}$

where

$$\sin(3\phi) = \frac{y}{h}$$
$$h = 2\lambda\delta^3 \tag{A.7}$$

We define the discriminant $D = -\delta^6 + \frac{y^2}{4\lambda^2}$. Then,

- if $D > 0$, there is only one real root

- if $D = 0$, all roots are real and at least two are equal

- if $D < 0$, all roots are real and unequal

Negative cosmological constant, AdS

For a negative cosmological constant, we write $\Lambda = -|\Lambda|$. Then:

$$\lambda = \frac{|\Lambda|}{3}, \; \delta = \frac{i}{\sqrt{|\Lambda|}}, \; y = -2M \tag{A.8}$$

Thus, we get for the discriminant:

$$D = \frac{1 + 9M^2|\Lambda|}{|\Lambda|^3} > 0$$

So, there is only one real root; only one horizon.

Substitution of (A.7) to (A.8), gives:

$$\phi = \frac{i}{3} sinh^{-1}\left(-3M\sqrt{|\Lambda|}\right) \tag{A.9}$$

where the transition from sin to sinh was made using the identity: $i\sin(ix) = -sinh(x)$. The real root corresponds to the first solution appeared in (A.6), giving us the location of the horizon at:

$$\boxed{r_h = \frac{2}{\sqrt{|\Lambda|}} \sinh\left[\frac{1}{3}\sinh^{-1}\left(3M\sqrt{|\Lambda|}\right)\right]} \tag{A.10}$$

Assuming small values of the cosmological constant and particularly $3M|\Lambda| << 1$ we can Taylor expand the above expression to find:

$$\boxed{r_h \approx 2M - \frac{8}{9}M^3|\Lambda|} \tag{A.11}$$

that is, smaller than the Schwarzschild radius.

Positive cosmological constant, dS

With a positive cosmological constant we have the following parameters:

$$\lambda = -\frac{\Lambda}{3}, \; \delta = \frac{1}{\sqrt{\Lambda}}, \; y = -2M \tag{A.12}$$

Thus, a discriminant:

$$D = \frac{9M^2\Lambda - 1}{\Lambda^3} \tag{A.13}$$

For $\Lambda < 1/9M^2$ the discriminant is negative, which means that there exist three real and unequal roots. However, the one root (x_3) is negative which means that is out of physical significance. The two roots correspond to the event and the cosmological horizon.

From (A.6), (A.7) and (A.8) it is trivial to get for the event, r_h, and the cosmological, r_c, horizons, respectively:

$$\boxed{\begin{aligned} r_h &= \frac{2}{\sqrt{\Lambda}} \sin\left[\frac{1}{3}\sin^{-1}\left(3M\sqrt{\Lambda}\right)\right] \\ r_c &= \frac{2}{\sqrt{\Lambda}} \sin\left[\frac{1}{3}\sin^{-1}\left(3M\sqrt{\Lambda}\right) + \frac{2\pi}{3}\right] \end{aligned}} \tag{A.14}$$

In the critical value of $\Lambda_{crit} = 1/9M^2$ the two horizons coincide to $1/\sqrt{\Lambda}$. We concern, although, small values for Λ. For $9M^2\Lambda << 1$ the event horizon is located at:

$$\boxed{r_h \approx 2M + \frac{8}{9}M^3\Lambda} \tag{A.15}$$

that is, larger than the Schwarzschild radius.

Appendix B

Horizons of the (A(d)S) Reissner–Nordström black hole

The Reissner–Nordström black hole is one with charge. The information of the black hole's charge is introduced to the Einstein's field equation, by the Electromagnetic energy-momentum tensor:

$$T_{\mu\nu} = F_{\mu\rho}F^{\rho}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{B.1}$$

where $F_{\mu\nu}$ the electromagnetic field tensor.

The field equations are (Einstein's and Maxwell's equations,respectively):

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu} \\ &\nabla^{\nu}F_{\nu\sigma} = 0 \\ &\partial_{\mu}F_{\nu\lambda} + \partial_{\lambda}F_{\mu\nu} + \partial_{\nu}F_{\lambda\mu} = 0 \end{aligned} \tag{B.2}$$

Contraction with $g^{\mu\nu}$ to the Einstein's equation provides $R = kT$. For the Electromagnetic field it is straightforward to show that $T = T^{\alpha}_{\alpha} = 0$. Hence, $R = 0$ and the Einstein's equation takes the following simplified form:

$$R_{\mu\nu} = k\left(F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \tag{B.3}$$

What we do is to compute the Christofell symbols for the spherical symmetric metric and from them the Riemann tensor, while finally the Ricci tensor. On the other hand, we find the components of the $F_{\mu\nu}$ with spherical symmetry concerned and then the $T_{\mu\nu}$. We combine the results via the above equation and the fields have been coupled, giving us a set of differential equations that determine the unknown functions of the metric, in respect of the black hole's charges.

Assuming spherical symmetry, only the radial components of the Electromagnetic field are non-vanishing:

$$\begin{aligned} &E_r = F_{tr} = \frac{Q}{r^2} \\ &F_{\theta\phi} = \frac{P}{r^2} \end{aligned} \tag{B.4}$$

where Q and P are the electric and magnetic charge of the black hole. However, we restrict ourselves to $P = 0$. Then, the result of the above procedure is the following metric:

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega} \tag{B.5}$$

Taking $g_{tt} = 0$, we get a second order polynomial for which we have to find the roots. There are easily found to be the followings:

$$\boxed{r_{\pm} = M \pm \sqrt{M^2 - Q^2}} \tag{B.6}$$

Depending on the values of M, Q, there might be two, one or zero real solutions, while $Q = 0$ provides the Schwarzschild event horizon (the "$+$" sign).

According to the sign of the under-square-root term we distinguish three cases:

1. $M^2 < Q^2$ (unphysical)
   - No event horizon
   - asymptotically flat
   - No Cauchy surface
2. $\boxed{M^2 > Q^2 \text{ (physical)}}$
   - $r_{\pm}$: coordinate singularities
   - asymptotically flat
   - r_+: event horizon like the Schwarzschild one
   - $r_- < r_+$: Cauchy horizon
3. $M^2 = Q^2$ (Extreme Solution)
   - $r = M$: the only horizon
   - asymptotically flat

Introducing a cosmological constant

In this case, if we introduce a cosmological constant the Einstein's field equations have the form of (A.1), albeit with a non-vanishing $T_{\mu\nu}$, but that of the Electromagnetic field. The result is the following metric:

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2} + r^2 d\Omega} \tag{B.7}$$

Taking $g_{tt} = 0$ the corresponding equation is a quartic one, which means that in general there are four roots.The exact solutions of this quartic equation is out of our goal at this moment. We restrict ourselves to an illustration of the situation.

A quartic equation can be written in the form of:

$$x^4 + ax^3 + bx^2 + cx + d = 0 \tag{B.8}$$

We assume that four real and different roots exist. Two relations of the roots important for us are the following:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= -a \\ x_1 x_2 x_3 x_4 &= d \end{aligned} \tag{B.9}$$

In the case of $\Delta_r = 0$, we have that $a = 0$ and $d = -3Q^2/\Lambda$. The $a = 0$ means the absence of third power in the polynomial. For d what concerns us is the sign of it and here is neccesary the distinction between de Sitter and Anti de Sitter spacetime. For $\Lambda < 0$ (AdS), $d > 0$, while for $\Lambda > 0$ (dS), $d < 0$. Looking the product of the roots we obtain the followings:

$\Lambda > 0 \quad \longrightarrow \quad$ One negative/Three positive roots or Three negative/One positive root

$\Lambda < 0 \quad \longrightarrow \quad$ All negative or All positive roots or Two negative/Two positive roots

Looking now to the sum of the roots the cases of "all positive" and "all negative" are forbidden. That is, two positive and two negative roots are allowed for the Anti de Sitter case. As the roots of the polynomial determine the horizons of the black hole, the negative ones are of no physical importance, while the two positive roots imply two horizons. The event and the Cauchy horizon.

$$\boxed{r_{cauchy} < r_h} \tag{B.10}$$

both depending on M, Q, Λ.

In the case of the de Sitter spacetime there might be three horizons. The two of them are the corresponding event and Cauchy horizons, while the third one is not a horizon attached with black hole. This horizon is called as the Cosmological horizon and is a characteristic of the spacetime itself. It is a consequence of the positive sign of the cosmological constant and consequently the positive sign of the scalar curvature. It is like an event horizon of a black hole, by means that no signal can be reached from above the horizon, but is a coordinate dependent horizon. The case of "Three negative/One positive" includes only that horizon, while not an event horizons for the black hole. Thus, according to the censorship conjecture, our concern is limited to the solution of "One negative/Three positive", in which two are the event and Cauchy horizon of the black hole and the third one is the cosmological horizon:

$$\boxed{r_{cauchy} < r_h < r_c} \tag{B.11}$$

all depending on M, Q, Λ.

Bibliography

[1] A. Einstein and N. Rosen. «The Particle Problem in the General Theory of Relativity». In: *Physical Review* 48 (1935), pp. 73–77 (page 1).

[2] Matt Visser. *Lorentzian wormholes: From Einstein to Hawking*. 1995. ISBN: 978-1-56396-653-8 (pages 1, 2, 18, 19).

[3] Sean M Carroll. *Spacetime and geometry*. Cambridge University Press, 2019 (pages 3, 6, 7, 17).

[4] Michael S. Morris and Kip S. Thorne. «Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity». In: *American Journal of Physics* 56.5 (1988), pp. 395–412. DOI: 10.1119/1.15620. eprint: https://doi.org/10.1119/1.15620. URL: https://doi.org/10.1119/1.15620 (pages 3, 5, 19, 23).

[5] Kirill A. Bronnikov and Sergey G. Rubin. *Black Holes, Cosmology and Extra Dimensions*. WSP, 2012. ISBN: 978-981-4374-20-0, 978-981-4440-02-8 (pages 4, 5, 8, 18, 19).

[6] Alex Simpson and Matt Visser. «Black-bounce to traversable wormhole». In: *JCAP* 02 (2019), p. 042. DOI: 10.1088/1475-7516/2019/02/042. arXiv: 1812.07114 [gr-qc] (pages 4, 27, 41, 42, 60).

[7] Jose' P. S. Lemos, Francisco S. N. Lobo, and Sergio Quinet de Oliveira. «Morris-Thorne wormholes with a cosmological constant». In: *Phys. Rev. D* 68 (2003), p. 064004. DOI: 10.1103/PhysRevD.68.064004. arXiv: gr-qc/0302049 (page 11).

[8] Miguel Alcubierre. *Wormholes, Warp Drives and Energy Conditions*. Ed. by Francisco S. N. Lobo. Vol. 189. Springer, 2017. DOI: 10.1007/978-3-319-55182-1. arXiv: 2103.05610 [gr-qc] (pages 11, 21).

[9] C. V. Vishveshwara. «Generalization of the "Schwarzschild Surface" to Arbitrary Static and Stationary Metrics». In: *Journal of Mathematical Physics* 9.8 (Aug. 1968), pp. 1319–1322. DOI: 10.1063/1.1664717 (page 14).

[10] C. W. Kilminster. «Gravitation—an Introduction to Current Research. Edited by Louis Witten. John Wiley; Sons, London. 1962. 481 pp. Illustrated. 113s.» In: *The Journal of the Royal Aeronautical Society* 67.630 (1963), pp. 388–388. DOI: 10.1017/S0001924000062886 (page 15).

[11] Robert M. Wald. *General Relativity*. Chicago, USA: Chicago Univ. Pr., 1984. DOI: 10.7208/chicago/9780226870373.001.0001 (page 43).

[12] Alex Simpson. «Traversable Wormholes, Regular Black Holes, and Black-Bounces». MA thesis. Victoria U., Wellington, 2019. arXiv: 2104.14055 [gr-qc] (pages 60, 65).

[13] Edgardo Franzin, Stefano Liberati, Jacopo Mazza, Alex Simpson, and Matt Visser. *Charged black-bounce spacetimes*. 2021. arXiv: 2104.11376 [gr-qc] (page 60).

[14] Kirill Bronnikov. «Scalar Fields as Sources for Wormholes and Regular Black Holes». In: *Particles* 1.1 (Mar. 2018), p. 5. ISSN: 2571-712X. DOI: 10.3390/particles1010005. URL: http://dx.doi.org/10.3390/particles1010005 (pages 60, 61).
[15] Tiberiu Harko, Francisco S. N. Lobo, M. K. Mak, and Sergey V. Sushkov. «Modified-gravity wormholes without exotic matter». In: *Physical Review D* 87.6 (Mar. 2013). ISSN: 1550-2368. DOI: 10.1103/physrevd.87.067504. URL: http://dx.doi.org/10.1103/PhysRevD.87.067504 (page 62).
[16] Alex Simpson, Prado Martin-Moruno, and Matt Visser. «Vaidya spacetimes, black-bounces, and traversable wormholes». In: *Class. Quant. Grav.* 36.14 (2019), p. 145007. DOI: 10.1088/1361-6382/ab28a5. arXiv: 1902.04232 [gr-qc] (pages 62–64).
[17] Jerry B. Griffiths and Jiri Podolsky. *Exact Space-Times in Einstein's General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge: Cambridge University Press, 2009. ISBN: 978-1-139-48116-8. DOI: 10.1017/CBO9780511635397 (page 63).
[18] R. Nickalls. «Viète, Descartes and the cubic equation». In: *The Mathematical Gazette* 90 (2006), pp. 203–208 (page 66).