# Geodesics, motion of a particle around a Horndeski black hole

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# Overview

1. Basic Horndeski Theory
2. Horndeski Black Hole
    - Timelike Radial Geodesics
    - Timelike Non-Radial Geodesics

# Section 1

1 Basic Horndeski Theory

2 Horndeski Black Hole
- Timelike Radial Geodesics
- Timelike Non-Radial Geodesics

- (**1971**) **Lovelock**: The most general metric theory to acquire second order field equations in an arbitrary number of dimensions
- (**1974**) **Horndeski**: Posed and answered the following important question:

**What is the most general scalar-tensor theory in 4-dimensional spacetime yielding second order field equations?**

Horndeski Theories belong to a general class of scalar-tensor theories with two basic properties:

- In four dimensions they give second-order field equations
- A class of them possesses a classical Galilean symmetry

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$$S_{Horndeski}[\chi, g] = \int d^4x\sqrt{-g}\Big[K(\chi, X) - G_3(\chi, X)\mathcal{E}_1 + G_4(\chi, X)R + G_{4,X}\mathcal{E}_2 + G_5(\chi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\chi - \frac{G_{5,X}}{6}\mathcal{E}_3\Big]$$

where

$$X = -\frac{1}{2}(\nabla\chi)^2$$
$$\mathcal{E}_n = n!\nabla_{[\mu_1}\nabla^{\mu_1}\chi\cdots\nabla_{\mu_n]}\nabla^{\mu_n}\chi$$

and

$$G_{4,X} = \frac{\partial G_4}{\partial X}$$

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We concentrate on the term:

$$I = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - (g^{\mu\nu} - G(\chi)G^{\mu\nu})\nabla_\mu\chi\nabla_\nu\chi\right]$$

*Cosmological applications* when $G(\chi)$ is a constant:

- Accelerated expansion without the need of any scalar potential (Amendola)
- Second-order field equations in accordance with Horndeski's theory (Sushkov)
- Inflationary phase (Sushkov, Germani, Kehagias)
- Late-time cosmology (Saridakis, Sushkov)
- Particle production after inflation (Koutsoumbas, Ntrekis, Papantonopoulos)
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We consider the Lagrangian

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}\left(g^{\mu\nu} - zG^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi = -F(r)\dot{t}^2 + G(r)\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2)$$

$$F(r) = \frac{3}{4} + \frac{r^2}{12z} - \frac{2M}{r} + \frac{\sqrt{z}}{4r}\arctan\left(\frac{r}{\sqrt{z}}\right), \qquad G(r) = \frac{(r^2+2z)^2}{4(r^2+z)^2F(r)}$$

Event Horizon at: $\nabla_\mu r \nabla^\mu r = 0 \rightarrow F(r) = 0$

Euler-Lagrange equations of motion: $\left(\dot{\Pi}_q - \frac{\partial\mathcal{L}}{\partial q} = 0\right)$,

- $E = F(r)\dot{t}$
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$$\mathcal{L} = \frac{E^2}{F(r)} - G(r)\dot{r}^2 - \frac{L^2}{r^2} \equiv h$$

- $h = 0 \rightarrow$ photons
- $h = 1 \rightarrow$ massive particles

Radial Equation

$$\dot{r}^2 = \frac{E^2}{F(r)G(r)} - \frac{1}{G(r)}\left(\frac{L^2}{r^2} + h\right)$$

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- $r_i = (96Mz)^{\frac{1}{3}}$
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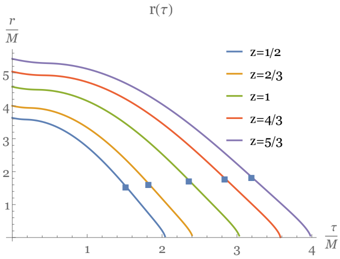


Figure: Orbits with respect to $\tau$ for different values of $z$

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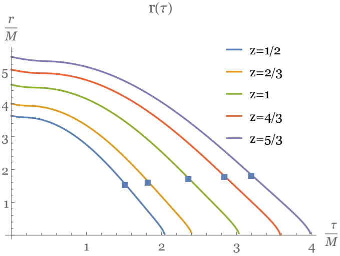
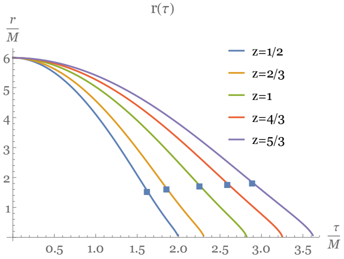



Figure: Orbits with respect to $\tau$ for different values of $z$

For fixed $r_i$

- $E = \frac{\sqrt{\frac{\left(z^{\frac{3}{2}} \arctan\left(\frac{6}{\sqrt{z}}\right) + 10z + 72\right)}{z}}}{2\sqrt{6}}$

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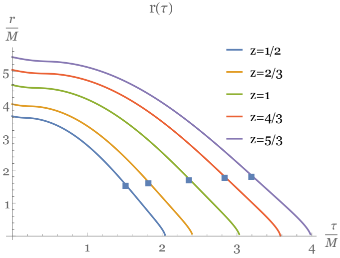


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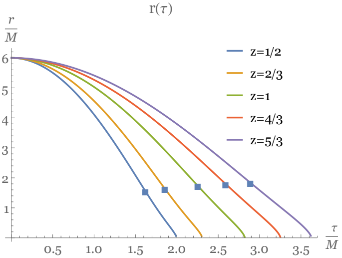


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For large z: $\frac{r\sqrt{z}}{4r\sqrt{z}} - \frac{2}{r} + \frac{3}{4} = 0 \Rightarrow r_{horizon} \to 2$

Timelike $(h = 1)$ non-radial $(L \neq 0)$ geodesics$\rightarrow$

$$\dot{r}^2 = \frac{E^2}{F(r)G(r)} - \frac{1}{G(r)}\left(\frac{L^2}{r^2} + 1\right)$$

For small z, the radial equation takes the form

$$\dot{r}^2 = 4E^2 - \left(3 + \frac{r^2}{3z} - \frac{8M}{r} + \frac{\sqrt{z}}{r}\frac{\pi}{2}\right)\left(\frac{L^2}{r^2} + 1\right)$$

By making the transformation $r = u^{-1}$,

$$\left(u\frac{du}{d\phi}\right)^2 = \left(8M - \sqrt{z}\frac{\pi}{2}\right)\left(u^5 - \frac{3}{8M - \sqrt{z}\frac{\pi}{2}}u^4 + \frac{1}{L^2}u^3 + \right.$$

$$\left. \underbrace{\frac{4E^2 - 3 - \frac{L^2}{3z}}{L^2\left(8M - \sqrt{z}\frac{\pi}{2}\right)}}_{\text{negative for bounded orbits}} u^2 - \frac{1}{3zL^2\left(8M - \sqrt{z}\frac{\pi}{2}\right)}\right)$$

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For fixed $r_i = 2.5$

$$\rightarrow E^2 = \frac{1}{24}\left(\frac{3\pi L^2\sqrt{z}}{r_i^3} - \frac{48L^2}{r_i^3} + \frac{18L^2}{r_i^2} + \frac{2L^2}{z} + \frac{2r_i^2}{z} + \frac{3\pi\sqrt{z}}{r_i} - \frac{48}{r_i} + 18\right)$$

$$\text{Bounded orbits when: } \frac{4E^2 - 3 - \frac{L^2}{3z}}{L^2\left(8M - \sqrt{z}\frac{\pi}{2}\right)} < 0$$

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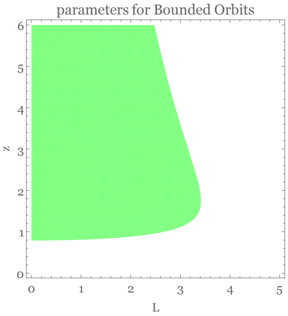


Figure: Parameter Space for bounded orbits

$$\phi = \int \frac{u du}{\sqrt{\left(8M - \sqrt{z}\frac{\pi}{2}\right)u^5 - 3u^4 + \frac{8M - \sqrt{z}\frac{\pi}{2}}{L^2}u^3 + \frac{4E^2 - 3 - \frac{L^2}{3z}}{L^2}u^2 - \frac{1}{3zL^2}}}$$

or $\phi = \int \frac{udu}{\sqrt{au^5 + bu^4 + gu^3 + eu^2 + d}}$, and if $z << 1$ then $e, d >> 1$

$$\phi = \int \frac{udu}{\sqrt{(eu^2 + d)\left(\frac{au^5 + bu^4 + gu^3}{eu^2 + d} + 1\right)}}$$

$$= -\frac{1}{2}\frac{3u(a(-15d^2 - 5deu^2 + 2e^2u^4) + 4eg(eu^2 + 3d)) - 8b(8d^2 + 4deu^2 - e^2u^4)}{24e^3\sqrt{eu^2 + d}}$$

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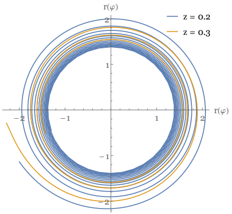


Figure: Unbounded orbits for different z

# Conclusions

- **Radial Geodesics**: As we increase $z$, the horizon radius becomes larger and approaches the value 2, and the time at which particles cross the horizon increases as well.
- **Non-Radial Geodesics**: There are unbounded orbits. As we decrease $z$, particles are falling with higher velocity towards the horizon. Their trajectory approaches asymptotically a circle of radius the horizon radius, too.

**To be Continued**...