Geodesics, motion of a particle around a Horndeski black hole

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Overview

1. Basic Horndeski Theory

2. Horndeski Black Hole
   - Timelike Radial Geodesics
   - Timelike Non-Radial Geodesics
Section 1

1. Basic Horndeski Theory

2. Horndeski Black Hole
   - Timelike Radial Geodesics
   - Timelike Non-Radial Geodesics
(1971) **Lovelock**: The most general metric theory to acquire second order field equations in an arbitrary number of dimensions

(1974) **Horndeski**: Posed and answered the following important question:

What is the most general scalar-tensor theory in 4-dimensional spacetime yielding second order field equations?

Horndeski Theories belong to a general class of scalar-tensor theories with two basic properties:

- In four dimensions they give second-order field equations
- A class of them possesses a classical Galilean symmetry

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- A class of them possesses a classical Galilean symmetry

(Deffayet, Esposito-Farese, Vikman)
\[ S_{\text{Horndeski}}[\chi, g] = \int d^4 x \sqrt{-g} \left[ K(\chi, X) - G_3(\chi, X) \mathcal{E}_1 \right. \\
+ G_4(\chi, X) R + G_{4,X} \mathcal{E}_2 + G_5(\chi, X) G_{\mu \nu} \nabla^\mu \nabla^\nu \chi - \frac{G_{5,X}}{6} \mathcal{E}_3 \right] \]

where

\[ X = -\frac{1}{2} \left( \nabla \chi \right)^2 \]

\[ \mathcal{E}_n = n! \nabla_{[\mu_1} \nabla^{\mu_1} \chi \cdots \nabla_{\mu_n]} \nabla^{\mu_n} \chi \]

and

\[ G_{4,X} = \frac{\partial G_4}{\partial X} \]

The Horndeski terms are also called generalized (arbitrary $G_i$) galileons.
Basic Horndeski Theory
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The Horndeski terms are also called generalized (arbitrary $G_i$) galileons.
We concentrate on the term:

$$I = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - (g^{\mu\nu} - G(\chi)G^{\mu\nu}) \nabla_\mu \chi \nabla_\nu \chi \right]$$

Cosmological applications when $G(\chi)$ is a constant:

- Accelerated expansion without the need of any scalar potential (Amendola)
- Second-order field equations in accordance with Horndeski’s theory (Sushkov)
- Inflationary phase (Sushkov, Germani, Kehagias)
- Late-time cosmology (Saridakis, Sushkov)
- Particle production after inflation (Koutsoumbas, Ntrekis, Papantonopoulos)
- Reheating with Derivative Coupling (Dalianis, Koutsoumbas, Ntrekis, Papantonopoulos)
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Section 2

1. Basic Horndeski Theory

2. Horndeski Black Hole
   - Timelike Radial Geodesics
   - Timelike Non-Radial Geodesics
We consider the Lagrangian

\[ \mathcal{L} = \frac{R}{2} - \frac{1}{2} (g^{\mu\nu} - z G^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi = -F(r)\dot{t}^2 + G(r)\dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 (\theta) \dot{\phi}^2) \]

\[ F(r) = \frac{3}{4} + \frac{r^2}{12z} - \frac{2M}{r} + \frac{\sqrt{z}}{4r} \arctan \left( \frac{r}{\sqrt{z}} \right), \quad G(r) = \frac{(r^2 + 2z)^2}{4(r^2 + z)^2 F(r)} \]

Event Horizon at: \( \nabla_\mu r \nabla^\mu r = 0 \rightarrow F(r) = 0 \)

Euler-Lagrange equations of motion: \( \dot{\Pi}_q - \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0 \),

- \( E = F(r)\dot{t} \)
- \( L = r^2 \dot{\phi} \)

Radial Equation

\[ \dot{r}^2 = \frac{E^2}{F(r)G(r)} - \frac{1}{G(r)} \left( \frac{L^2}{r^2} + h \right) \]

- \( h = 0 \rightarrow \) photons
- \( h = 1 \rightarrow \) massive particles
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Timelike \((h = 1)\) radial \((L = 0)\) geodesics: 
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For large \(r\)

- \(r_i = (96Mz)^{\frac{1}{3}}\)

- \(E^2 = \frac{3}{4} + \left(\frac{9M^2}{4z}\right)^\frac{1}{3}\)

**Figure:** Orbits with respect to \(\tau\) for different values of \(z\)
Timelike ($h = 1$) radial ($L = 0$) geodesics: $\dot{r}^2 = \frac{E^2}{F(r)G(r)} - \frac{1}{G(r)}$

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For fixed $r_i$

- $E = \sqrt{\frac{\left( \frac{3}{2} \frac{3}{2} \arctan \left( \frac{6}{\sqrt{z}} \right) + 10z + 72 \right)}{2\sqrt{6}}}$

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For large \(z\):

\[
\frac{r\sqrt{z}}{4r\sqrt{z}} - \frac{2}{r} + \frac{3}{4} = 0 \Rightarrow r_{\text{horizon}} \rightarrow 2\]
Timelike \((h = 1)\) non-radial \((L \neq 0)\) geodesics→

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\dot{r}^2 = \frac{E^2}{F(r)G(r)} - \frac{1}{G(r)} \left( \frac{L^2}{r^2} + 1 \right)
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For small \(z\), the radial equation takes the form

\[
\dot{r}^2 = 4E^2 - \left( 3 + \frac{r^2}{3z} - \frac{8M}{r} + \frac{\sqrt{z} \pi}{2r} \right) \left( \frac{L^2}{r^2} + 1 \right)
\]

By making the transformation \(r = u^{-1}\),

\[
\left( u \left[ \frac{du}{d\phi} \right] \right)^2 = \left( 8M - \frac{\sqrt{z} \pi}{2} \right) \left( u^5 - \frac{3}{8M - \sqrt{z} \pi} u^4 + \frac{1}{L^2} u^3 + \frac{4E^2 - 3 - \frac{L^2}{3z}}{L^2 \left( 8M - \sqrt{z} \frac{\pi}{2} \right)} u^2 - \frac{1}{3zL^2 \left( 8M - \sqrt{z} \frac{\pi}{2} \right)} \right)
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negative for bounded orbits
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negative for bounded orbits
For fixed $r_i = 2.5$

$$E^2 = \frac{1}{24} \left( \frac{3\pi L^2 \sqrt{z}}{r_i^3} - \frac{48L^2}{r_i^3} + \frac{18L^2}{r_i^2} + \frac{2L^2}{z} + \frac{2r_i^2}{z} + \frac{3\pi \sqrt{z}}{r_i} - \frac{48}{r_i} + 18 \right)$$

Bounded orbits when:

$$\frac{4E^2 - 3 - \frac{L^2}{3z}}{L^2 \left( 8M - \sqrt{z} \frac{\pi}{2} \right)} < 0$$
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Bounded orbits when:

$$\frac{4E^2 - 3 - \frac{L^2}{3z}}{L^2 \left( 8M - \sqrt{z} \frac{\pi}{2} \right)} < 0$$

**Figure:** Parameter Space for bounded orbits

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*Geodesics, motion of a particle around a Horndeski black hole*
\[
\phi = \int \frac{udu}{\sqrt{\left(8M - \sqrt{z} \frac{\pi}{2}\right) u^5 - 3u^4 + \frac{8M - \sqrt{z} \frac{\pi}{2}}{L^2} u^3 + \frac{4E^2 - 3 - \frac{L^2}{3z}}{L^2} u^2 - \frac{1}{3zL^2}}}
\]

or
\[
\phi = \int \frac{udu}{\sqrt{a u^5 + b u^4 + g u^3 + e u^2 + d}}, \quad \text{and if } z << 1 \quad \text{then} \quad e, d >> 1
\]

\[
\phi = \int \frac{udu}{\sqrt{(e u^2 + d) \left( \frac{a u^5 + b u^4 + g u^3}{e u^2 + d} + 1 \right)}}
\]

\[
= - \frac{1}{2} \frac{3u(a(-15d^2 - 5deu^2 + 2e^2u^4) + 4eg(eu^2 + 3d)) - 8b(8d^2 + 4deu^2 - e^2u^4)}{24e^3 \sqrt{e u^2 + d}}
\]

\[
+ \frac{1}{2} \frac{3d(5ad - 4eg) \log(\sqrt{e \sqrt{e u^2 + d} + eu})}{8e^{7/2}} + \frac{\sqrt{e u^2 + d}}{e}
\]
\[ \phi = \int \frac{u \, du}{\sqrt{\left( 8M - \sqrt{z} \frac{\pi}{2} \right) u^5 - 3u^4 + \frac{8M - \sqrt{z} \frac{\pi}{2}}{L^2} u^3 + \frac{4E^2 - 3 - \frac{L^2}{3z}}{L^2} u^2 - \frac{1}{3zL^2}}} \]

or
\[ \phi = \int \frac{u \, du}{\sqrt{au^5 + bu^4 + gu^3 + eu^2 + d}} \]

and if \( z \ll 1 \) then \( e, d \gg 1 \)

\[ \phi = \int \frac{u \, du}{\sqrt{(eu^2 + d) \left( \frac{au^5 + bu^4 + gu^3}{eu^2 + d} + 1 \right)}} \]

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\[ + \frac{1}{2} \left[ 3d(5ad - 4eg) \log(\sqrt{e \sqrt{eu^2 + d} + eu}) + \frac{\sqrt{eu^2 + d}}{8e^{7/2}} \right] + \frac{\sqrt{eu^2 + d}}{e} \]

**Figure:** Unbounded orbits for different \( z \)

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Conclusions

- **Radial Geodesics**: As we increase $z$, the horizon radius becomes larger and approaches the value 2, and the time at which particles cross the horizon increases as well.

- **Non-Radial Geodesics**: There are unbounded orbits. As we decrease $z$, particles are falling with higher velocity towards the horizon. Their trajectory approaches asymptotically a circle of radius the horizon radius, too.

To be Continued...