From Compact Objects to Quasi-Normal Modes and Back

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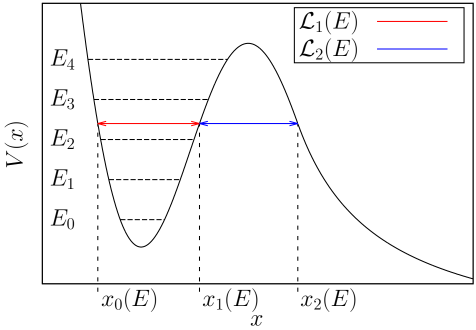
Overview

1. Introduction
2. Direct Problem
3. Surprise?!
4. Detectability
5. Conclusions



COMPACT OBJECTS

- Neutron stars and black holes are “well known”
- Other exotic objects: boson stars, gravastars, wormholes ...
- Their properties are different from black holes and neutron stars [1]
- **What are their properties and how can we learn about them?**

[1] New review: Cardoso and Pani (2017)

QUASI-NORMAL MODES (QNM)

- General kind of perturbation for dissipative systems
- Well studied for black holes and compact stars [2]

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu} \tag{1}$$

- Now we focus on the so-called axial modes

$$\frac{\mathrm{d}^2}{\mathrm{d}r^{*2}}\psi + \left(\omega_n^2 - V(r)\right)\psi = 0 \tag{2}$$

- QNM ω_n is different for many **exotic compact objects (ECOs)**

[2] Reviews: Kokkotas and Schmidt (1999); Nollert (1999); Berti et al. (2009)

POTENTIAL SCATTERING

- The QNM spectrum can be understood from potential scattering
- Different type of **potentials** and **boundary conditions** produce different spectra
- Schwarzschild black hole: single potential barrier
- Stars: repulsive barrier
- **Ultra compact stars**: repulsive barrier + single potential barrier

BH VS ECO

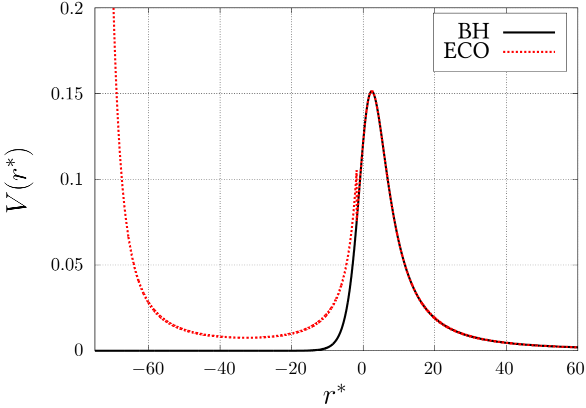


FIGURE 1: Schwarzschild BH potential vs ECO potential for constant density star $R/M = 2.26$, both for $l = 2$.

SOLVE WAVE EQUATION

- Solve wave equation to get QNM spectrum ω_n
- Analytic, semi-analytic and numerical methods are available
- WKB method is valuable for analytic and semi-analytic work
- For analytic studies now: **Bohr-Sommerfeld (BS) rule**

BOHR-SOMMERFELD RULES

- Well known for potential wells (**bound states**)

$$\int_{x_0}^{x_1} \sqrt{\omega_n^2 - V(x)}\,\mathrm{d}x = \pi\left(n + \frac{1}{2}\right) \tag{3}$$

- Generalized rule for **quasi-stationary states** [3]

$$\int_{x_0}^{x_1} \sqrt{\omega_n^2 - V(x)}\,\mathrm{d}x = \pi\left(n + \frac{1}{2}\right) - \frac{i}{4}\exp\left(2i\int_{x_1}^{x_2} \sqrt{\omega_n^2 - V(x)}\,\mathrm{d}x\right), \tag{4}$$

with turning points x_0, x_1, x_2 and $\omega_n^2 \equiv E_n$

[3] Popov et al. (1991)

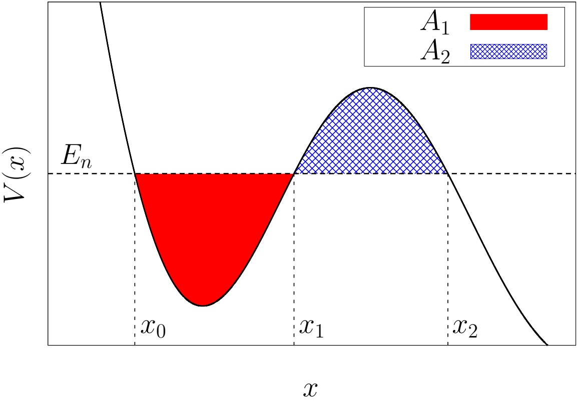


FIGURE 2: Typical potential containing quasi-stationary states, taken from Völkel and Kokkotas (2017a).

CONSTANT DENSITY STARS

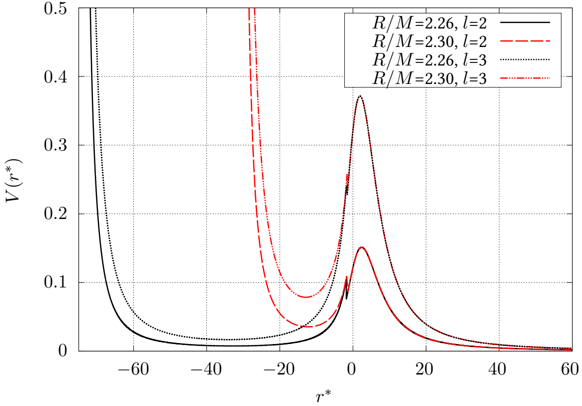


FIGURE 3: Effective potential $V(r^*)$ for axial perturbations for non-rotating constant density stars, taken from Völkel and Kokkotas (2017a).

RESULTS

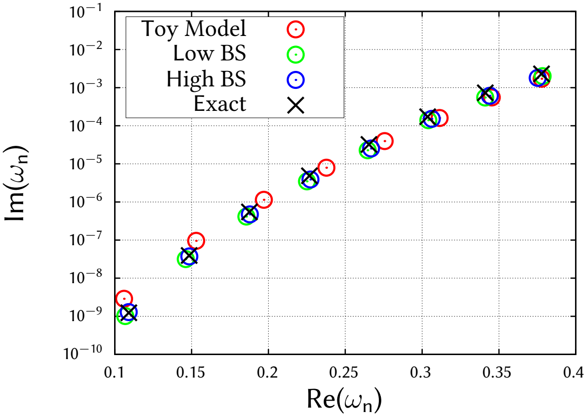
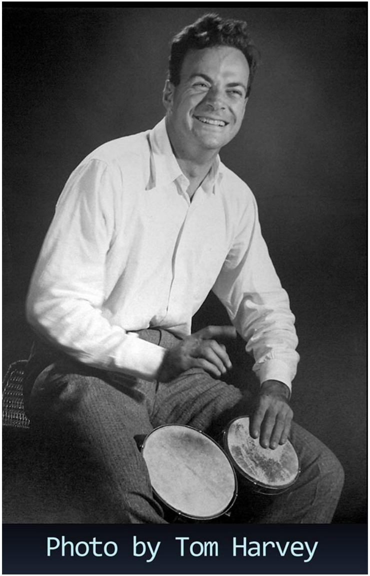


FIGURE 4: Comparison of different Bohr-Sommerfeld results for the QNM spectrum ω_n for a constant density star with $R/M = 2.26, l = 2$ with full numerical results from Kokkotas (1994).

CAN ONE HEAR THE SHAPE OF A DRUM?



FROM SPECTRUM TO POTENTIAL

Can one hear the shape of a drum? [4]

- GW observation will provide one with QNM spectrum ω_n, not $V(x)$
- **Can we reconstruct $V(x)$ from ω_n?**
- Great value of WKB/BS: **Yes, but** ...
- Depends on qualitative type of potential and validity of WKB/BS

[4] Kac (1966); Gordon et al. (1992)

INVERSE METHOD

How can we reconstruct the potential?

- We invert WKB/BS integral approach
- Method known for individual potential wells and barriers, respectively
- We will generalize these results for our type of potential
- **Method in principle applicable for any potential of that kind**

Reconstruct Potential

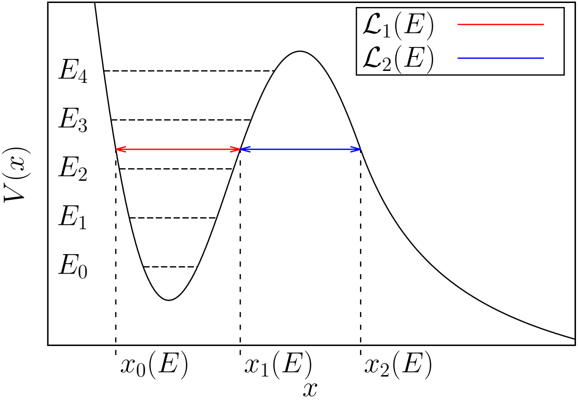


Figure 5: Reconstruction of width $\mathcal{L}_1(E)$ and $\mathcal{L}_2(E)$, taken from Völkel and Kokkotas (2017b).

Inverting Integrals

- Reconstruction of **widths** $\mathcal{L}_1(E)$ and $\mathcal{L}_2(E)$ possible [5]

$$\mathcal{L}_1(E) = x_1 - x_0 = 2\frac{\partial}{\partial E}\int_{E_{\min}}^{E} \frac{n(E') + 1/2}{\sqrt{E - E'}} \mathbf{d}E', \tag{5}$$

$$\mathcal{L}_2(E) = x_2 - x_1 = -\frac{1}{\pi}\int_{E}^{E_{\max}} \frac{(\mathbf{d}T(E')/\mathbf{d}E')}{T(E')\sqrt{E' - E}} \mathbf{d}E'. \tag{6}$$

with **turning points** $x_0(E), x_1(E), x_2(E)$, **spectrum** $n(E)$ and **transmission** $T(E)$

- **Connection** between transmission $T(E)$ and QNM spectrum $\omega_n^2 = E_n$ found within WKB/BS [6]

[5] Wheeler (1976); Lazenby and Griffiths (1980)
[6] Völkel and Kokkotas (2017b)

RESULTS I

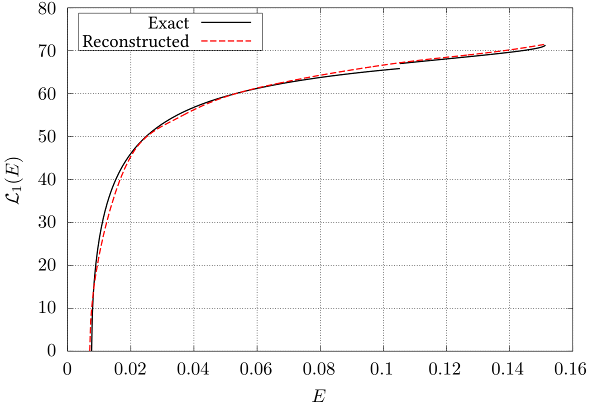


FIGURE 6: Reconstruction of width $\mathcal{L}_1(E)$ for a constant density star with $R/M = 2.26, l = 2$, taken from Völkel and Kokkotas (2017b).

RESULTS II

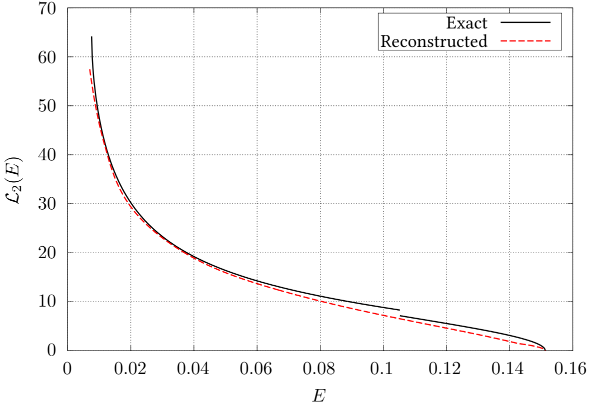


FIGURE 7: Reconstruction of width $\mathcal{L}_2(E)$ for a constant density star with $R/M = 2.26, l = 2$, taken from Völkel and Kokkotas (2017b).

RESULTS III

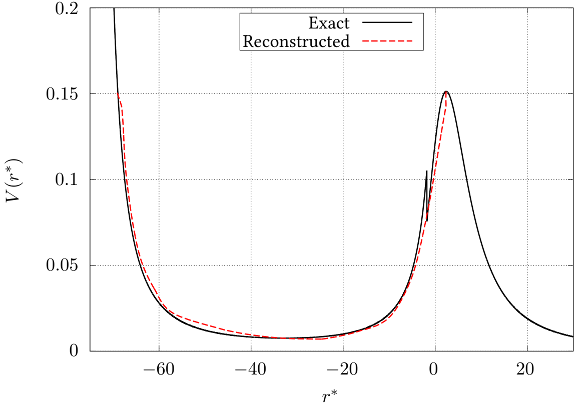


FIGURE 8: Reconstruction of potential $V(r^*)$ for a constant density star with $R/M = 2.26, l = 2$, taken from Völkel and Kokkotas (2017b).

DISCUSSION

- Method works in principle for any potential of the type shown (between $E_{\text{min}}, E_{\text{max}}$)
- Results are approximative: WKB/BS, (inter-/extrapolation) for $n(E)$
- Accuracy significantly increases with the number of trapped modes
- **Birkhoff's theorem used to find unique solution**
- More complicated for rotating systems

DETECTABILITY

Could current gravitational wave detectors detect exotic compact objects?

- Project with A. Maselli [7] presents first simple **parameter estimation** for phenomenological "echo"[8] templates
- If such objects exist and the time evolution can be sufficiently described by the templates, they should partially already be detectable with current detectors
- Exotic compact objects refer to a large class of objects, more realistic and detailed work is necessary for any serious conclusions, many unknowns

[7] Maselli, Völkel, and Kokkotas (2017)
[8] Abedi et al. (2016); Ashton et al. (2016); Abedi et al. (2017)

CONCLUSIONS

- **WKB/BS methods are great tools to study gravitational waves of compact objects:**
  - $\Rightarrow$ There are analytic and semi-analytic ways to **calculate the QNM spectrum** ω_n of the object
  - $\Leftarrow$ WKB/BS methods can be used to **solve the inverse problem**, to reconstruct the potential $V(x)$ from the spectrum ω_n
- **Parameter estimation** of exotic compact objects from GWs should be possible within the next years
- **The presented work can be found in:**

1) Völkel and Kokkotas, *Class. and Quant. Grav.*, 34(12):125006, 2017
2) Völkel and Kokkotas, *Class. and Quant. Grav.*, 34(17):175015, 2017
3) Maselli, Völkel and Kokkotas, acc. in *Phys. Rev. D*, arXiv:1708.02217

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