Canonical Exorcism for Cosmological Ghosts

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Canonical Exorcism for Cosmological Ghosts

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Cosmological Ghosts

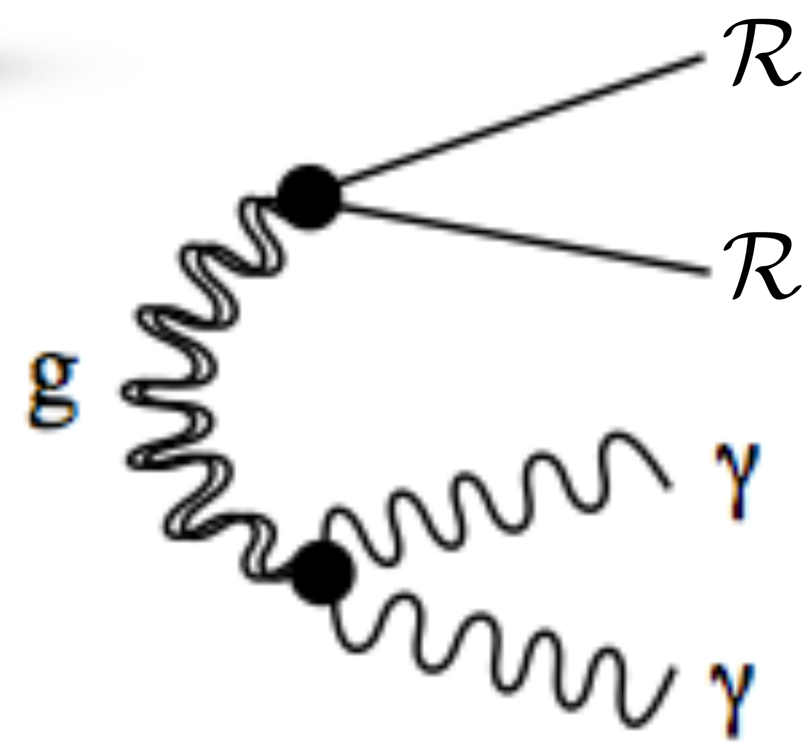
$$S[\mathcal{R}] = \frac{1}{2}\int d\tau d^3\mathbf{x}\, Z\left(\left(\mathcal{R}'\right)^2 - c_{\mathrm{S}}^2\left(\partial_i\mathcal{R}\right)^2\right) \qquad \mathcal{R} = \Phi + H\frac{\delta\varphi}{\dot{\varphi}},$$

Ghost $Z(t) < 0$

$$H_{\mathbf{k}} = \frac{\left|P_{\mathbf{k}}\right|^2}{2Z} + \frac{Zc_S^2k^2\left|\mathcal{R}_{\mathbf{k}}\right|^2}{2} < 0$$

ghosts - modes (oscillators)

*with the **negative mass***



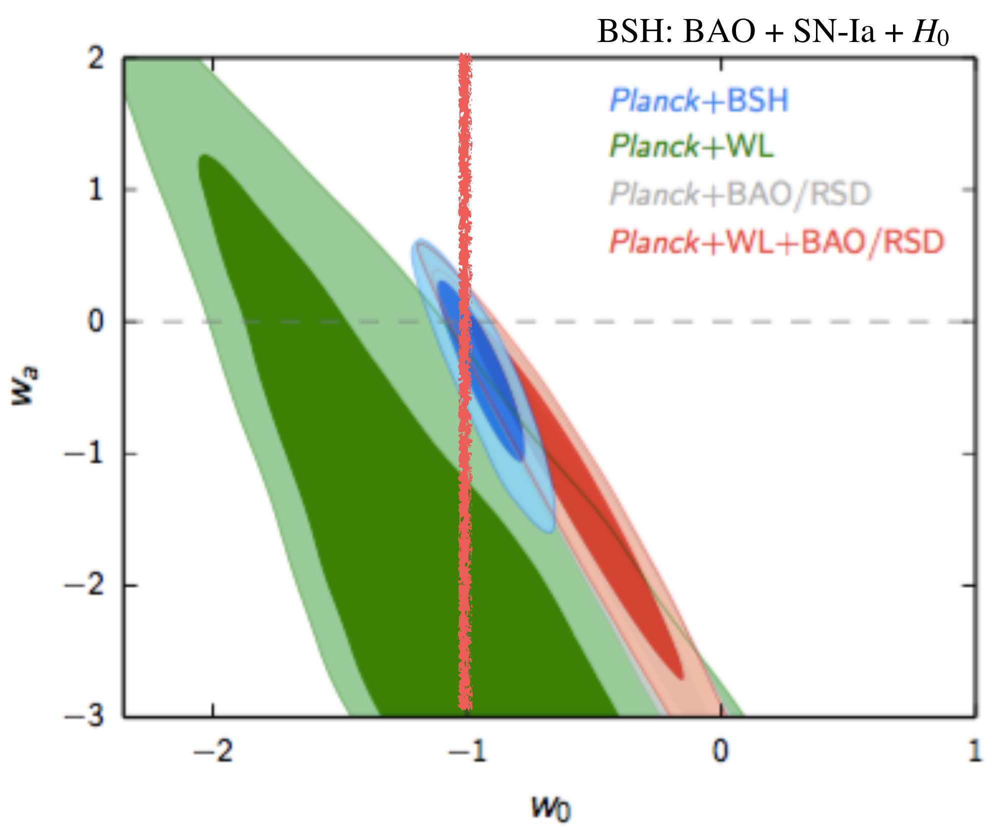
$$\Gamma_{0\to 2\gamma 2\phi} \sim \frac{\Lambda^8}{M_{\mathrm{Pl}}^4}$$

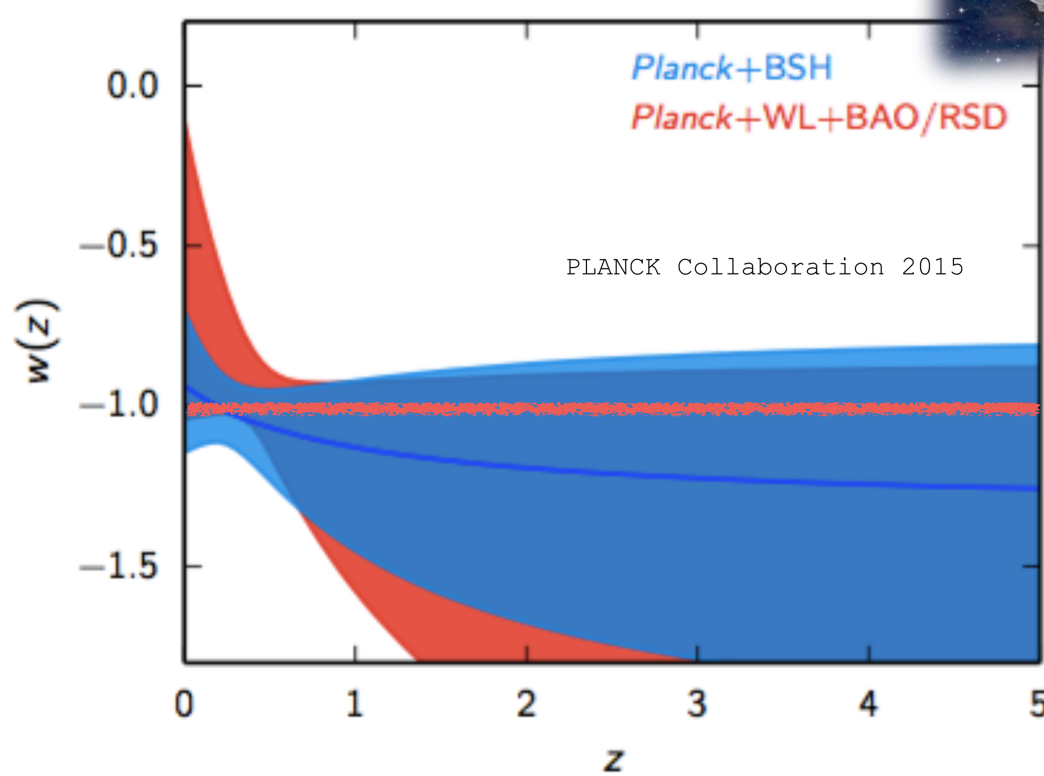
Cline, Jeon, Moore, (2003)

Cosmological Ghosts

often, ***but not necessarily***, appear if one tries to violate the ***Null Energy Condition***

- in gravity modifications and Dark Energy models, especially in the *Phantom* models with w<-1
- in bouncing scenarios of the early universe
- in inflationary models with blue-tilted spectrum of the gravitational waves





Can one change
the sign of the Hamiltonian
by a canonical transformation?

Mukhanov-Sasaki / canonical variable

$$S\left[\mathcal{R}\right] = \frac{1}{2}\int d\tau d^3\mathbf{x}\, z^2 \left(\left(\mathcal{R}'\right)^2 - c_{\mathrm{S}}^2 \left(\partial_i \mathcal{R}\right)^2\right)$$

time-dependent field redefinition $v = z\left(\tau\right)\mathcal{R}$

$$S\left[v\right] = \frac{1}{2}\int d\tau d^3\mathbf{x} \left(\left(v'\right)^2 - c_{\mathrm{S}}^2 \left(\partial_i v\right)^2 + \frac{z''}{z} v^2\right)$$

new canonical momentum $\pi = z'\mathcal{R} + \frac{\mathcal{P}}{z}$

Transforming from positive-definite to negative energy

Positive-definite but time-dependent Hamiltonian for each mode

$$H_{\rm old}\left(\mathbf{k}\right) = \frac{\left|\mathcal{P}_{\mathbf{k}}\right|^2}{2z^2} + \frac{c_{\rm S}^2 z^2 \mathbf{k}^2 \left|\mathcal{R}_{\mathbf{k}}\right|^2}{2}$$

$$H_{\rm new}\left(\mathbf{k}\right) = \frac{1}{2}\left|\pi_{\mathbf{k}}\right|^2 + \frac{1}{2}\left(c_{\rm S}^2 \mathbf{k}^2 - \frac{z''}{z}\right)\left|v_{\mathbf{k}}\right|^2$$

new time-dependent Hamiltonian for each mode, is unbounded from below on "super-horizon" scales

Canonical Transformations

$$(q, p, H) \to (\theta, \pi, \mathcal{H})$$

preserve Poincaré-Cartan integral invariant:

$$I = \oint pdq - Hdt = \oint \pi d\theta - \mathcal{H}dt$$

generating function: $pdq - Hdt - (\pi d\theta - \mathcal{H}dt) = dF$

$$p = \frac{\partial F}{\partial q}, \quad \pi = -\frac{\partial F}{\partial \theta}, \quad \mathcal{H} = H + \frac{\partial F}{\partial t}$$

preserve Poisson brackets: $\{q, p\} = \{\theta, \pi\} = 1$

Motion -canonical transformation

$$(q, p, H) \to (Q_0, P_0, 0)$$

with the generating function which is the on-shell action- i.e. Hamilton principal function $S\left(q, Q_0\right)$

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q, t\right) = 0$$

Hamilton-Jacobi equation

Transformation in the Heisenberg picture

$$\hat{\mathcal{O}}' = \hat{U}\hat{\mathcal{O}}\hat{U}^\dagger$$

unitary time-dependent transformation of the ***canonical variables***

$$\hat{H}'(\hat{\mathcal{O}}', t) = \hat{U}^\dagger\,\hat{H}(\hat{\mathcal{O}}', t)\,\hat{U} - i\hat{U}^\dagger\left(\frac{\partial\hat{U}}{\partial t}\right)_{\mathcal{O}'}$$

the Hamiltonian transforms as a
connection in the non-Abelian field theories!

Time-Dependent Linear **Bogolyubov** Transformations

$$\widehat{q} = \alpha\widehat{\theta} + \beta\widehat{\pi}\,,$$
$$\widehat{p} = \gamma\widehat{\theta} + \delta\widehat{\pi}\,,$$

$$\alpha\left(t\right),\ \beta\left(t\right),\ \gamma\left(t\right),\ \delta\left(t\right)$$

canonical: $\alpha\delta - \beta\gamma = 1$

$$F\left(q,\theta,t\right) = -\frac{1}{\beta}\left(\theta q - \frac{\delta}{2}q^2 - \frac{\alpha}{2}\theta^2\right)$$

Change in Hamiltonian for an Oscillator

$$H_- = -\frac{1}{2}\left(p^2 + \omega^2 q^2\right) \Rightarrow H_n = \frac{1}{2}\pi^2 A + \pi\theta C + \frac{1}{2}\theta^2 B$$

$$A = \beta^2 \left(\frac{d}{dt}\left(\frac{\delta}{\beta}\right) - \left(\frac{\delta}{\beta}\right)^2 - \omega^2\right)$$

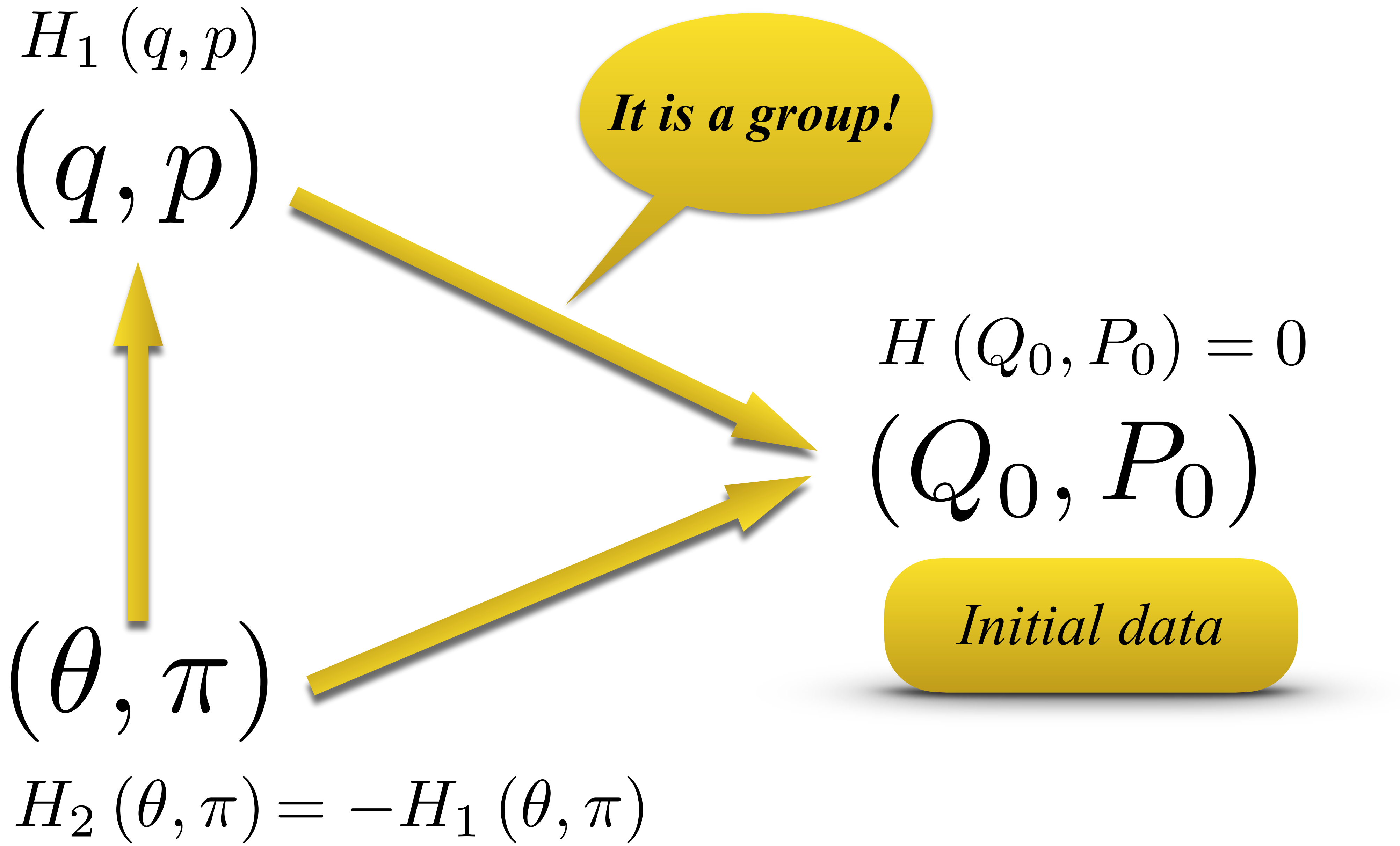
$$C = \frac{1}{\beta}\left[\frac{d\beta}{dt} + \delta + \alpha A\right]$$

$$B = \alpha^2 \left[\frac{d}{dt}\left(\frac{\gamma}{\alpha}\right) - \left(\frac{\gamma}{\alpha}\right)^2 - \omega^2\right],$$

Strong time dependence

Can one always solve these nonlinear equations and change the sign of energy?

Canonical map through zero Hamiltonian



Harmonic Oscillator Example

$$H_- = -\frac{1}{2}\left(p^2 + \omega^2 q^2\right) \quad \Rightarrow \quad H_+ = \frac{1}{2}\left(\pi^2 + \omega^2 \theta^2\right)$$

$$q\left(t\right) = q_0 \cos \omega t - \frac{p_0}{\omega} \sin \omega t$$

$$p\left(t\right) = q_0 \omega \sin \omega t + p_0 \cos \omega t\,,$$

$$p = -\dot{q}$$

$$\theta\left(t\right) = q_0 \cos \omega t + \frac{p_0}{\omega} \sin \omega t$$

$$\pi\left(t\right) = -q_0 \omega \sin \omega t + p_0 \cos \omega t\,,$$

$$\pi = \dot{\theta}$$

$$q = \theta \cos 2\omega t - \frac{\pi}{\omega} \sin 2\omega t\,,$$

$$p = \theta\omega \sin 2\omega t + \pi \cos 2\omega t\,.$$

$$\{q, p\} = \{\theta, \pi\} = 1$$

Harmonic Oscillator: Generating Function

$$H_- = -\frac{1}{2}\left(p^2 + \omega^2 q^2\right) \quad \Rightarrow \quad H_+ = \frac{1}{2}\left(\pi^2 + \omega^2\theta^2\right)$$

$$q = \theta \cos 2\omega t - \frac{\pi}{\omega} \sin 2\omega t \,,$$

$$p = \theta\omega \sin 2\omega t + \pi \cos 2\omega t \,.$$

$$F\left(q, \theta, t\right) = -\frac{\omega}{2 \sin 2\omega t}\left(\cos 2\omega t\left(q^2 + \theta^2\right) - 2\theta q\right)$$

Generating Function:

$$= S_{2t}^{-}\left(q, \theta\right) = -S_{2t}^{+}\left(q, \theta\right)$$

From Harmonic Oscillator to Field Modes

$$q = \theta \cos 2\omega t - \frac{\pi}{\omega} \sin 2\omega t \,,$$
$$p = \theta\omega \sin 2\omega t + \pi \cos 2\omega t \,.$$

$$q_{\mathbf{k}} = \theta_{\mathbf{k}} \cos 2\omega_k t - \frac{\pi_{\mathbf{k}}}{\omega_k} \sin 2\omega_k t$$
$$p_{\mathbf{k}} = \theta_{\mathbf{k}}\omega_k \sin 2\omega_k t + \pi_{\mathbf{k}} \cos 2\omega_k t$$

time-dependent
and nonlocal in *space*
canonical transformation

Typical Interactions: $\mathscr{L} = p(\varphi, \partial\varphi)$ k-essence example

$$S[\mathcal{R}] = \frac{1}{2}\int d\tau d^3\mathbf{x}\, z^2 \left((\mathcal{R}')^2 - c_{\mathrm{S}}^2 (\partial_i \mathcal{R})^2\right)$$

$$\mathcal{R} = \Phi + \mathcal{H}\frac{\delta\varphi}{\varphi'}$$

expressed through canonical momentum: $\mathcal{R}' = \frac{\mathcal{P}}{z^2} = \frac{c_S^2 \mathcal{H}^2}{a^4(\varepsilon + p)}\mathcal{P}$

direct interaction with other fields: $h_{\mu\nu} T^{\mu\nu}_{\mathrm{other}}$

from perturbed Einstein equations: $\mathcal{R}' = \frac{c_S^2 \mathcal{H}}{4\pi G_N a^2 (\varepsilon + p)} \Delta\Phi$

$$\Phi T^{00}_{\mathrm{other}} = \frac{4\pi G_N \mathcal{H}}{a^2}\left(\frac{1}{\Delta}\mathcal{P}\right) T^{00}_{\mathrm{other}}$$

Harmonic Oscillator: Interactions

$$H_- = -\frac{1}{2}\left(p^2 + \omega^2 q^2\right)$$ + other normal matter $$H_m = \frac{1}{2}\left(P^2 + \Omega^2 Q^2\right)$$

interaction

$$H_I = q\left(\lambda_1 Q^2 + \lambda_2 P^2\right) + p\left(\lambda_3 Q^2 + \lambda_4 P^2\right)$$

$$H_I = \theta\left(\lambda_1' Q^2 + \lambda_2' P^2\right) + \pi\left(\lambda_3' Q^2 + \lambda_4' P^2\right)$$

$$\lambda_1' = \lambda_1 \cos 2\omega t + \lambda_3 \omega \sin 2\omega t$$
$$\lambda_2' = \lambda_2 \cos 2\omega t + \lambda_4 \omega \sin 2\omega t$$
$$\lambda_3' = \lambda_3 \cos 2\omega t - \lambda_1 \omega^{-1} \sin 2\omega t$$
$$\lambda_4' = \lambda_4 \cos 2\omega t - \lambda_2 \omega^{-1} \sin 2\omega t$$

Different variables: k-essence example

$\mathcal{R}$ is good, but why not to use other fields with a clear physical meaning?

$$\Phi = \frac{4\pi G_N \mathcal{H}}{a^2}\left(\frac{1}{\Delta}\mathcal{P}\right)$$

$$\delta\varphi = \frac{\varphi'}{\mathcal{H}}\left(\mathcal{R} - \frac{4\pi G_N \mathcal{H}}{a^2}\left(\frac{1}{\Delta}\mathcal{P}\right)\right)$$

$$\delta\varepsilon = \frac{\mathcal{H}}{a^2}\left(\frac{1}{a^2} + 12\pi G_N\left(\varepsilon + p\right)\frac{1}{\Delta}\right)\mathcal{P} - 3\left(\varepsilon + p\right)\mathcal{R}$$

…

Action for Newton potential: k-essence example

$$S\left[\Phi\right]=\frac{1}{2}\int d\tau d^{3}x\ \left(\theta\gamma\right)^{2}\left(\Phi'\left(-\Delta\right)\Phi'-c_{s}^{2}\Phi\Delta^{2}\Phi+\frac{\gamma\left(\theta^{2}\gamma'\right)'}{\theta^{2}\gamma^{2}}\Phi\Delta\Phi\right).$$

$$\gamma=\frac{a^{2}}{4\pi G\mathcal{H}}$$

$$\theta=\frac{1}{zc_{s}}=\frac{\mathcal{H}}{a^{2}\left(\varepsilon+p\right)^{1/2}}$$

Conclusions

- There are ***many canonical*** variables for cosmological perturbations
- Time-dependent linear canonical, unitary (at least mode by mode) transformations always allow to find such ***canonical*** variables which are ***not ghosty***