Cosmological and astrophysical applications of vector-tensor theories

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Collaboration with
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There have been many attempts for constructing dark energy models in the framework of scalar-tensor theories.

Spin 0

Many of them belong to the so-called Horndeski theories.

$$S = \int d^4x \sqrt{-g}\, L$$

Most general scalar-tensor theories with second-order equations of motion

$$L = G_2(\phi, X) + G_3(\phi, X)\Box\phi + G_4(\phi, X)R - 2G_{4,X}(\phi, X)\left[(\Box\phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}\right]$$
$$+G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} + \frac{1}{3}G_{5,X}(\phi, X)[(\Box\phi)^3 - 3(\Box\phi)\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}{}_{;\sigma}]$$

Single scalar field ϕ with $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

Horndeski derived this action at the age of 25 (1973).

R and $G_{\mu\nu}$ are the 4-dimensional Ricci scalar and the Einstein tensors, respectively.

- General Relativity corresponds to $G_4 = M_{\rm pl}^2/2$.
- Horndeski theories accommodate a wide variety of gravitational theories like Brans-Dicke theory, $f(R)$ gravity, and covariant Galileons.

What happens for a vector field instead of a scalar field ?

Spin 1

(i) Maxwell field (massless)

Lagrangian: $\mathcal{L}_F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

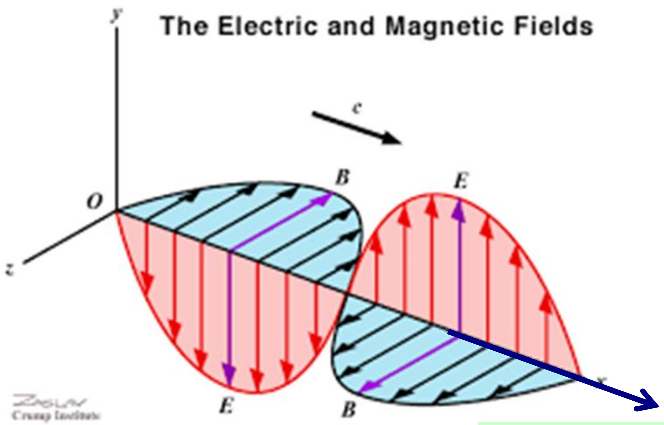
There are two transverse polarizations (electric and magnetic fields).

(ii) Proca field (massive)

Lagrangian: $\mathcal{L}_F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu$

Introduction of the mass m of the vector field A_μ allows the propagation in the longitudinal direction due to the breaking of $U(1)$ gauge invariance.

2 transverse and 1 longitudinal = 3 DOFs



Longitudinal propagation

Generalized Proca (GP) theories

On general curved backgrounds, it is possible to extend the massive Proca theories to those containing three DOFs (besides two tensor polarizations).

Heisenberg Lagrangian (2014)

L. Heisenberg (2014), G. Tasinato (2014),
J. Beltran Jimenez and L. Heisenberg (2016)

$$\mathcal{L}_2 = G_2(X, F, Y)\,,$$
$$\mathcal{L}_3 = G_3(X)\nabla_\mu A^\mu\,,$$
$$\mathcal{L}_4 = G_4(X)R + G_{4,X}(X)\left[(\nabla_\mu A^\mu)^2 - \nabla_\rho A_\sigma \nabla^\sigma A^\rho\right]\,,$$
$$\mathcal{L}_5 = G_5(X)G_{\mu\nu}\nabla^\mu A^\nu - \frac{1}{6}G_{5,X}(X)[(\nabla_\mu A^\mu)^3 - 3\nabla_\mu A^\mu \nabla_\rho A_\sigma \nabla^\sigma A^\rho + 2\nabla_\rho A_\sigma \nabla^\gamma A^\rho \nabla^\sigma A_\gamma]$$
$$-g_5(X)\tilde{F}^{\alpha\mu}\tilde{F}^\beta{}_\mu \nabla_\alpha A_\beta\,,$$
$$\mathcal{L}_6 = G_6(X)L^{\mu\nu\alpha\beta}\nabla_\mu A_\nu \nabla_\alpha A_\beta + \frac{1}{2}G_{6,X}(X)\tilde{F}^{\alpha\beta}\tilde{F}^{\mu\nu}\nabla_\alpha A_\mu \nabla_\beta A_\nu\,,$$

} Intrinsic vector mode

where

$$X = -\frac{1}{2}A_\mu A^\mu\,, \qquad F = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \qquad Y = A^\mu A^\nu F_\mu{}^\alpha F_{\nu\alpha}$$
$$L^{\mu\nu\alpha\beta} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}R_{\rho\sigma\gamma\delta}\,, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

1 scalar,
2 vector,
2 tensor DOFs

The non-minimal derivatives couplings like $G_4(X)R$ are required to keep the equations of motion up to second order.

Taking the scalar limit $A^\mu \to \nabla^\mu \pi$, the above Lagrangian recovers a sub-class of Horndeski theories (with $\mathcal{L}_6$ vanishing).

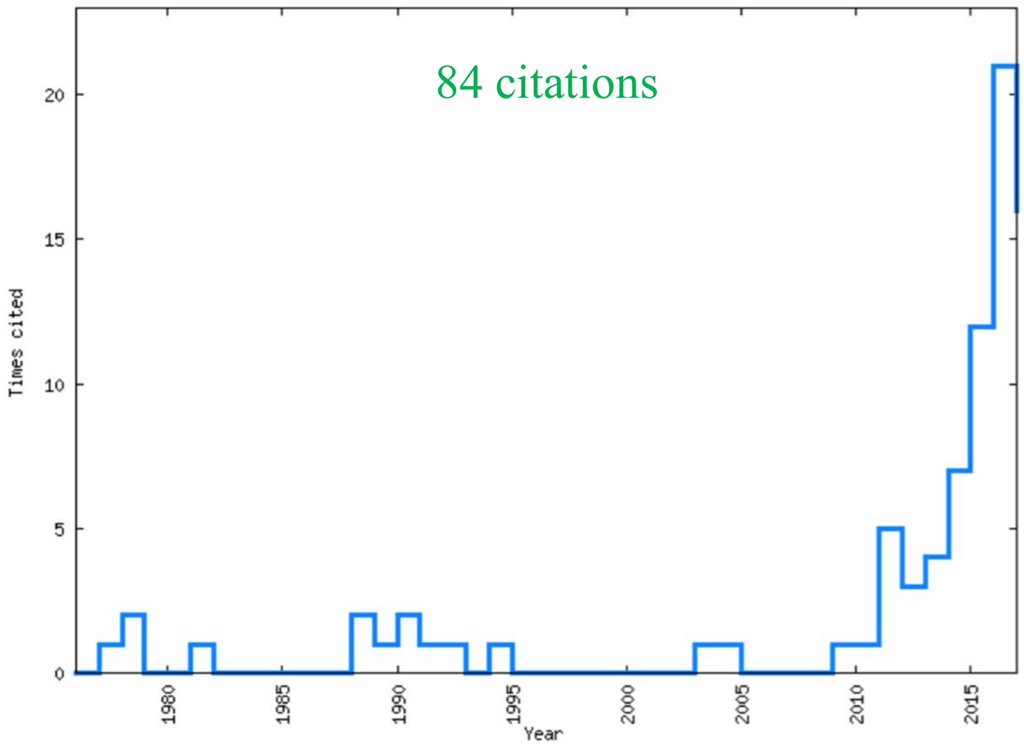
U(1) gauge-invariant case: constant G_6

Conservation of Charge and the Einstein-Maxwell Field Equations

G.W. Horndeski (Waterloo U.)

1976 - 8 pages

J.Math.Phys. 17 (1976) 1980-1987
DOI: 10.1063/1.522837



Cosmology in GP theories

De Felice et al, JCAP 1606, 048 (2016)

Can we realize a viable cosmology with the late-time acceleration?

Vector field: $A^\mu = (\phi(t), 0, 0, 0)$ (which does not break spatial isotropy)

Variation of the Heisenberg action with respect to $g_{\mu\nu}$ on the flat FLRW background leads to

$$G_2 - G_{2,X}\phi^2 - 3G_{3,X}H\phi^3 + 6G_4H^2 - 6(2G_{4,X} + G_{4,XX}\phi^2)H^2\phi^2 + G_{5,XX}H^3\phi^5 + 5G_{5,X}H^3\phi^3 = \rho_M\,,$$

$$G_2 - \dot{\phi}\phi^2 G_{3,X} + 2G_4\,(3H^2 + 2\dot{H}) - 2G_{4,X}\phi\,(3H^2\phi + 2H\dot{\phi} + 2\dot{H}\phi) - 4G_{4,XX}H\dot{\phi}\phi^3$$
$$+ G_{5,XX}H^2\dot{\phi}\phi^4 + G_{5,X}H\phi^2(2\dot{H}\phi + 2H^2\phi + 3H\dot{\phi}) = -P_M\,.$$

The matter density ρ_M and the pressure P_M obey the continuity equation

$$\dot{\rho}_M + 3H(\rho_M + P_M) = 0$$

Variation of the action with respect to A^μ leads to

$$\phi\left(G_{2,X} + 3G_{3,X}H\phi + 6G_{4,X}H^2 + 6G_{4,XX}H^2\phi^2 - 3G_{5,X}H^3\phi - G_{5,XX}H^3\phi^3\right) = 0\,.$$

The branch $\phi \neq 0$ gives the solution where ϕ depends on H alone, which allows the existence of de Sitter solutions with constant ϕ and H.

Vector Galileons

The Lagrangian of vector Galileons which recover the Galilean symmetry in the scalar limit ($A_\mu \to \partial_\mu \pi$) on the flat space-time is given by

$$G_2(X) = b_2 X\,, \qquad G_3(X) = b_3 X\,, \qquad G_4(X) = \frac{M_{\rm pl}^2}{2} + b_4 X^2\,, \qquad G_5(X) = b_5 X^2.$$

We substitute these functions into the vector-field equation:

$$G_{2,X} + 3G_{3,X} H\phi + 6G_{4,X} H^2 + 6G_{4,XX} H^2\phi^2 - 3G_{5,X} H^3\phi - G_{5,XX} H^3\phi^3 = 0$$

Taking note that $X = \phi^2/2$, the background EOM admits the solution

$$\phi H = \text{constant}.$$

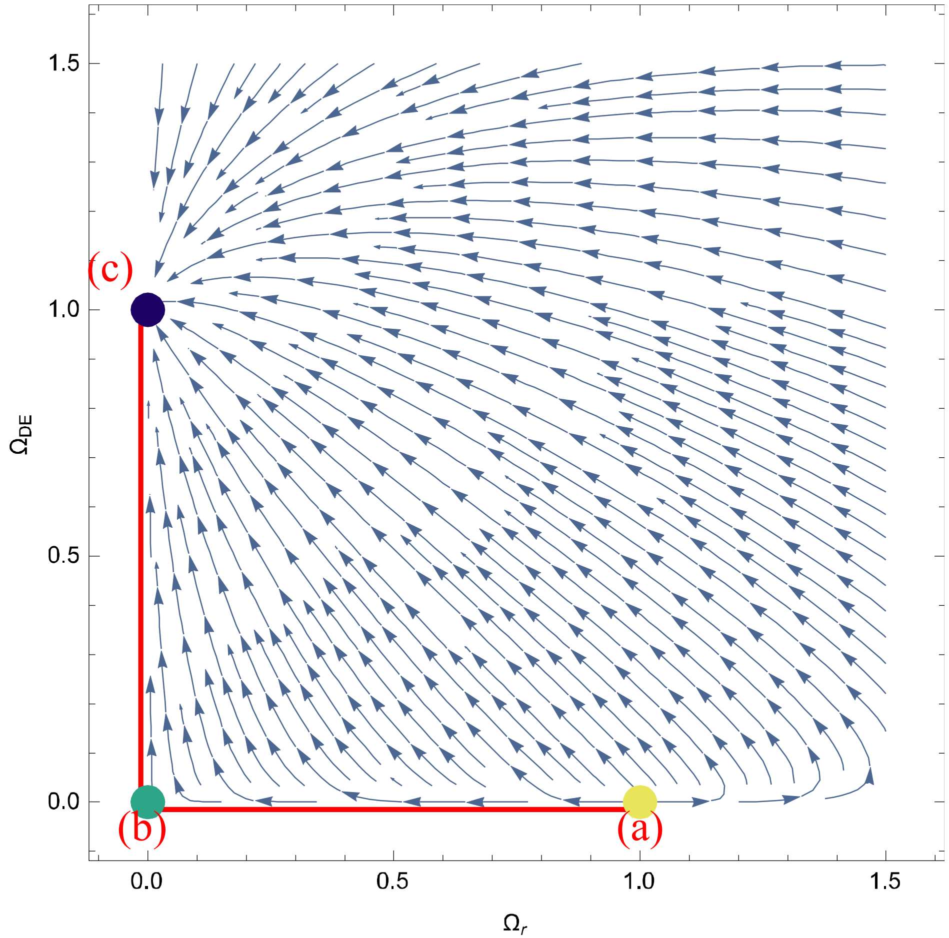
⇩

The temporal component ϕ is small in the early cosmological epoch, but it grows with the decrease of H.

The solution finally approaches the de Sitter attractor characterized by

$$\phi = \text{constant}, \qquad H = \text{constant}.$$

Phase-space trajectories for vector Galileons



(a) Radiation point: $w_{\rm DE} = -7/3$
(b) Matter point: $w_{\rm DE} = -2$
(c) De Sitter point: $w_{\rm DE} = -1$

The de Sitter fixed point (c) is always stable against homogeneous perturbations, so it corresponds to the late-time attractor.

The dark energy equation of state $w_{\rm DE}$ is -2 during the matter era.

This case is excluded from the joint data analysis of SN Ia, CMB, and BAO.

Generalizations of vector Galileons

Let us consider the case in which ϕ is related with H according to

$$\phi^p \propto H^{-1} \qquad (p > 0)$$

This solution can be realized for

$$G_2(X) = b_2 X^{p_2}\,, \qquad G_3(X) = b_3 X^{p_3}\,, \qquad G_4(X) = \frac{M_{\rm pl}^2}{2} + b_4 X^{p_4}\,, \qquad G_5(X) = b_5 X^{p_5}\,,$$

where

$$p_3 = \frac{1}{2}(p + 2p_2 - 1)\,, \qquad p_4 = p + p_2\,, \qquad p_5 = \frac{1}{2}(3p + 2p_2 - 1)\,.$$

⇒ The vector Galileon corresponds to $p_2 = p = 1$.

The dark energy and radiation density parameters obey

$$\Omega'_{\rm DE} = \frac{(1+s)\Omega_{\rm DE}(3 + \Omega_r - 3\Omega_{\rm DE})}{1 + s\,\Omega_{\rm DE}}\,,$$

$$\Omega'_r = -\frac{\Omega_r[1 - \Omega_r + (3+4s)\Omega_{\rm DE}]}{1 + s\,\Omega_{\rm DE}}\,,$$

where $s \equiv \dfrac{p_2}{p}\,.$

⇒ There are 3 fixed points:

(a) $(\Omega_{\rm DE}, \Omega_r) = (0, 1)$

(b) $(\Omega_{\rm DE}, \Omega_r) = (0, 0)$

(c) $(\Omega_{\rm DE}, \Omega_r) = (1, 0)$

Dark energy equation of state

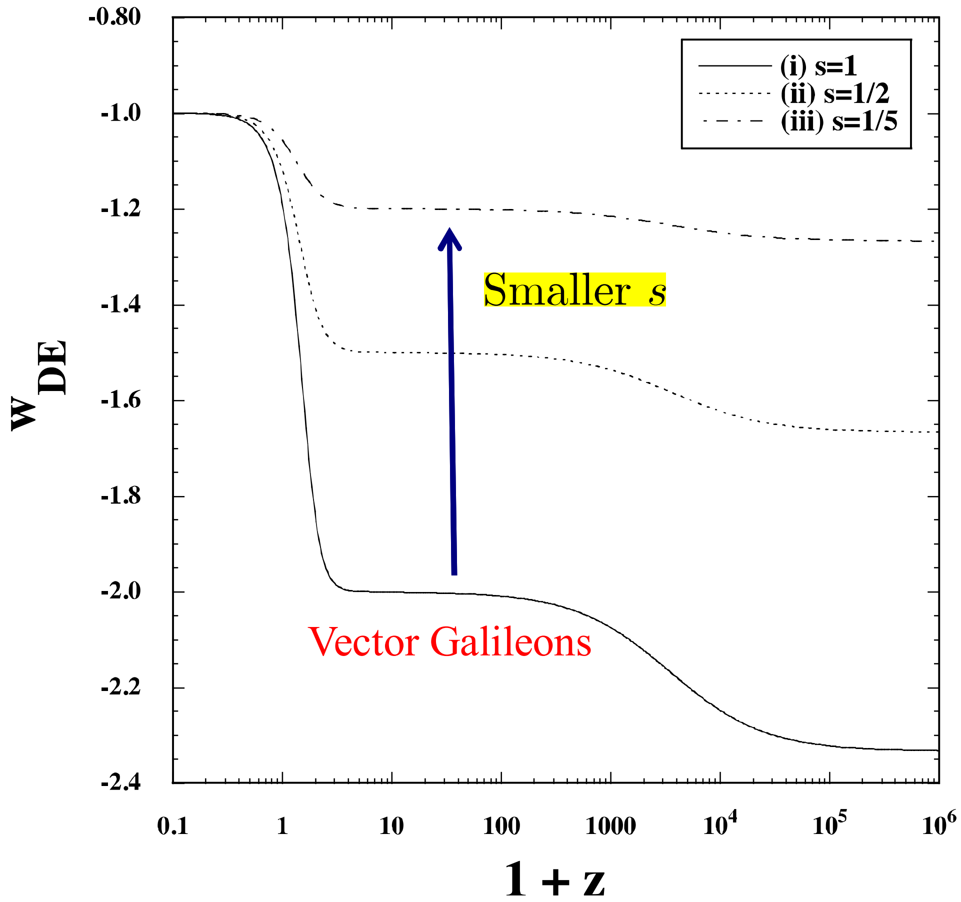
$$w_{\rm DE} = -\frac{3(1+s)+s\,\Omega_r}{3(1+s\,\Omega_{\rm DE})}\,.$$

(a) $w_{\rm DE} = -1 - 4s/3$ in the radiation era,
(b) $w_{\rm DE} = -1 - s$ in the matter era,
(c) $w_{\rm DE} = -1$ in the de Sitter era

The vector Galileon corresponds to the case $p_2 = p = 1$, i.e., $s = 1$.

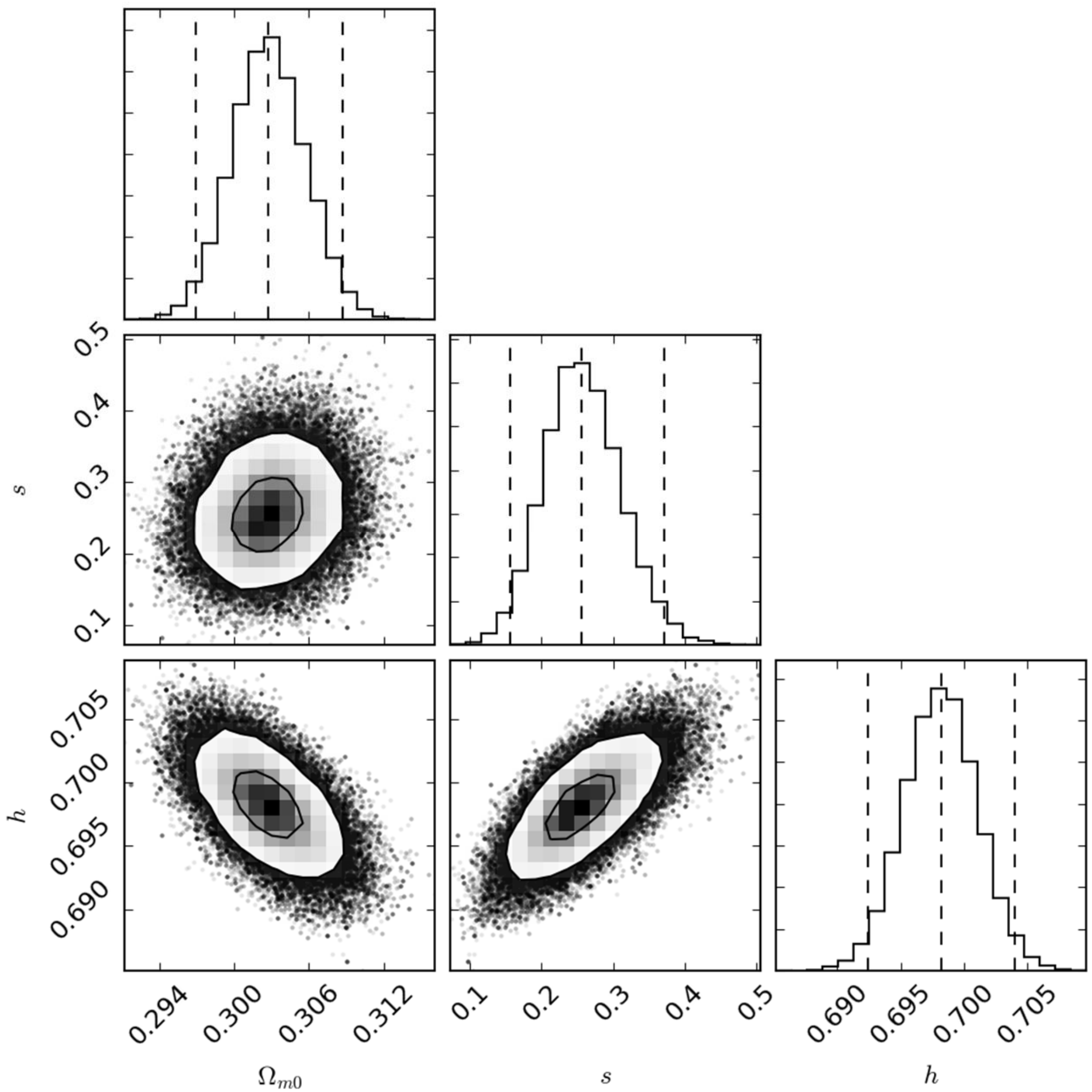
For smaller $s = p_2/p$ close to 0, $w_{\rm DE} = -1 - s$ approaches -1.

For larger p the field ϕ evolves more slowly as $\phi \propto H^{-1/p}$, so $w_{\rm DE}$ approaches -1.



Observational constraints

A. De Felice, L. Heisenberg, ST, 1703.09573.



The joint data analysis of SN Ia, CMB shift parameter, BAO, and H0 give the bound

$$\Omega_{m0} = 0.3027^{+0.0060}_{-0.0057}\,,$$
$$h = 0.6981^{+0.0059}_{-0.0057}\,,$$
$$s = 0.254^{+0.118}_{-0.097}\,.$$

(95 %CL)

The model fits the data better than the LCDM at the background level.

Cosmological perturbations in GP theories

We need to study perturbations on the flat FLRW background to study

(i) Conditions for avoiding ghosts and instabilities,
(ii) Observational signatures for the matter distribution in the Universe.

In doing so, let us consider the perturbed metric in the flat gauge:

$$ds^2 = -(1+2\alpha)\,dt^2 + 2\,(\partial_i\chi + V_i)\,dt\,dx^i + a^2(t)\,(\delta_{ij} + h_{ij})\,dx^i dx^j\,,$$

where α, χ are scalar perturbations, V_i and h_{ij} are the vector and tensor perturbations, respectively, obeying

$$\partial^i V_i = 0\,,$$
$$\partial^i h_{ij} = 0\,, \qquad {h_i}^i = 0\,.$$

We also consider the perturbations of the vector field, as

$$A^0 = \phi(t) + \underline{\delta\phi}\,,$$
$$A^i = \frac{1}{a^2}\delta^{ij}\left(\underline{\partial_j\chi_V} + \underline{E_j}\right)$$

where $\delta\phi$ and χ_V are scalar perturbations, while E_j is the vector perturbation obeying $\partial^j E_j = 0$.

Theoretical consistency and observational signatures

- There are 6 theoretically consistent conditions associated with tensor, vector, and scalar perturbations:

No ghosts: $q_t > 0, \; q_v > 0, \; q_s > 0$

No instabilities: $c_t^2 > 0, \; c_v^2 > 0, \; c_s^2 > 0$

See arXiv:1603.05806 for details.

There exists a wide range of parameter space consistent with these conditions.

- The effective gravitational coupling associated with the growth of large-scale structures can be smaller than the Newton constant.

The existence of the intrinsic vector mode can lead to

$$G_{\rm eff} < G$$

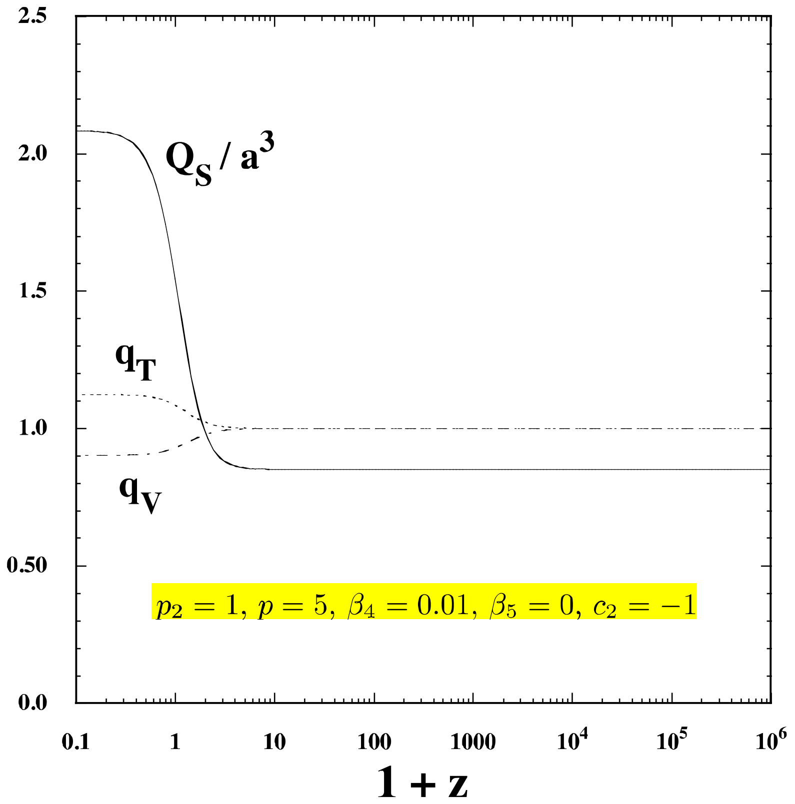
See arXiv:1605.05066 for details.

A model consistent with no-ghost and stability conditions

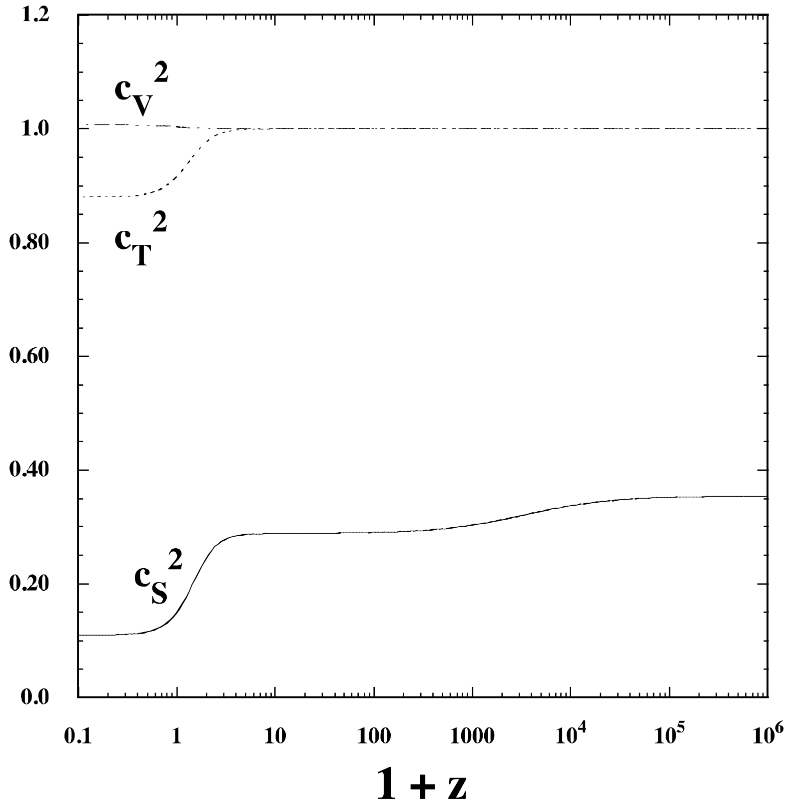
$$G_2(X) = b_2 X\,, \qquad G_3(X) = b_3 X^{p_3}\,, \qquad G_4(X) = \frac{M_{\rm pl}^2}{2} + b_4 X^{p_4}\,, \qquad G_5(X) = 0.$$

Provided that $0 < \beta_4 < 1/[6(2p+1)]$, there exists the parameter space in which all the theoretically consistent conditions are satisfied.

$q_T > 0, q_V > 0, Q_S > 0$



$c_T^2 > 0, c_V^2 > 0, c_S^2 > 0$



Cosmic growth in GP theories

De Felice et al,
arXiv: 1605.05066

Under the quasi-static approximation on sub-horizon scales, the matter perturbation obeys

$$\ddot{\delta}_M + 2H\dot{\delta}_M - 4\pi G_{\rm eff}\,\rho_M \delta_M \simeq 0$$

where the effective gravitational coupling is

$$G_{\rm eff} = \frac{\xi_2 + \xi_3}{\xi_1}$$

$$\xi_1 = 4\pi\phi^2 \left(w_2 + 2Hq_T\right)^2 \,,$$

$$\xi_2 = \left[H\left(w_2 + 2Hq_T\right) - \dot{w}_1 + 2\dot{w}_2 + \rho_M\right]\phi^2 - \frac{w_2^2}{q_V}\,,$$

$$\xi_3 = \frac{1}{8H^2\phi^2 q_S^3 q_T c_S^2}\Big[2\phi^2\left\{q_S[w_2\dot{w}_1 - (w_2 - 2Hq_T)\dot{w}_2] + \rho_M w_2[3w_2(w_2 + 2Hq_T) - q_S]\right\} - \frac{q_S}{q_V} w_2\left\{w_6\phi(w_2 + 2Hq_T) - w_2(w_2 - 2Hq_T)\right\}\Big]^2 .$$

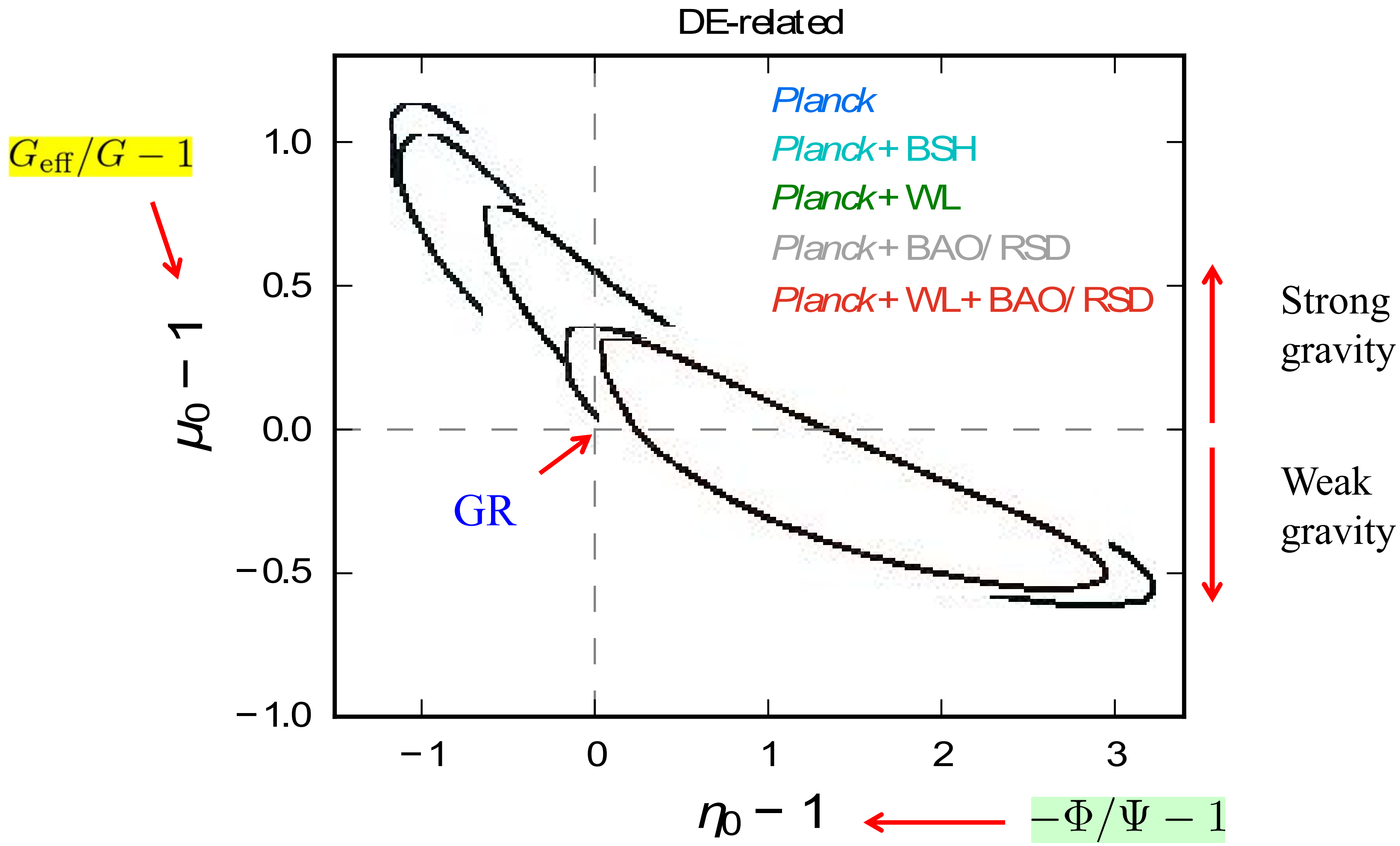
ξ_3 is positive under the no-ghost and stability conditions (which enhances the gravitational attraction).

For smaller q_V close to 0, there is a tendency that $G_{\rm eff}$ decreases.

Planck constraints on the effective gravitational coupling and the gravitational slip parameter

Ade et al (2015)

$G_{\rm eff}/G$ and Φ/Ψ are assumed to be constant.



Weak gravity in generalized Proca theories

De Felice et al, 1605.05066 (2016)

$G_{\rm eff}$ is modified through the intrinsic vector mode through the quantity q_V.

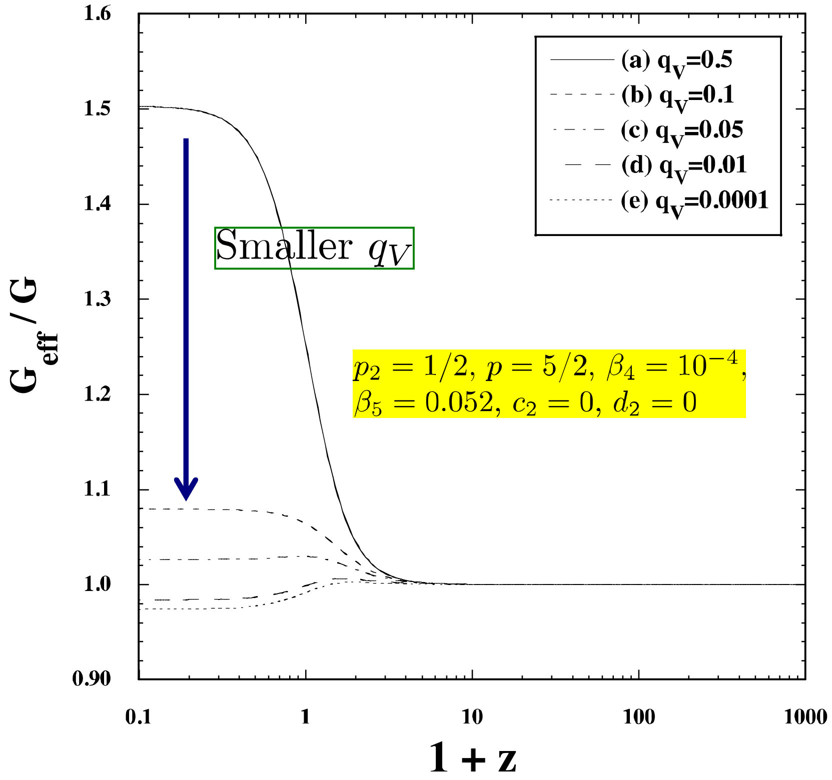
For a massive vector field with $G_2 = F + m^2 X$ we have

$$q_V = 1 - 4g_5 H\phi + 2G_6 H^2 + 2G_{6,X} H^2 \phi^2$$

Effect of the intrinsic vector mode

For smaller q_V approaching 0, the effect of the vector field tends to reduce the gravitational attraction.

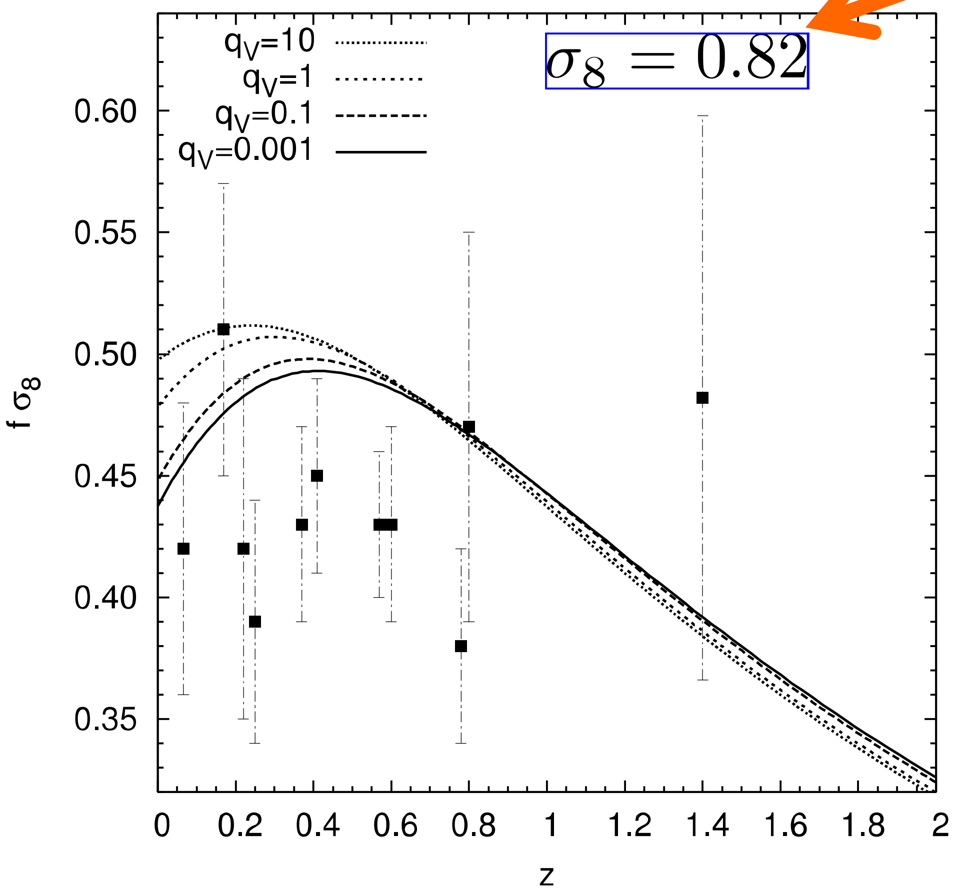
It is possible to see signatures of the intrinsic vector mode in redshift-space distortion measurements.



Observational signatures in red-shift space distortions (RSD)

From the RSD measurement we can constrain the growth rate of matter perturbations: $f = \dot{\delta}_m/(H\delta_m)$.

Planck best-fit value



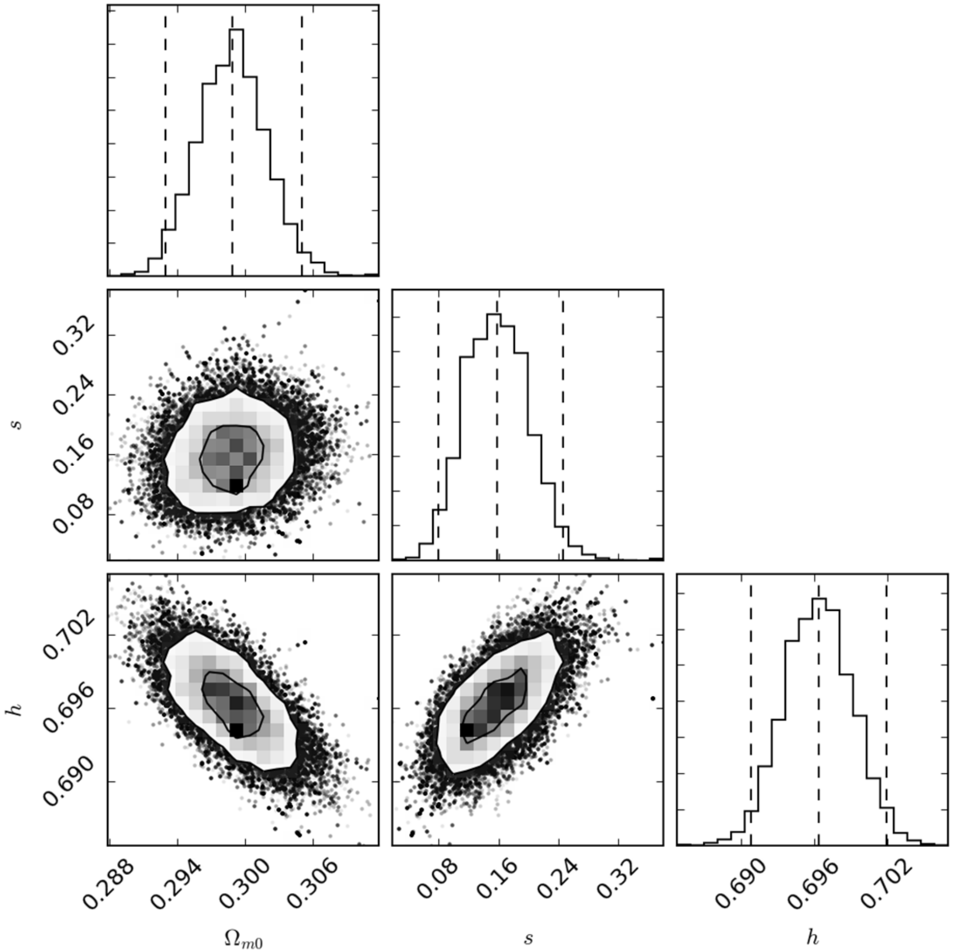
For smaller q_V, the values of $f\sigma_8$ tend to be smaller.

The present $f\sigma_8$ data alone are not sufficient to distinguish between the models with different q_V.

The relization of weak gravity is also limited down to the value around $G_{\rm eff} \simeq 0.95G$.

Observational constraints including the RSD data

A. De Felice, L. Heisenberg, ST, 1703.09573.



The joint analysis including the RSD data give the bound

$$\Omega_{m0} = 0.299^{+0.006}_{-0.006}\,,$$
$$h = 0.696^{+0.006}_{-0.005}\,,$$
$$s = 0.16^{+0.08}_{-0.08}\,,$$

(95 % CL)

The model with $s > 0$ stills fits the data better than the LCDM. However, the case of weak gravity is not necessarily the best fit.

Best-fit: $\chi^2_{\rm min} = 625.6$

ΛCDM: $\chi^2_{\rm min} = 642.7$

Healthy extension of GP theories

Heisenberg, Kase, ST, PLB (2016)

The Heisenberg Lagrangian contains the Galileon-like contributions:

$$\mathcal{L}_{i+2}^{\rm Ga} = g_{i+2}\,\hat{\delta}_{\alpha_1\cdots\alpha_i\gamma_{i+1}\cdots\gamma_4}^{\beta_1\cdots\beta_i\gamma_{i+1}\cdots\gamma_4}\nabla_{\beta_1}A^{\alpha_1}\cdots\nabla_{\beta_i}A^{\alpha_i}$$

where

$$\hat{\delta}_{\alpha_1\cdots\alpha_i\gamma_{i+1}\cdots\gamma_4}^{\beta_1\cdots\beta_i\gamma_{i+1}\cdots\gamma_4} = \mathcal{E}_{\alpha_1\cdots\alpha_i\gamma_{i+1}\cdots\gamma_4}\mathcal{E}^{\beta_1\cdots\beta_i\gamma_{i+1}\cdots\gamma_4}$$

We can consider the generalized Lagrangians like

$$\mathcal{L}_4^{\rm N} = f_4\hat{\delta}_{\alpha_1\alpha_2\alpha_3\gamma_4}^{\beta_1\beta_2\beta_3\gamma_4}A^{\alpha_1}A_{\beta_1}\nabla^{\alpha_2}A_{\beta_2}\nabla^{\alpha_3}A_{\beta_3}\,,$$

$$\mathcal{L}_5^{\rm N} = f_5\hat{\delta}_{\alpha_1\alpha_2\alpha_3\alpha_4}^{\beta_1\beta_2\beta_3\beta_4}A^{\alpha_1}A_{\beta_1}\nabla^{\alpha_2}A_{\beta_2}\nabla^{\alpha_3}A_{\beta_3}\nabla^{\alpha_4}A_{\beta_4}$$

In the scalar limit $A^\mu \to \nabla^\mu \pi$, these recover the Lagrangains of Gleyzes-Langlois-Piazza-Verinizz theories.
(healthy extension of second-order Horndeski theories)

The analysis of linear perturbations on the flat FLRW background and on the anisotropic cosmological background shows that there are no additional ghostly DOFs even with these new Lagrangians.

Anisotropic cosmology in beyond-generalized Proca (BGP) theories

Four new Lagrangians :

Heisenberg, Kase, ST, arXiv/1607.03175

$$\mathcal{L}_4^{\rm N} = f_4(X)\hat{\delta}^{\beta_1\beta_2\beta_3\gamma_4}_{\alpha_1\alpha_2\alpha_3\gamma_4} A^{\alpha_1} A_{\beta_1} \nabla^{\alpha_2} A_{\beta_2} \nabla^{\alpha_3} A_{\beta_3} \,,$$
$$\mathcal{L}_5^{\rm N} = f_5(X)\hat{\delta}^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} A^{\alpha_1} A_{\beta_1} \nabla^{\alpha_2} A_{\beta_2} \nabla^{\alpha_3} A_{\beta_3} \nabla^{\alpha_4} A_{\beta_4}$$
$$\tilde{\mathcal{L}}_5^{\rm N} = \tilde{f}_5(X)\hat{\delta}^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} A^{\alpha_1} A_{\beta_1} \nabla^{\alpha_2} A^{\alpha_3} \nabla_{\beta_2} A_{\beta_3} \nabla^{\alpha_4} A_{\beta_4}$$
$$\mathcal{L}_6^{\rm N} = f_6(X)\hat{\delta}^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} \nabla_{\beta_1} A_{\beta_2} \nabla^{\alpha_1} A^{\alpha_2} \nabla_{\beta_3} A^{\alpha_3} \nabla_{\beta_4} A^{\alpha_4}$$

Anisotropic background:

$$ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}\left[e^{-4\sigma(t)}dx^2 + e^{2\sigma(t)}\left(dy^2+dz^2\right)\right]$$

with the vector field $A^\mu = \left(\frac{\phi(t)}{N(t)}, e^{-\alpha(t)+2\sigma(t)}\, v(t), 0, 0\right)$

The Hamiltonian constraint is

$$\frac{\partial L}{\partial N} = -\frac{\mathcal{H}}{N} = 0 \quad \Rightarrow \quad \mathcal{H} = 0$$

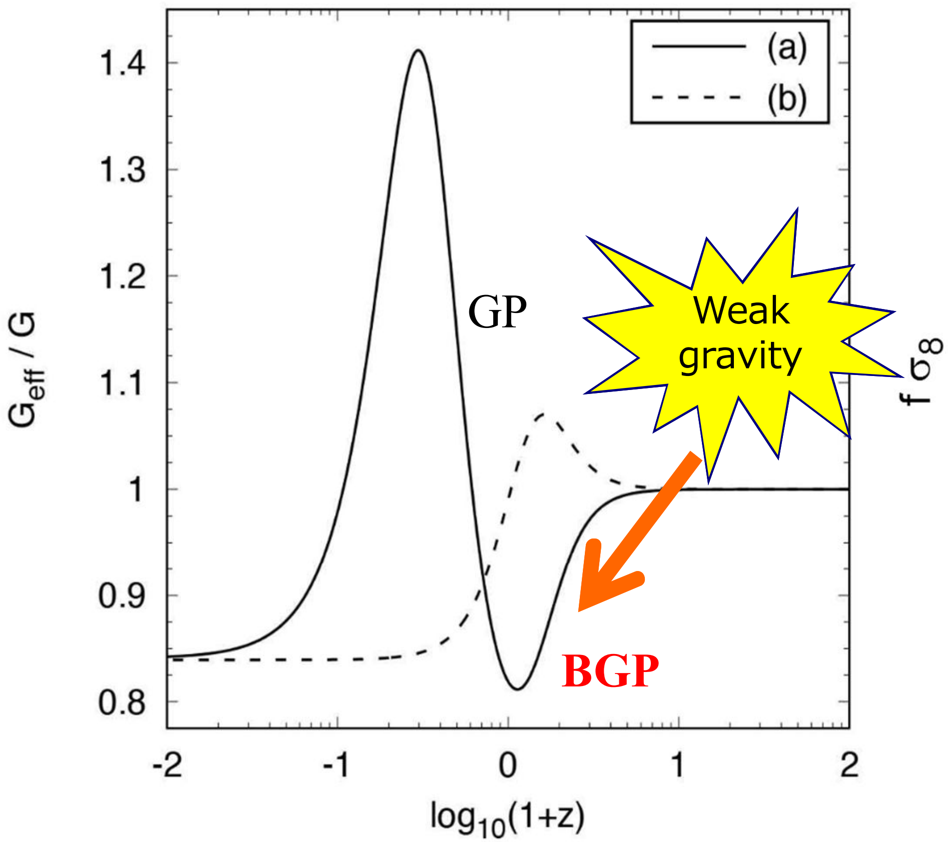
(bounded from below)

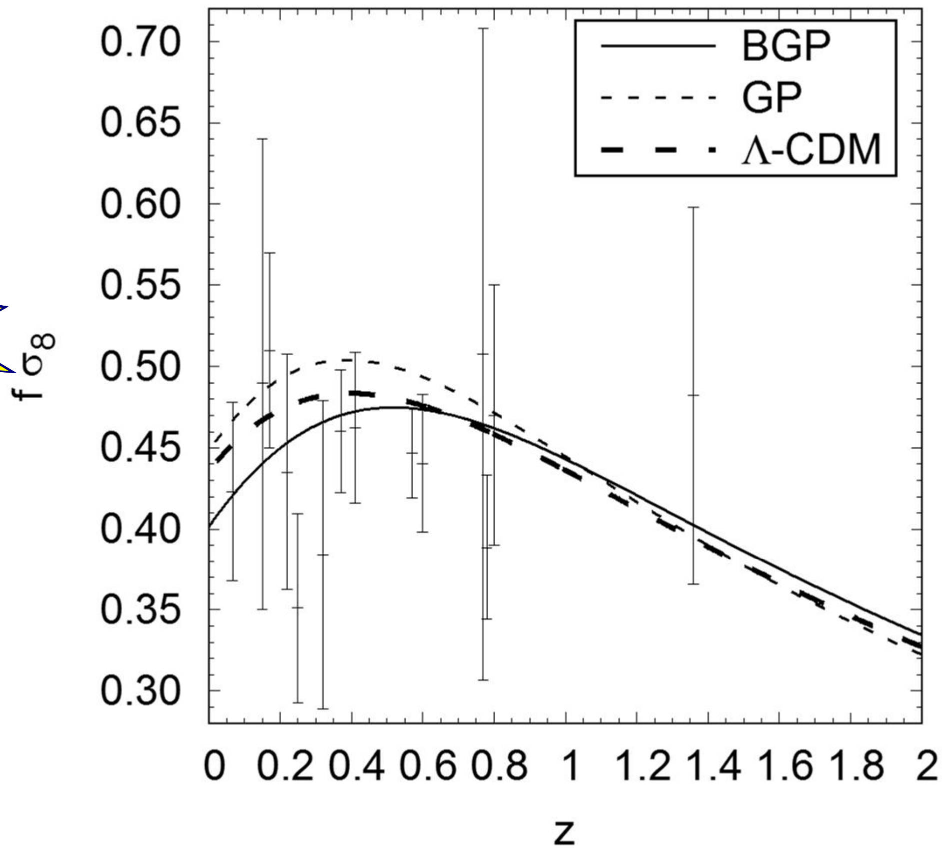
No ghost-like Ostrogradski instability

Observational signatures of BGP theories

Nakamura, Kase, ST, arXiv: 1702.08610

The relization of weak gravity like the value $G_{\rm eff} \simeq 0.8G$ is possible.





It remains to be seen whether the BGP theories fit the data better than the LCDM.

Black holes in GP theories

For the U(1)-invariant massive vector field ($G_2 = m^2 X$), Bekenstein showed that there are no hairy BH solutions (1972).

However, with derivative self interaction $G_4(X)$, the first exact hairy BH solution was found by Chagoya et al. (2016) for the Lagrangian

$$\mathcal{L}_4 = G_4(X)R + G_{4,X}(X)\left[(\nabla_\mu A^\mu)^2 - \nabla_\mu A_\nu \nabla^\nu A^\mu\right]$$

$$G_4(X) = \frac{1}{2}M_{\rm pl}^2 + \frac{1}{4}X \quad \text{where} \quad X = -\frac{1}{2}A^\mu A_\mu$$

On a static and spherically symmetric background given by the metric

$$ds^2 = -f(r)dt^2 + h^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

and the vector $A_\mu = (A_0, A_1, 0, 0)$, there exists the exact BH solution

$$f = h = 1 - \frac{2M}{r}$$

$$A_0 = P + \frac{Q}{r}, \qquad A_1 = \pm\frac{\sqrt{2P(MP+Q)r + Q^2}}{r - 2M}$$

$$\Rightarrow \quad X = \frac{A_0^2}{2f} - \frac{hA_1^2}{2} = \frac{P^2}{2} = \text{constant}.$$

Difference between scalar-tensor theories and GP theories

- In shift-symmetric Horndeski theories like

$$\mathcal{L}_4 = G_4(X)R + G_{4,X}(X)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right]$$

where $X = -\partial^\mu\phi\partial_\mu\phi/2$

the solution to the scalar field ϕ can be generally written in the form

$$\phi'\mathcal{F}(\phi'; g, g', g'') = 0$$

Hui and Nicolis (2012)

⇒ The no-hair solution $\phi' = 0$ follows unless we tune the function $\mathcal{F}$, e.g., negative powers of ϕ'.

e.g., the Gauss-Bonnet coupling $\alpha\phi\mathcal{G}$ realized by $G_5(X) = -4\alpha \ln X$

Sotiriou and Zhou (2014)

See also Babichev and Charmousis (2014) for a time-dependent Galileon.

- In GP theories, the vector field $A_\mu = (A_0, A_1, 0, 0)$ obeys

$$\mathcal{F}(A_1, A_0, A_0'; g, g', g'') = 0$$

⇒ The existence of the temporal component A_0 allows the existence of hairy BH solutions with $A_1 \neq 0$ without tuning the models.

Structure of the EOM of the longitudinal mode

$$A_1\left[r^2 f G_{2,X} - 2(rf'h + fh - f)G_{4,X} + 4h(rA_0A_0' - rf'X - fX_1)G_{4,XX} - hA_0'^2(3h-1)G_{6,X} - 2h^2X_1A_0'^2G_{6,XX}\right]$$
$$= r[r(f'X - A_0A_0') + 4fX_1]G_{3,X} + 2f'hX_1G_{5,X} + (A_0A_0' - f'X)\left[(1-h)G_{5,X} - 2hX_1G_{5,XX}\right]$$

(i) Theories with even subscripts: $G_{2,4,6}(X)$ ⇒ The rhs vanishes.

The above EOM is written in the form

$$A_1\tilde{\mathcal{F}}(A_1, A_0, A_0'; g, g') = 0$$

⇒ There are two branches

$A_1 = 0$, or $\tilde{\mathcal{F}} = 0$ (non-vanishing A_1)

(ii) Theories with odd subscripts: $G_{3,5}(X)$ ⇒ The lhs vanishes.

The EOM is of the form

$$\mathcal{F}(A_1, A_0, A_0'; g, g') = 0$$

⇒ The general solution is

$A_1 \neq 0$

Searches for exact BH solutions

The BH solution found by Chagoya et al. for specific $G_4(X)$ satisfies

$$\begin{cases} f = h\,, \\ X = \dfrac{A_0^2}{2f} - \dfrac{hA_1^2}{2} = X_c = \text{constant}. \end{cases} \quad \Rightarrow \quad A_1 = \pm \frac{\sqrt{A_0^2 - 2fX_c}}{f}$$

Let us first impose these conditions for other interactions $G_{3,5,6}(X)$ for the purpose of searching for exact solutions.

Consider the cubic coupling $G_3(X)$. The longitudinal mode satisfies

$$G_{3,X}\left[f^2(rf' + 4f)A_1^2 + r(2fA_0' - f'A_0)A_0\right] = 0$$

(i) $G_{3,X}(X_c) = 0$ ⇒ Reissner-Nordstrom (RN) solution with $A_1 \neq 0$

(ii) $G_{3,X}(X_c) \neq 0$ ⇒ Extremal RN solution with $A_1 = 0$

Exact solutions for the cubic interactions

(i) For the branch $G_{3,X}(X_c) = 0$, the equations of motion for f, h, A_0 give

$$rA_0'' + 2A_0' = 0\,, \quad \Longrightarrow \quad A_0 = P + \frac{Q}{r}$$

$$f' = -\frac{r^2 A_0'^2 + 2M_{\rm pl}^2(f-1)}{2M_{\rm pl}^2 r} \quad \Longrightarrow \quad f = h = 1 - \frac{2M}{r} + \frac{Q^2}{2M_{\rm pl}^2 r^2}$$

with the non-vanishing longitudinal mode $$A_1 = \pm\frac{\sqrt{A_0^2 - 2fX_c}}{f}$$

The constant P, which is independent of mass M and charge Q, is regarded as the primary Proca hair.

This RN solution exists for the Lagrangian $G_3(X) = G_3(X_c) + \sum_{n=2} b_n\,(X - X_c)^n$.

(ii) For the branch $G_{3,X}(X_c) \neq 0$, including $G_3 = \beta_3 X$, the solution is

$$f = h = \left(1 - \frac{M}{r}\right)^2, \qquad A_0 = \pm\sqrt{2}M_{\rm pl}\left(1 - \frac{M}{r}\right), \qquad A_1 = 0$$

(Extremal RN solution)

Exact solutions for other interactions

If we impose the two conditions $f = h$ and $X = X_c$, the resulting BH solutions are either Schwarzschild, RN, or extremal RN solutions.

The models giving rise to exact solutions are

(1) $$G_4(X) = G_4(X_c) + \frac{1}{4}(X - X_c) + \sum_{n=3} b_n (X - X_c)^n$$ where $X_c = \frac{P^2}{2}$

Schwarzschild solution with $A_0 = P + \frac{Q}{r}$, $A_1 = \pm \frac{\sqrt{2P(MP+Q)r + Q^2}}{r - 2M}$

This includes the solution of Chagoya et al (2016).

(2) $$G_5(X) = G_5(X_c) + \sum_{n=2} b_n (X - X_c)^n$$ where $X_c = M_{\rm pl}^2$

RN solution with $A_0 = -\frac{2MM_{\rm pl}^2}{Q} + \frac{Q}{r}$, $A_1 = \pm \frac{2M_{\rm pl}^3 \sqrt{2(2M^2M_{\rm pl}^2 - Q^2)}\, r^2}{Q[2M_{\rm pl}^2 r(2M - r) - Q^2]}$.

(3) $$G_6(X) = \sum_{n=2} b_n (X - X_c)^n$$ where $X_c = M_{\rm pl}^2$

Extremal RN solution with $A_0 = \pm\sqrt{2} M_{\rm pl}\left(1 - \frac{M}{r}\right)$, $A_1 = 0$

Geneal non-exact solutions

We would like to derive more general solutions without imposing the conditions $f = h$ and $X = X_c$.

For example, consider the cubic vector Galileon given by

$$G_3(X) = \beta_3 X$$

In this case, the longitudinal mode A_1 is related to A_0, f, h, as

$$A_1 = \pm\sqrt{\frac{rA_0(f'A_0 - 2fA_0')}{fh(rf' + 4f)}}$$

Around the BH horizon characterized by the distance r_h, we expand f, h, A_0 in the form

$$f = \sum_{i=1}^{\infty} f_i (r - r_h)^i\,, \qquad h = \sum_{i=1}^{\infty} h_i (r - r_h)^i\,, \qquad A_0 = a_0 + \sum_{i=1}^{\infty} a_i (r - r_h)^i$$

The effect of the coupling β_3 works as corrections to the RN metric

$$f_{\rm RN} = h_{\rm RN} = \left(1 - \frac{r_h}{r}\right)\left(1 - \mu\frac{r_h}{r}\right).$$

μ is a constant between $0 < \mu < 1$.

Corrections to the RN metric induced by the cubic coupling

The solution expanded around the horizon for cubic Galileons is given by

$$f = \sum_{i=1}^{\infty} f_i (r - r_h)^i \,, \qquad h = \sum_{i=1}^{\infty} h_i (r - r_h)^i \,, \qquad A_0 = a_0 + \sum_{i=1}^{\infty} a_i (r - r_h)^i$$

where
$$f_1 = h_1 = \frac{1-\mu}{r_h} \,, \qquad a_1 = \frac{\sqrt{2\mu} M_{\rm pl}}{r_h}$$

$$f_2 = \frac{2\mu - 1}{r_h^2} + \mathcal{F}_2 \beta_3 \,, \qquad h_2 = \frac{2\mu - 1}{r_h^2} + \mathcal{H}_2 \beta_3 \,, \qquad a_2 = -\frac{\sqrt{2\mu} M_{\rm pl}}{r_h^2} + \alpha_2 \beta_3$$

$\mathcal{F}_2, \mathcal{H}_2, \alpha_2$ contain μ and a_0.

The effect of the coupling β_3 leads to the difference between the two metrics f and h.

The solution around the horizon is expressed in terms of three constants:

$$\mu, a_0, r_h .$$

The longitudinal mode around the horizon behaves as

$$A_1 = \frac{a_0}{f_1 (r - r_h)}$$

The scalar product is regular:

$$A_\mu dx^\mu = A_0 (dt + dr_*) = A_0 du_+$$

u_+ is the advanced null coordinate.

Solutions at spatial infinity

We expand the solutions at spatial infinity, as

$$f = 1 + \sum_{i=1}^{\infty} \frac{\tilde{f}_i}{r^i}, \qquad h = 1 + \sum_{i=1}^{\infty} \frac{\tilde{h}_i}{r^i}, \qquad A_0 = P + \sum_{i=1}^{\infty} \frac{\tilde{a}_i}{r^i}.$$

The iterative solutions for cubic Galileons are given by

$$f = 1 - \frac{2M}{r} - \frac{P^2 M^3}{6M_{\rm pl}^2 r^3} + \cdots$$

$$h = 1 - \frac{2M}{r} - \frac{P^2 M^2}{2M_{\rm pl}^2 r^2} - \frac{P^2 M^3}{2M_{\rm pl}^2 r^3} + \cdots$$

The two metrics are not identical.

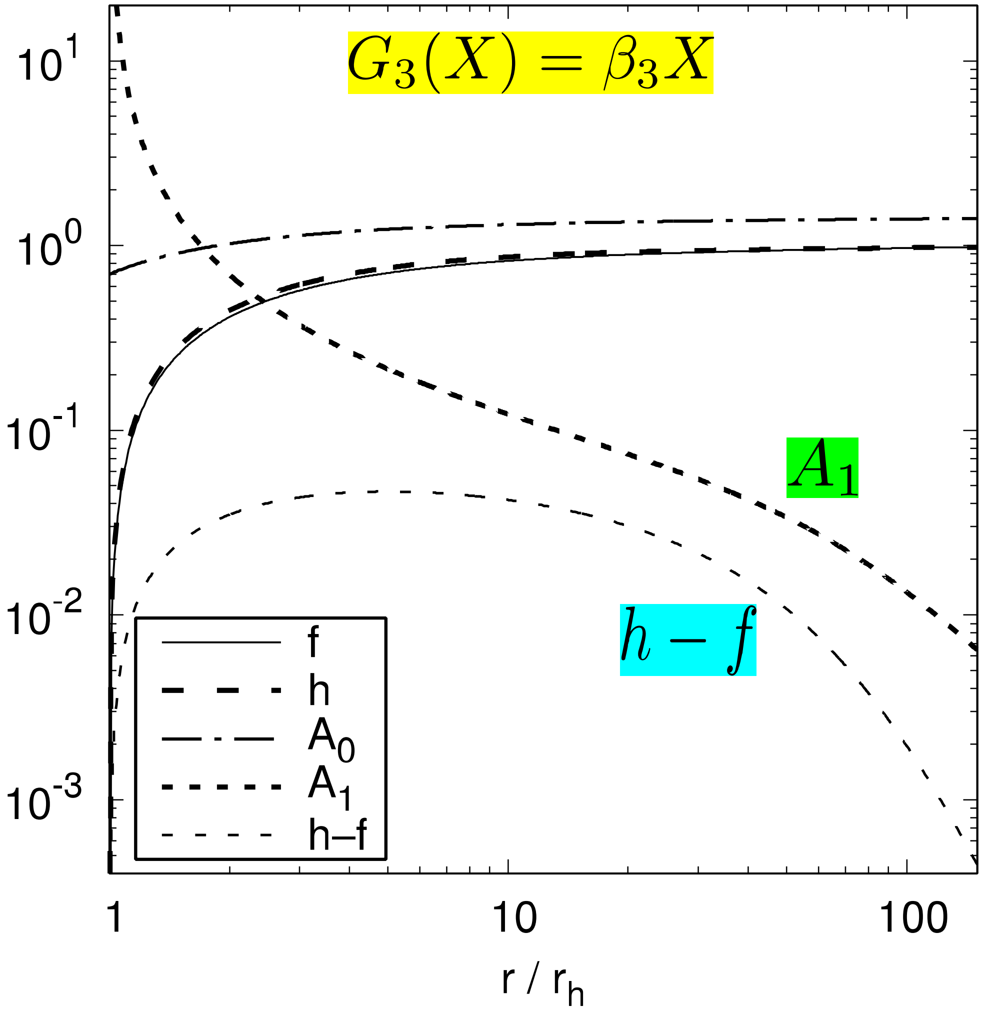
$$A_0 = P - \frac{PM}{r} - \frac{PM^2}{2r^2} + \cdots$$

$$A_1 = \frac{\tilde{b}_2}{r^2} + \frac{M(M + 2\tilde{b}_2 \beta_3)}{\beta_3 r^3} + \cdots$$

The solution is expressed in terms of three constants:

$M, P, \tilde{b}_2$ ⇒ Related to the three parameters μ, a_0, r_h around the horizon.

Numerical solutions



The solutions in the two regimes $r \simeq r_h$ and $r \gg r_h$ are connected to each other.

The constant P cannot be fixed by M and $\tilde{b}_2$, so it is a primary hair.

The difference between the two metrics f and h is large around the horizon.

This can be potentially probed in future measurements of gravitational waves.

Non-exact solutions for other derivative couplings

There are also a bunch of hairy BH solutions for other power-law couplings:

$$G_i = \beta_i X^n \qquad \text{where} \quad i = 3, 4, 6 \quad \text{and} \quad n \geq 0$$

For the massive vector with $G_2 = m^2 X$, there is a no-hair theorem proved by Bekenstein (1972).

For the quintic coupling $G_5 = \beta_5 X^n$ $(n \geq 1)$, the two solutions in the regimes $r \simeq r_h$ and $r \gg r_h$ are disconnected due to the divergence of A_1 at $h = 1/(2n+1)$.

For the couplings $G_4 = \beta_4 X^n$ and $G_6 = \beta_6 X^n$, there exists the branch $A_1 = 0$ besides $A_1 \neq 0$. Even for the branch $A_1 = 0$, the difference between f and h arises around the horizon.

The BH solution for the $U(1)$-invariant sixth-order Lagrangian $G_6 = \beta_6$ (i.e., $n = 0$) was discussed by Horndeski in 1978 (before he became an artist).

Horndeski's paper in 1978

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Static spherically symmetric solutions to a system of generalized Einstein-Maxwell field equations

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In this paper I investigate the static, spherically symmetric, pure electric, source-free solutions to the most general second-order vector-tensor theory of gravitation and electromagnetism which is such that its field equations are (i) derivable from a variational principle, (ii) consistent with the notion of conservation of charge, (iii) in agreement with Einstein's equations in the absence of electromagnetic fields, and (iv) compatible with Maxwell's equations in a flat space. These solutions (which are given in series form) bear a strong resemblance to the Reissner-Nordström solution (of the Einstein-Maxwell field equations) in the asymptotic domain; however, they differ quite radically from the Reissner-Nordström solution in the vicinity of the source. In addition it appears as though many of these solutions are only compatible with electric monopoles of finite extent.

Expansion at spatial infinity done by Horndeski

$$e^{2a} = 1 - \frac{2M}{r} + \frac{C^2}{r^2} + \frac{kC^2}{2r^4} - \frac{kMC^2}{2r^5} + O(r^{-6}),$$

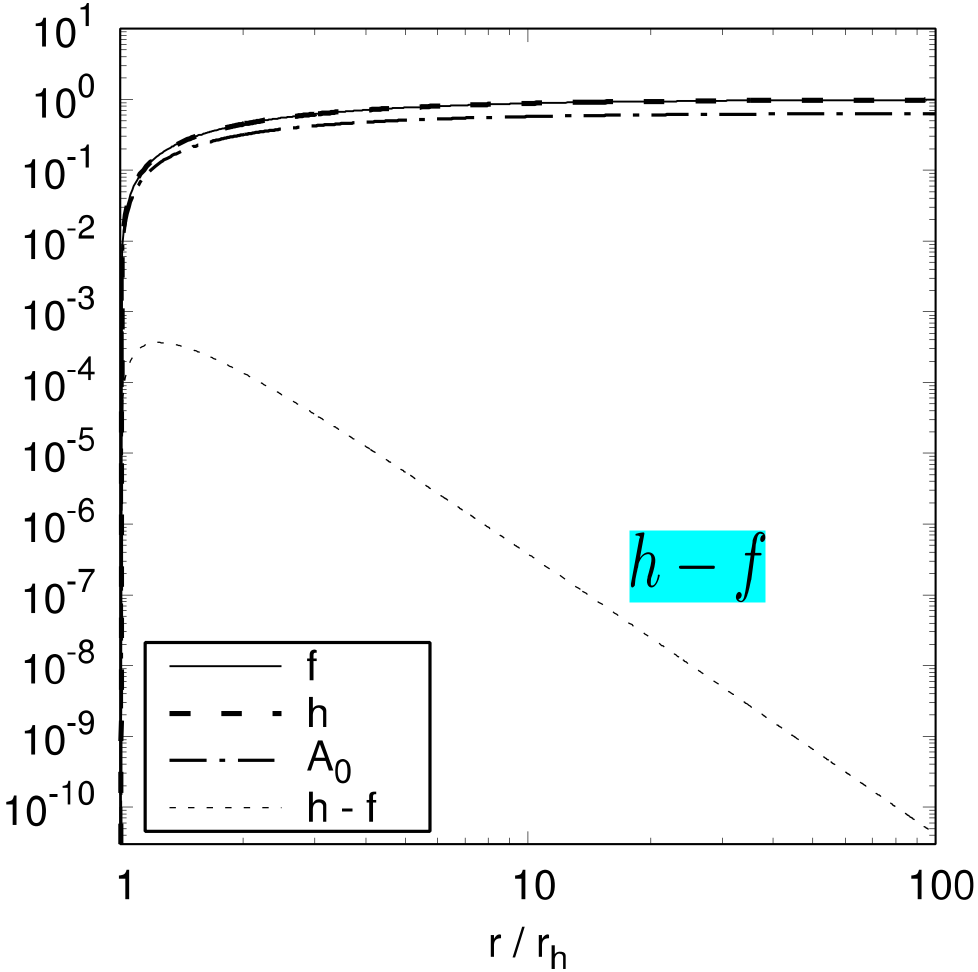
$$e^{2b} = 1 + \frac{2M}{r} + \frac{(4M^2 - C^2)}{r^2} + \frac{4M(2M^2 - C^2)}{r^3} + \frac{(C^4 - 12M^2C^2 + 16M^4)}{r^4} + \frac{M(12C^4 - kC^2 - 64M^2C^2 + 64M^4)}{2r^5} + O(r^{-6})$$

These coincide with our results.

Sixth-order couplings

We also obtained BH solutions for the coupling $G_6 = \beta_6 X^n$ with $n \geq 1$.

$G_6(X) = \beta_6 X$ $\quad$ $A_1 = 0$



Even for the branch $A_1 = 0$, there is the difference between f and h around the horizon.

The difference $h - f$ rapidly approaches 0 for $r \gg r_h$ to recover the general relativistic behavior.

For $n \geq 1$ the constant P at spatial infinity depends on M, Q, so it is of the secondary type.

If we can precisely measure the deviation from GR around the horizon, it is possible to distinguish between different couplings.

Conclusions

1. Generalized Proca theories give rise to interesting cosmological solutions with a late-time de Sitter attractor.

2. We constructed a class of dark energy models in which all the stability conditions are satisfied during the cosmic expansion history.

3. The joint data analysis of SN Ia, CMB, BAO, H_0, and RSD show that the model in GP theories is favored over the LCDM.

4. The healthy extension of GP theories allows the realization of weak gravity which should be consistent with both RSD and CMB data.

5. There exist a bunch of hairy black hole solutions in generalized Proca theories.

Let's see whether future observations show the signature of vector-tensor theories.