
# Black Holes, Integrable Systems and Soft Hair

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**based on arXiv: 1605.04490 [hep-th]**

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# Introduction

- **Integrable systems :**
  - **Field theories in 2D (up to just a few exceptions)**
  - **Nonnecessary relativistic** $u = u(t, x)$
- **Example : Korteweg-de Vries (KdV) eq.**

$$\dot{u} = 3uu' - 2u'''$$

  - **Originally : dynamics of solitons on shallow water channels**
  - **Nowadays : condensed matter, quantum optics, plasma physics, nonlinear acoustics, atmospheric and oceanographic science, …**
  - **Integrable system : infinite number of conserved charges**
  - $H_{(j)}$ **: nonlinear functionals of u & their spacelike derivatives**
  - **In involution :** $\{H_{(i)}, H_{(j)}\} = 0$ **(Poisson bracket algebra is abelian)**

# Summary

- **Wide class of integrable systems in 2D (e.g., KdV): fully geometrized !**
  - **Evolution of spacelike surfaces embedded in constant curvature spacetimes**

- **GR in 3D with a suitable set of bc's**
  - **Einstein equations reduce to the ones of the integrable systems**

- **Symmetries of integrable systems (non Noetherian in 2D)**
  - **diffemorphisms preserving asympt. form of the metric**
  - **manifestly become Noetherian in the geometrc picture !**

- **Infinite set of conserved charges (in involution)**
  - **Recovered in canonical approach: surf. integrals spanning the asympt. symms.**

- **Links between soliton dynamics & black hole physics unveiled**

# Motivation

- **Asymptotic structure of spacetime :**
  - **ADM : lapse and shift functions (deformations of spacelike surfaces)**
  - **Lagrange multipliers (Hamiltonian constrains)**
  - **Assumed to be fixed to constants at infinity**

  - **Observables (like the energy): measured w.r.t. fixed time and length scales**
  - **Reasonable and useful practice**
  - **Strictly, not a necessary one !**

- **Non standard choices of time and length scales at the boundary**
  - **Lagrange multipliers: fixed at infinity by a precise dependence on the dynamical fields.**

- **Focus in GR in 3D with negative $\Lambda$**
  - **Extending the standard analysis of Brown and Henneaux**
  - **Both, metric formalism & gauge fields**

# Outline

- **Generic choice of boundary conditions**
  - Chern-Simons approach (simpler)
  - Asymptotic symmetries and conserved charges
- **Specific choices of boundary conditions**
  - Sensible criteria
  - $k = 0$: chiral movers (Brown–Henneaux)
  - $k = 1$: KdV movers
  - Generic k: KdV hierarchy [ Lifshitz scaling : $z = 2k + 1$ ]
- **BTZ black hole with selected boundary conditions**
  - global charges and thermodynamics
  - Energy, temperature and entropy: expected Lifshitz-like scalings
- **Anisotropic modular invariance**
  - Asymptotic growth of the number of states
  - Generalization of Cardy formula [ depends on z, and ground state energy ]
  - Recovering black hole entropy

# Outline

- **Results in terms of the spacetime metric**
  - **General solution of Einstein eqs. with** $\Lambda$
  - **Spacetime metrics of constant curvature & Lifshitz scaling**

- **Discussion**
  - **Geometrization of KdV : from KdV to gravitation and viceversa**
  - **Nonrelativistic holography without Lifshitz spacetimes**
  - **Link with “soft hair” & a fractional extension of the KdV hierarchy**
  - **Flat limit & a new hierarchy of integrable systems**
  - **Generalization to higher spin gravity in 3D**
  - **Spin-3 fields & the Bousinesq hierarchy**
  - **Integrable systems with Poisson structures given by “Flat W-algebras”**

# Gravitation on AdS3

$$I = I_{CS}\left[A^{+}\right] - I_{CS}\left[A^{-}\right]$$

$$I_{CS}\left[A\right] = \frac{k}{4\pi}\int_{M}\left\langle AdA + \frac{2}{3}A^{3}\right\rangle \qquad k = \frac{l}{4G}$$

$$\mathfrak{g} = sl\left(2,\mathbb{R}\right)\oplus sl\left(2,\mathbb{R}\right) \qquad L_i : i = -1, 0, 1$$

$$\langle \cdots \rangle \; = \; \mathrm{tr}\left(\cdots\right) \qquad A^{\pm} = \omega \pm \frac{e}{l}$$

$$g_{\mu\nu} = \frac{\ell^2}{2}\left\langle \left(A_{\mu}^{+} - A_{\mu}^{-}\right)\left(A_{\nu}^{+} - A_{\nu}^{-}\right)\right\rangle$$

# Generic asymptotic fall-off

**Coussaert, Henneaux, van Driel, gr-qc/9506019**

**Henneaux, Pérez, Tempo, Troncoso, arXiv:1309.4362 [hep-th]**

**Bunster, Henneaux, Pérez, Tempo, Troncoso, arXiv:1404.3305 [hep-th]**

$$A^{\pm} = g_{\pm}^{-1}\left(d + a^{\pm}\right) g_{\pm} \qquad g_{\pm} = e^{\pm \log(r/\ell) L_0}$$

$$a^{\pm} = a_{\varphi}^{\pm} d\varphi + a_t^{\pm} dt$$

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$$a^{\pm} = a^{\pm}_{\varphi} d\varphi + a^{\pm}_{t} dt$$

$$a^{\pm}_{\varphi} = L_{\pm 1} - \frac{1}{4}\mathcal{L}_{\pm} L_{\mp 1} \;\; ; \;\; a^{\pm}_{t} = \pm \Lambda^{\pm}\left[\mu^{\pm}\right]$$

$$\Lambda^{\pm}\left[\mu^{\pm}\right] = \mu^{\pm}\left(L_{\pm 1} - \frac{1}{4}\mathcal{L}_{\pm} L_{\mp 1}\right) \mp \mu^{\pm\prime} L_0 + \frac{1}{2}\mu^{\pm\prime\prime} L_{\mp 1}$$

$\mathcal{L}, \mu$ **: arbitrary functions of** $t, \varphi$

**Field eqs. :** $\dot{\mathcal{L}}_{\pm} := \pm \mathcal{D}^{\pm} \mu^{\pm}$

$$\mathcal{D}^{\pm} := \left(\partial_{\varphi} \mathcal{L}_{\pm}\right) + 2\mathcal{L}_{\pm}\partial_{\varphi} - 2\partial_{\varphi}^{3}$$

# Generic choice of boundary conditions

**Boundary conditions :**

**specified only once $\mu^{\pm}$ are precisely chosen at the boundary**

**Standard choice: $\mu^{\pm} = 1$ (Brown-Henneaux)**

**Let's explore the set of different possible choices of $\mu^{\pm}$**
**(consistent with the action principle)**

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**Chern-Simons action: already in Hamiltonian form:**

$$I = I_{CS}\left[A^{+}\right] - I_{CS}\left[A^{-}\right] \qquad A^{\pm} = A_i^{\pm} dx^i + A_t^{\pm} dt$$

$$I_{CS}\left[A^{\pm}\right] = -\frac{\kappa}{4\pi}\int dt d^2x \varepsilon^{ij}\left\langle A_i^{\pm}\dot{A}_j^{\pm} - A_t^{\pm}F_{ij}^{\pm}\right\rangle + B_{\infty}^{\pm}$$

$B_{\infty}^{\pm}$ **: suitable boundary terms (action principle has to be well-defined)**

**Action attains an extremum everywhere, provided field eqs. are fulfilled &**

$$\delta B_{\infty}^{\pm} = \mp\frac{\kappa}{8\pi}\int dt d\varphi \mu^{\pm}\delta\mathcal{L}_{\pm}$$

# Generic choice of boundary conditions

$$I_{CS}\left[A^{\pm}\right] = -\frac{\kappa}{4\pi}\int dt d^2x \varepsilon^{ij}\left\langle A_i^{\pm}\dot{A}_j^{\pm} - A_t^{\pm}F_{ij}^{\pm}\right\rangle + B_{\infty}^{\pm}$$

$$\delta B_{\infty}^{\pm} = \mp\frac{\kappa}{8\pi}\int dt d\varphi \mu^{\pm}\delta\mathcal{L}_{\pm}$$

**Integrability conditions:**

$$\delta^2 B_{\infty}^{\pm} = \mp\frac{\kappa}{8\pi}\int dt d\varphi \delta\mu^{\pm}\wedge\delta\mathcal{L}_{\pm} = 0$$

**solved by** $\mu^{\pm} = \frac{\delta H^{\pm}}{\delta\mathcal{L}_{\pm}}$

**with** $H^{\pm} = \int d\phi \mathcal{H}^{\pm}\left[\mathcal{L}_{\pm}, \mathcal{L}'_{\pm}, \mathcal{L}''_{\pm}, \cdots\right]$

**assumed to be arbitrary functionals of** $\mathcal{L}_{\pm}$ **and their derivatives**

# Generic choice of boundary conditions

$$I_{CS}\left[A^{\pm}\right] = -\frac{\kappa}{4\pi}\int dt d^2x \varepsilon^{ij}\left\langle A_i^{\pm}\dot{A}_j^{\pm} - A_t^{\pm}F_{ij}^{\pm}\right\rangle + B_{\infty}^{\pm}$$

**Boundary terms integrate as:**

$$B_{\infty}^{\pm} = \mp\frac{\kappa}{8\pi}\int dt d\varphi \mathcal{H}^{\pm}$$

**Boundary conditions completely determined once the functionals**

$H^{\pm} = \int d\phi \mathcal{H}^{\pm}\left[\mathcal{L}_{\pm}, \mathcal{L}'_{\pm}, \mathcal{L}''_{\pm}, \cdots\right]$ **are specified (at the boundary)**

**Choice of Lagrange multipliers** $\mu^{\pm} = \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}}$

**guarantees the integrability of the boundary terms**

**[ required by consistency of the action principle ]**

# Asymptotic symmetries

**By virtue of :** $A^{\pm} = g_{\pm}^{-1}\left(d + a^{\pm}\right) g_{\pm}$ $\qquad g_{\pm} = e^{\pm \log(r/\ell) L_0}$

**The analisys can be performed in terms of** $a^{\pm} = a_{\varphi}^{\pm} d\varphi + a_t^{\pm} dt$

---

**Gauge transformations** $\delta a^{\pm} = d\eta^{\pm} + [a^{\pm}, \eta^{\pm}]$

**that preserve the form of** $a^{\pm}$ **:**

$$a_{\varphi}^{\pm} = L_{\pm 1} - \frac{1}{4}\mathcal{L}_{\pm} L_{\mp 1} \;\; ; \;\; a_t^{\pm} = \pm\Lambda^{\pm}\left[\mu^{\pm}\right]$$

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---

$a^{\pm}_{\varphi}$ **is preserved for** $\eta^{\pm} = \Lambda^{\pm}\left[\varepsilon^{\pm}\right]$ **with** $\varepsilon^{\pm} = \varepsilon^{\pm}(t, \varphi)$

**Provided** $\delta \mathcal{L}_{\pm} = \mathcal{D}^{\pm} \varepsilon^{\pm}$

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**Provided** $\delta \mathcal{L}_{\pm} = \mathcal{D}^{\pm} \varepsilon^{\pm}$

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$$a_t^{\pm} \;:\; \delta\mu^{\pm} = \pm\dot{\varepsilon}^{\pm} + \varepsilon^{\pm}\mu^{\pm\prime} - \mu^{\pm}\varepsilon^{\pm\prime} \qquad \dot{\mathcal{L}}_{\pm} := \pm\mathcal{D}^{\pm}\mu^{\pm}$$

$$\mu^{\pm} = \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}} \qquad \dot{\varepsilon}^{\pm} = \pm\frac{\delta}{\delta \mathcal{L}_{\pm}}\int d\phi \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}} \mathcal{D}^{\pm}\varepsilon^{\pm}$$

## Conserved charges

$$\delta\mathcal{L}_\pm = \mathcal{D}^\pm \varepsilon^\pm \qquad\qquad \dot{\varepsilon}^\pm = \pm \frac{\delta}{\delta\mathcal{L}_\pm}\int d\phi \frac{\delta H^\pm}{\delta\mathcal{L}_\pm}\mathcal{D}^\pm\varepsilon^\pm$$

$\varepsilon^\pm$**: generically acquire a nontrivial dependence on** $\mathcal{L}_\pm$ **and their derivatives**

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**Canonical generators :**

$$\delta Q^\pm\left[\varepsilon^\pm\right] = -\frac{\kappa}{8\pi}\int d\varphi\, \varepsilon^\pm \delta\mathcal{L}_\pm$$

**Conservation in time** $(\delta\dot{Q}^\pm = 0)$ **is guaranteed for** $\dot{\varepsilon}^\pm = \pm \frac{\delta}{\delta\mathcal{L}_\pm}\int d\phi \frac{\delta H^\pm}{\delta\mathcal{L}_\pm}\mathcal{D}^\pm\varepsilon^\pm$ **on-shell**

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**In order to integrate** $\delta Q^\pm\left[\varepsilon^\pm\right]$ **one needs to know the general solution of**

$$\dot{\varepsilon}^\pm = \pm \frac{\delta}{\delta\mathcal{L}_\pm}\int d\phi \frac{\delta H^\pm}{\delta\mathcal{L}_\pm}\mathcal{D}^\pm\varepsilon^\pm \qquad \text{with} \quad \dot{\mathcal{L}}_\pm := \pm\mathcal{D}^\pm\mu^\pm$$

**for a generic choice of boundary conditions, specified by** $H^\pm$
**this is a very hard task !**

# Conserved charges

**Nonetheless, if $H^{\pm}$ are indep. of time and the angle the asymptotic Killing vectors $\partial_\varphi$ & $\partial_t$ belong to the asympt. symms., one can integrate their generators:**

**angular momentum :**

$$J = Q\left[\partial_\varphi\right] = \frac{\kappa}{8\pi}\int d\varphi \left(\mathcal{L}_+ - \mathcal{L}_-\right)$$

**variation of the total energy :**

$$\delta E = \delta Q\left[\partial_t\right] = \frac{\kappa}{8\pi}\int d\varphi \left(\mu^+\delta\mathcal{L}_+ + \mu^-\delta\mathcal{L}_-\right)$$

**by virtue of** $\mu^{\pm} = \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}}$ **integrates as**

$$E = \frac{\kappa}{8\pi}\left(H^+ + H^-\right)$$

**Complete analysis of the asymptotic structure: concrete choices of boundary conditions (precise form of $H^{\pm}$ ) have to be given**

$$\delta Q^{\pm}\left[\varepsilon^{\pm}\right] = -\frac{\kappa}{8\pi}\int d\varphi \varepsilon^{\pm}\delta\mathcal{L}_{\pm}$$

# Specific choices of boundary conditions

**Sensible criteria to fix the form of $H^{\pm}$**

- **Allowing as much asymptotic symmetries as possible:**

  **knowing the general solution of** $\dot{\varepsilon}^{\pm} = \pm \frac{\delta}{\delta \mathcal{L}_{\pm}} \int d\phi \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}} \mathcal{D}^{\pm} \varepsilon^{\pm}$

  **for arbitrary values of** $\mathcal{L}_{\pm}$ **and their derivatives**

- **An infinite number of asymptotic symmetries is welcome:**

  **helps in order to explicitly find the space of solutions that fulfill the boundary conditions.**

**These criteria are met in the cases that** $H^{\pm}$ **define integrable systems**

**Let's see a few ( but still infinite ! ) number of explicit examples with the desired features**

# k = 0 : chiral movers (Brown–Henneaux)

**Boundary conditions specified by** $\mu^{\pm}_{(0)} = 1$

$\mu^{\pm} = \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}}$ **amounts to set** $H^{\pm}_{(0)} = \int d\varphi \mathcal{H}^{\pm}_{(0)}$

$$\mathcal{H}^{\pm}_{(0)} := \mathcal{L}_{\pm}$$

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$$\mathcal{H}^{\pm}_{(0)} := \mathcal{L}_{\pm}$$

---

**Field eqs. :** $\dot{\mathcal{L}}_{\pm} := \pm \mathcal{D}^{\pm} \mu^{\pm}$ **:** $\dot{\mathcal{L}}_{\pm} = \pm \mathcal{L}'_{\pm}$

$\dot{\varepsilon}^{\pm} = \pm \frac{\delta}{\delta \mathcal{L}_{\pm}} \int d\phi \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}} \mathcal{D}^{\pm} \varepsilon^{\pm}$ **:** $\dot{\varepsilon}^{\pm} = \pm \varepsilon^{\pm\prime}$

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$$\mu^{\pm} = \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}}$$ **amounts to set** $H^{\pm}_{(0)} = \int d\varphi \mathcal{H}^{\pm}_{(0)}$

$$\mathcal{H}^{\pm}_{(0)} := \mathcal{L}_{\pm}$$

**Field eqs. :** $\dot{\mathcal{L}}_{\pm} := \pm \mathcal{D}^{\pm} \mu^{\pm}$ **:** $\dot{\mathcal{L}}_{\pm} = \pm \mathcal{L}'_{\pm}$

$$\dot{\varepsilon}^{\pm} = \pm \frac{\delta}{\delta \mathcal{L}_{\pm}} \int d\phi \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}} \mathcal{D}^{\pm} \varepsilon^{\pm}$$ **:** $\dot{\varepsilon}^{\pm} = \pm \varepsilon^{\pm\prime}$

**Variation of global charges integrates as**

$$\delta Q^{\pm}\left[\varepsilon^{\pm}\right] = -\frac{\kappa}{8\pi} \int d\varphi \varepsilon^{\pm} \delta \mathcal{L}_{\pm}$$ **:** $Q^{\pm}\left[\varepsilon^{\pm}\right] = -\frac{\kappa}{8\pi} \int d\varphi \varepsilon^{\pm} \mathcal{L}_{\pm}$

$$\{Q\left[\varepsilon_1\right], Q\left[\varepsilon_2\right]\} = \delta_{\varepsilon_2} Q\left[\varepsilon_1\right]$$

$$\delta \mathcal{L}_{\pm} = \mathcal{D}^{\pm} \varepsilon^{\pm}$$

$$\mathcal{D}^{\pm} := (\partial_\varphi \mathcal{L}_{\pm}) + 2\mathcal{L}_{\pm} \partial_\varphi - 2\partial_\varphi^3$$

**Algebra: 2 copies of Virasoro with the Brown-Henneaux central extension**

# k = 1 : KdV movers

**Boundary conditions given by** $\mu^{\pm}_{(1)} = \mathcal{L}_{\pm}$

$\mu^{\pm} = \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}}$ **amounts to set** $H^{\pm}_{(1)} = \int d\varphi \mathcal{H}^{\pm}_{(1)}$

$$\mathcal{H}^{\pm}_{(1)} := \frac{1}{2}\mathcal{L}^{2}_{\pm}$$

---

**Field eqs. :** $\dot{\mathcal{L}}_{\pm} := \pm \mathcal{D}^{\pm}\mu^{\pm}$ **reduce to KdV :**

$$\dot{\mathcal{L}}_{\pm} = \pm\left(3\mathcal{L}_{\pm}\mathcal{L}'_{\pm} - 2\mathcal{L}'''_{\pm}\right)$$

---

$\dot{\varepsilon}^{\pm} = \pm\frac{\delta}{\delta \mathcal{L}_{\pm}}\int d\phi \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}}\mathcal{D}^{\pm}\varepsilon^{\pm}$ **reduce to**

$$\dot{\varepsilon}^{\pm} = \pm\left(3\mathcal{L}_{\pm}\partial_{\varphi}\varepsilon^{\pm} - 2\partial^{3}_{\varphi}\varepsilon^{\pm}\right)$$

## k = 1 : KdV movers

We want to solve : $$\dot{\varepsilon}^{\pm} = \pm\left(3\mathcal{L}_{\pm}\partial_{\varphi}\varepsilon^{\pm} - 2\partial_{\varphi}^{3}\varepsilon^{\pm}\right)$$

with : $$\dot{\mathcal{L}}_{\pm} = \pm\left(3\mathcal{L}_{\pm}\mathcal{L}'_{\pm} - 2\mathcal{L}'''_{\pm}\right)$$

---

KdV is an integrable system : we know the general solution for $\varepsilon^{\pm}$

Assuming that they are local functions of $\mathcal{L}_{\pm}$ and their derivatives :

$$\varepsilon^{\pm} = \sum_{j=0}^{\infty} \eta^{\pm}_{(j)} R^{\pm}_{(j)}$$

$\eta^{\pm}_{(j)}$ : constants

$R^{\pm}_{(j)}$ : Gelfand-Dikii polynomials

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Defined through :

$$\partial_{\varphi} R^{\pm}_{(j+1)} = \frac{j+1}{2j+1}\mathcal{D}^{\pm} R^{\pm}_{(j)}$$

They fulfill :

$$R^{\pm}_{(j)} = \frac{\delta H^{\pm}_{(j)}}{\delta \mathcal{L}_{\pm}}$$

[ Normalization such that $R_{(j)} = \mathcal{L}^{j} +$ derivatives ]

## k = 1 : KdV movers

$$\varepsilon^{\pm} = \sum_{j=0}^{\infty} \eta^{\pm}_{(j)} R^{\pm}_{(j)}$$

$$\partial_{\varphi} R^{\pm}_{(j+1)} = \frac{j+1}{2j+1} \mathcal{D}^{\pm} R^{\pm}_{(j)} \qquad R^{\pm}_{(j)} = \frac{\delta H^{\pm}_{(j)}}{\delta \mathcal{L}_{\pm}}$$

**Hence,** $R^{\pm}_{(0)} = \mu^{\pm}_{(0)} = 1$ **,** $R^{\pm}_{(1)} = \mu^{\pm}_{(1)} = \mathcal{L}_{\pm}$ **, etc.**

**Variation of global charges integrates as**

$$\delta Q^{\pm}\left[\varepsilon^{\pm}\right] = -\frac{\kappa}{8\pi} \int d\varphi \varepsilon^{\pm} \delta \mathcal{L}_{\pm} \;: \qquad Q^{\pm}\left[\varepsilon^{\pm}\right] = -\frac{\kappa}{8\pi} \sum_{j=0}^{\infty} \eta^{\pm}_{(j)} H^{\pm}_{(j)}$$

**Their algebra is abelian with no central extensions**

$\{H^{\pm}_{(k)}, H^{\pm}_{(j)}\} = 0$ **[ Integrable systems : conserved charges are in involution ]**

# k = 1 : KdV movers

**Remarks :**

**First four conserved charges of the series :**

$$H^{\pm}_{(0)} = \int d\varphi \mathcal{L}_{\pm} \quad , \quad H^{\pm}_{(1)} = \int d\varphi \frac{1}{2} \mathcal{L}^2_{\pm} \quad , \quad H^{\pm}_{(2)} = \int d\varphi \frac{1}{3} \left( \mathcal{L}^3_{\pm} + 2\mathcal{L}'^2_{\pm} \right)$$

$$H^{\pm}_{(3)} = \int d\varphi \frac{1}{4} \left( \mathcal{L}^4_{\pm} + 8\mathcal{L}_{\pm}\mathcal{L}'^2_{\pm} + \frac{16}{5} \mathcal{L}''^2_{\pm} \right) ,$$

**Total energy of a gravitational configuration :**

**sum of the energies of left & right KdV movers**

$$E = \frac{\kappa}{16\pi} \int d\varphi \left( \mathcal{L}^2_{+} + \mathcal{L}^2_{-} \right)$$

# Generic k : KdV hierarchy

**Boundary conditions extended as :** $$\mu^{\pm}_{(k)} = R^{\pm}_{(k)} = \frac{\delta H^{\pm}_{(k)}}{\delta \mathcal{L}_{\pm}}$$

**[ k = 0 : Brown-Henneaux , k = 1 : KdV , … etc. ]**

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**Field eqs. :**

$$\dot{\mathcal{L}}_{\pm} = \pm \mathcal{D}^{\pm} R^{\pm}_{(k)}$$

**Left and right movers :**
**evolve according to the k-th representative of the KdV hierarchy:**

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**Asympt. symm. parameters :**

$$\dot{\varepsilon}^{\pm} = \pm \frac{\delta}{\delta \mathcal{L}_{\pm}} \int d\phi \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}} \mathcal{D}^{\pm} \varepsilon^{\pm}$$

**with** $H^{\pm} = H^{\pm}_{(k)}$.

# Generic k : KdV hierarchy

$$\dot{\mathcal{L}}_\pm = \pm \mathcal{D}^\pm R^\pm_{(k)}$$

$$\dot{\varepsilon}^\pm = \pm \frac{\delta}{\delta \mathcal{L}_\pm} \int d\phi \frac{\delta H^\pm}{\delta \mathcal{L}_\pm} \mathcal{D}^\pm \varepsilon^\pm$$ **with** $H^\pm = H^\pm_{(k)}$.

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**k > 1 : field eqs. & contidions on $\varepsilon^\pm$ are severely modified**
**[ compared with k = 1 ]**

**Remarkable properties of the Gelfand–Dikii polynomials**
**imply that for k > 1, we have the same series for $\varepsilon^\pm$ (as for k = 1)**

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**Assuming that they are local functions of $\mathcal{L}_\pm$ and their derivatives :**

$$\varepsilon^\pm = \sum_{j=0}^{\infty} \eta^\pm_{(j)} R^\pm_{(j)}$$

$\eta^\pm_{(j)}$ **: constants**

$R^\pm_{(j)}$ **: Gelfand-Dikii polynomials**

# Generic k : KdV hierarchy

$$\dot{\mathcal{L}}_\pm = \pm \mathcal{D}^\pm R^\pm_{(k)}$$

$$\mu^\pm_{(k)} = R^\pm_{(k)} = \frac{\delta H^\pm_{(k)}}{\delta \mathcal{L}_\pm}$$

---

**Variation of global charges then again integrates as :**

$$Q^\pm\left[\varepsilon^\pm\right] = -\frac{\kappa}{8\pi}\sum_{j=0}^{\infty} \eta^\pm_{(j)} H^\pm_{(j)}$$

**Abelian algebra with no central extensions**

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**[ Generic k ] Total energy of a gravitational configuration :**

**sum of the energies of left & right movers (k-th KdV eq.)**

$$E = E_+ + E_-$$ **with**

$$E_\pm = \frac{\kappa}{8\pi} H^\pm_{(k)}$$

## Generic k : KdV hierarchy

$$\dot{\mathcal{L}}_\pm = \pm \mathcal{D}^\pm R^\pm_{(k)} \qquad \mu^\pm_{(k)} = R^\pm_{(k)} = \frac{\delta H^\pm_{(k)}}{\delta \mathcal{L}_\pm}$$

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## "boundary gravitons" : anisotropic Lifshitz scaling

**dynamical exponent :** $z = 2k + 1$

---

**Boundary conditions make the field eqs. inv. under :**

$$t \to \lambda^z t \ , \quad \varphi \to \lambda \varphi \ , \quad \mathcal{L}_\pm \to \lambda^{-2} \mathcal{L}_\pm$$

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**Exact solutions : locally AdS spacetimes**

**Inherit anisotropic scaling [ induced by the choice of boundary conditions ]**

**Metrics are manifestly inv. under the anisotropic scaling provided**

$$r \to \lambda^{-1} r$$

**[explicitly seen later]**

# BTZ black hole with selected boundary conditions

**BTZ fits within our boundary conditions:**

$\mathcal{L}_{\pm}$ **constants, trivially solve the field eqs.** $\dot{\mathcal{L}}_{\pm} = \pm \mathcal{D}^{\pm} R^{\pm}_{(k)}$

---

**ADM form : similar than the standard one**

**Laspe and shift determined by** $\mu^{\pm}_{(k)} = \mathcal{L}^{k}_{\pm}$ **[explicitly shown later]**

**AdS spacetime recovered for** $\mathcal{L}_{\pm} = -1$

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**Energy of left and right movers :** $E_{\pm} = \frac{\kappa}{8\pi} H^{\pm}_{(k)}$

**[ in terms of** $z = 2k + 1$ **]**

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**BTZ :** $$E_{\pm} = \frac{\kappa}{2} \frac{1}{z+1} \mathcal{L}_{\pm}^{\frac{z+1}{2}}$$

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**AdS :** $$E_{0}^{\pm}[z] = \frac{\kappa}{2} \frac{1}{z+1} (-1)^{\frac{z+1}{2}}$$

**[ depends on z ]**

# BTZ black hole with selected boundary conditions

**Bekenstein-Hawking entropy :**

$$S = \frac{A}{4G} = \pi\kappa\left(\sqrt{\mathcal{L}_+} + \sqrt{\mathcal{L}_-}\right)$$

**In terms of extensive variables :**

$$S = \pi\kappa\left(\frac{2}{\kappa}(z+1)\right)^{\frac{1}{z+1}}\left(E_+^{\frac{1}{z+1}} + E_-^{\frac{1}{z+1}}\right)$$

**Left & right temperatures** $T_\pm = \beta_\pm^{-1}$ **:** $\beta_\pm = \frac{\partial S}{\partial E_\pm} = 2\pi\left(\frac{2}{\kappa}(z+1)\,E_\pm\right)^{-\frac{z}{z+1}}$

$$S = \frac{\kappa}{2}(2\pi)^{1+\frac{1}{z}}\left(T_+^{\frac{1}{z}} + T_-^{\frac{1}{z}}\right)$$

**Expected dependence on the energy and temperature for noninteracting left & right movers of a field theory with Lifshitz scaling in 2D**

**Even better : S is precisely recovered from a generalization of the Cardy formula in the anisotropic case !**

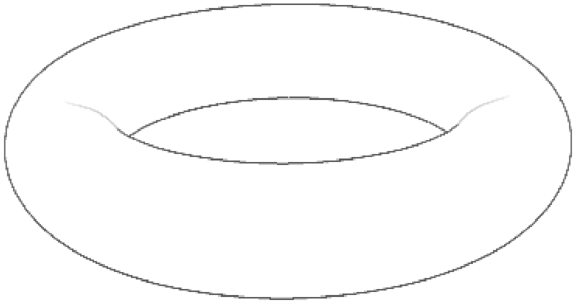
# Anisotropic modular invariance

González, Tempo, Troncoso, arXiv:1107.3647 [hep-th]

**Thermal field theories with Lifshitz scaling in 2D**

**defined on a solid torus with** $0 \leq \varphi < 2\pi$

$0 \leq t_E < \beta$



**Duality between low and high temperatures :**

$$\frac{\beta}{2\pi} \to \left(\frac{2\pi}{\beta}\right)^{\frac{1}{z}}$$

**[ anisotropic S-duality ]**

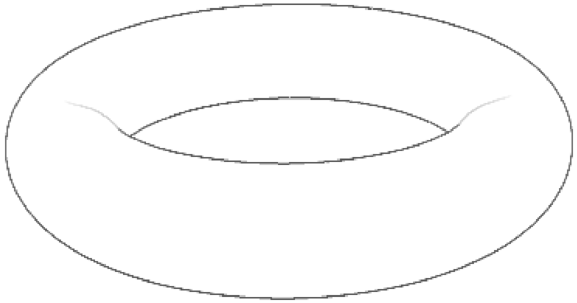
**The partition function can then be assumed to be invariant under :**

$$Z\left[\beta; z\right] = Z\left[\frac{(2\pi)^{1+\frac{1}{z}}}{\beta^{\frac{1}{z}}}; \frac{1}{z}\right]$$

# Anisotropic modular invariance

**Independent noninteracting left and right movers with the same z**

**torus with modular parameter :** $$\tau = i\frac{\beta}{2\pi}$$



$\beta$ **: complexification of** $\beta_{\pm}$

**high/low temperature duality now reads :**

$$\tau \to \frac{i^{1+\frac{1}{z}}}{\tau^{\frac{1}{z}}}$$

**[ anisotropic S-duality ]**

**It can be assumed that :**

$$Z\left[\tau; z\right] = Z\left[i^{1+\frac{1}{z}}\tau^{-\frac{1}{z}}; z^{-1}\right]$$

**z = 1 : results reduce to the standard ones in CFT**

# Anisotropic modular invariance

**Asymptotic growth of the number of states at fixed left & right energies $\Delta_{\pm}$**

---

**Assuming that the spectrum of left & right movers possesses a gap :**

**At low T, the partition function becomes dominated by the ground state**

**assumed to be nondegenerate, with left & right energies given by** $-\Delta_0^{\pm}[z]$

**[ depend on z ]**

---

**At low T :**

$$Z[\tau;z] \approx e^{-2\pi i\left(\tau\Delta_0[z]-\bar{\tau}\bar{\Delta}_0[z]\right)}$$

---

**by virtue of**

$$Z\,[\tau;z] = Z\left[i^{1+\frac{1}{z}}\tau^{-\frac{1}{z}};z^{-1}\right]$$

---

**high T regime :**

$$Z[\tau;z] \approx e^{2\pi\left((-i\tau)^{-\frac{1}{z}}\Delta_0\left[z^{-1}\right]+(i\bar{\tau})^{-\frac{1}{z}}\bar{\Delta}_0\left[z^{-1}\right]\right)}$$

# Anisotropic modular invariance

high T regime :

$$Z[\tau;z] \approx e^{2\pi\left(\left(-i\tau\right)^{-\frac{1}{z}}\Delta_0\left[z^{-1}\right]+\left(i\bar{\tau}\right)^{-\frac{1}{z}}\bar{\Delta}_0\left[z^{-1}\right]\right)}$$

Hence, at fixed energies $\Delta_\pm \gg \Delta_0^\pm\left[z\right]$

asymptotic growth of the number of states obtained evaluating Z in the saddle point approximation : described by an entropy given by

$$S = 2\pi\left(z+1\right)\left[\left(\frac{\Delta_0\left[z^{-1}\right]}{z}\right)^z\Delta\right]^{\frac{1}{z+1}} + 2\pi\left(z+1\right)\left[\left(\frac{\bar{\Delta}_0\left[z^{-1}\right]}{z}\right)^z\bar{\Delta}\right]^{\frac{1}{z+1}}$$

Note that Cardy formula is recovered for z = 1

role of the central charges played by lowest eigenvalues of the shifted

Virasoro operators $L_0 \to L_0 - \frac{c}{24}$

# Asymptotic growth of the number of states

$$S = 2\pi\,(z+1)\left[\left(\frac{\Delta_0\left[z^{-1}\right]}{z}\right)^{z}\Delta\right]^{\frac{1}{z+1}} + 2\pi\,(z+1)\left[\left(\frac{\bar{\Delta}_0\left[z^{-1}\right]}{z}\right)^{z}\bar{\Delta}\right]^{\frac{1}{z+1}}$$

---

**In terms of the (Lorentzian) left and right energies :**

$$S = 2\pi\,(z+1)\left[\left(\frac{\left|\Delta_0^+\left[z^{-1}\right]\right|}{z}\right)^{z}\Delta_+\right]^{\frac{1}{z+1}} + 2\pi\,(z+1)\left[\left(\frac{\left|\Delta_0^-\left[z^{-1}\right]\right|}{z}\right)^{z}\Delta_-\right]^{\frac{1}{z+1}}$$

---

**1st law (Canonical ensemble) :** $dS = \beta_+ d\Delta_+ + \beta_- d\Delta_-$

$$\Delta_\pm = \frac{1}{z}\,(2\pi)^{1+\frac{1}{z}}\left|\Delta_0^\pm\left[z^{-1}\right]\right| T_\pm^{1+\frac{1}{z}}$$

**[ anisotropic version of Stefan-Boltzmann law ]**

---

$$S = (2\pi)^{1+\frac{1}{z}}\left(1+\frac{1}{z}\right)\left(\left|\Delta_0^+\left[z^{-1}\right]\right| T_+^{\frac{1}{z}} + \left|\Delta_0^-\left[z^{-1}\right]\right| T_-^{\frac{1}{z}}\right)$$

**[ formulae reduce to standard results for z = 1 ]**

# Recovering the black hole entropy

**Black hole entropy, in terms of left & right energies of k-th KdV movers :**

$$S = \pi\kappa\left(\frac{2}{\kappa}(z+1)\right)^{\frac{1}{z+1}}\left(E_{+}^{\frac{1}{z+1}} + E_{-}^{\frac{1}{z+1}}\right)$$

---

**Precisely recovered form S of a 2D field theory with anisotropic scaling**

$$S = 2\pi(z+1)\left[\left(\frac{\left|\Delta_0^{+}\left[z^{-1}\right]\right|}{z}\right)^{z}\Delta_{+}\right]^{\frac{1}{z+1}} + 2\pi(z+1)\left[\left(\frac{\left|\Delta_0^{-}\left[z^{-1}\right]\right|}{z}\right)^{z}\Delta_{-}\right]^{\frac{1}{z+1}}$$

---

**Provided one identifies :**

$$\Delta_{\pm} = E_{\pm}$$

$$E_0^{\pm}[z] = \frac{\kappa}{2}\frac{1}{z+1}(-1)^{\frac{z+1}{2}}$$

$$\Delta_0^{\pm}[z] = -E_0^{\pm}[z]$$

# Recovering the black hole entropy

**Identifying :**

$$\Delta_{\pm} = E_{\pm}$$

$$\Delta_0^{\pm}[z] = -E_0^{\pm}[z]$$

---

**Analogously :**

$$\Delta_{\pm} = \frac{1}{z}(2\pi)^{1+\frac{1}{z}}\left|\Delta_0^{\pm}[z^{-1}]\right| T_{\pm}^{1+\frac{1}{z}}$$

$$S = (2\pi)^{1+\frac{1}{z}}\left(1+\frac{1}{z}\right)\left(\left|\Delta_0^{+}[z^{-1}]\right| T_{+}^{\frac{1}{z}} + \left|\Delta_0^{-}[z^{-1}]\right| T_{-}^{\frac{1}{z}}\right)$$

---

**Reduce to the ones found for the black hole :**

$$\beta_{\pm} = 2\pi\left(\frac{2}{\kappa}(z+1)E_{\pm}\right)^{-\frac{z}{z+1}}$$

$$S = \frac{\kappa}{2}(2\pi)^{1+\frac{1}{z}}\left(T_{+}^{\frac{1}{z}} + T_{-}^{\frac{1}{z}}\right)$$

# Spacetime metric: generic fall-off

**Generic choice of boundary conditions :** $\mu^{\pm} = \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}}$

**Asymptotic structure of spacetime : reconstructed from** $sl\left(2,\mathbb{R}\right)$ **gauge fields**

$$g_{\mu\nu} = \frac{\ell^2}{2}\left\langle\left(A_\mu^+ - A_\mu^-\right)\left(A_\nu^+ - A_\nu^-\right)\right\rangle$$

---

$$g_{tt} = -\left(\mathcal{N}^2 - \ell^2 \mathcal{N}^{\varphi 2}\right)\frac{r^2}{\ell^2} + f_{tt} + \mathcal{O}\left(r^{-1}\right)$$

$$g_{tr} = -\mathcal{N}^{\varphi\prime}\frac{\ell^2}{r} + \mathcal{O}\left(r^{-4}\right) \ ,$$

$$g_{t\varphi} = \mathcal{N}^{\varphi} r^2 + f_{t\varphi} + \mathcal{O}\left(r^{-1}\right) \ ,$$

$$g_{rr} = \frac{\ell^2}{r^2} + O\left(r^{-5}\right) \ ,$$

$$g_{\varphi\varphi} = r^2 + f_{\varphi\varphi} + \mathcal{O}\left(r^{-1}\right) \ ,$$

$$g_{r\varphi} = O\left(r^{-3}\right) \ ,$$

**with :** $\mu^{\pm} = \mathcal{N}\ell^{-1} \pm \mathcal{N}^{\varphi}$

# Spacetime metric: generic fall-off

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$$g_{\varphi\varphi} = r^2 + f_{\varphi\varphi} + \mathcal{O}\left(r^{-1}\right) \ ,$$

$$g_{r\varphi} = O\left(r^{-3}\right) \ ,$$

**ADM, lapse and shift :**

$$N^{\perp} = \frac{r}{\ell}\mathcal{N} + \mathcal{O}\left(r^{-1}\right) \ , \ N^{r} = -r\mathcal{N}^{\varphi\prime} + \mathcal{O}\left(r^{-1}\right) \ , \ N^{\varphi} = \mathcal{N}^{\varphi} + \mathcal{O}\left(r^{-2}\right)$$

$$f_{\varphi\varphi} = \frac{\ell^2}{4}\left(\mathcal{L}_+ + \mathcal{L}_-\right) \ ,$$

$$f_{t\varphi} = -\frac{\ell^2}{2}\mathcal{N}^{\varphi\prime\prime} + f_{\varphi\varphi}\mathcal{N}^{\varphi} + \frac{\ell}{4}\left(\mathcal{L}_+ - \mathcal{L}_-\right)\mathcal{N} \ ,$$

$$f_{tt} = \left(\frac{1}{\ell^2}\mathcal{N}^2 - \mathcal{N}^{\varphi 2}\right) f_{\varphi\varphi} + 2 f_{t\varphi}\mathcal{N}^{\varphi} + \ell^2 \mathcal{N}^{\varphi\prime 2} - \mathcal{N}\mathcal{N}^{\prime\prime}$$

# Results in terms of spacetime metric

$$g_{tt} = -\left(\mathcal{N}^2 - \ell^2 \mathcal{N}^{\varphi 2}\right)\frac{r^2}{\ell^2} + f_{tt} + \mathcal{O}\left(r^{-1}\right)$$

$$g_{tr} = -\mathcal{N}^{\varphi\prime}\frac{\ell^2}{r} + \mathcal{O}\left(r^{-4}\right) \ ,$$

$$g_{t\varphi} = \mathcal{N}^{\varphi} r^2 + f_{t\varphi} + \mathcal{O}\left(r^{-1}\right) \ ,$$

with : $\mu^{\pm} = \mathcal{N}\ell^{-1} \pm \mathcal{N}^{\varphi}$

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$$g_{r\varphi} = O\left(r^{-3}\right) \ ,$$

---

**Einstein field eqs. with negative $\Lambda$ in vacuum are fulfilled provided**

$$\dot{\mathcal{L}}_{\pm} = \pm \mathcal{D}^{\pm}\mu^{\pm}$$

**[ same as before ]**

---

**Asymptotic form of the metric : mapped into itself under asymptotic K.V.'s**

$$\delta_{\xi} g_{\mu\nu} = \mathscr{L}_{\xi} g_{\mu\nu}$$

# Results in terms of spacetime metric

**Asymptotic form of the metric : mapped into itself under asymptotic K.V.'s**

$$\delta_\xi g_{\mu\nu} = \mathscr{L}_\xi g_{\mu\nu}$$

$$\xi^t = \frac{\ell}{2\mathcal{N}}\left[\varepsilon^+ + \varepsilon^- + \frac{\ell^2}{2\mathcal{N}r^2}\left(\mathcal{N}\left(\varepsilon^+ + \varepsilon^-\right)'' - \mathcal{N}''\left(\varepsilon^+ + \varepsilon^-\right)\right)\right] + \mathcal{O}\left(r^{-4}\right) ,$$

$$\xi^r = -\frac{1}{2\mathcal{N}}\left[\left(\varepsilon^+ - \varepsilon^-\right)'\mathcal{N} - \ell\mathcal{N}^{\varphi\prime}\left(\varepsilon^+ + \varepsilon^-\right)\right] r + \frac{\ell^3\mathcal{N}^{\varphi\prime}}{4\mathcal{N}r}\left[\left(\varepsilon^+ + \varepsilon^-\right)'' - \left(\varepsilon^+ + \varepsilon^-\right)\frac{\mathcal{N}''}{\mathcal{N}}\right]\frac{1}{r} + \mathcal{O}\left(r^{-2}\right) ,$$

$$\xi^\varphi = \frac{1}{2\mathcal{N}}\left[\left(\varepsilon^+ - \varepsilon^-\right)\mathcal{N} - \ell\left(\varepsilon^+ + \varepsilon^-\right)\mathcal{N}^\varphi\right] - \frac{\ell^2}{2\mathcal{N}r^2}\left[\left(\varepsilon^+ + \varepsilon^-\right)''\mathcal{N} + \ell\left(\varepsilon^+ - \varepsilon^-\right)''\mathcal{N}^\varphi - \frac{\ell}{\mathcal{N}}\left(\varepsilon^+ + \varepsilon^-\right)\left(\mathcal{N}\mathcal{N}^{\varphi\prime\prime} + \mathcal{N}''\mathcal{N}^\varphi\right)\right] + \mathcal{O}\left(r^{-4}\right) ,$$

$$\text{with } \varepsilon^\pm = \varepsilon^\pm(t,\varphi)$$

**Provided :**

$$\delta\mathcal{L}_\pm = \mathcal{D}^\pm\varepsilon^\pm \qquad \dot{\varepsilon}^\pm = \pm\frac{\delta}{\delta\mathcal{L}_\pm}\int d\phi\,\frac{\delta H^\pm}{\delta\mathcal{L}_\pm}\mathcal{D}^\pm\varepsilon^\pm$$

**[ same as before, including surface integrals (global charges) ]**

# Results in terms of spacetime metric

**General solution of the Einstein equations with generic b.c.'s (ADM) :**

$$ds^2 = -\left(N^{\perp}\right)^2 dt^2 + g_{ij}\left(N^i dt + dx^i\right)\left(N^j dt + dx^j\right)$$

**spacelike geometry :**

$$dl^2 = g_{ij} dx^i dx^j = \frac{\ell^2}{r^2}\left[dr^2 + \ell^2\left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_+\right)\left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_-\right)d\varphi^2\right]$$

**lapse and shift :**

$$N^r = -r\mathcal{N}^{\varphi\prime}\,,\ N^{\varphi} = \mathcal{N}^{\varphi} + \frac{\left(\frac{r^2}{\ell^2}\mathcal{N} + \frac{1}{4}\mathcal{N}''\right)\left(\mathcal{L}_+ - \mathcal{L}_-\right) - 2\left(\frac{r^2}{\ell^2} + \frac{1}{8}\left(\mathcal{L}_+ + \mathcal{L}_-\right)\right)\mathcal{N}^{\varphi\prime\prime}}{4\ell\left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_+\right)\left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_-\right)}$$

$$N^{\perp} = \frac{\ell\left[\left(\frac{r^4}{\ell^4} - \frac{1}{16}\mathcal{L}_+\mathcal{L}_-\right)\mathcal{N} + \frac{1}{2}\left(\frac{r^2}{\ell^2} + \frac{1}{8}\left(\mathcal{L}_+ + \mathcal{L}_-\right)\right)\mathcal{N}'' - \frac{\ell}{16}\left(\mathcal{L}_+ - \mathcal{L}_-\right)\mathcal{N}^{\varphi\prime\prime}\right]}{r\sqrt{\left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_+\right)\left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_-\right)}}$$

**with** $\dot{\mathcal{L}}_{\pm} = \pm\mathcal{D}^{\pm}\mu^{\pm}$

# Results in terms of spacetime metric

**For the specific choice of b.c.'s (k-th KdV) :** $\mu^{\pm}_{(k)} = R^{\pm}_{(k)} = \frac{\delta H^{\pm}_{(k)}}{\delta \mathcal{L}_{\pm}}$

**spacetime metrics becomes invariant under :**

$$t \to \lambda^z t \ , \quad \varphi \to \lambda \varphi \ , \quad \mathcal{L}_{\pm} \to \lambda^{-2} \mathcal{L}_{\pm} \quad \text{with} \quad r \to \lambda^{-1} r$$

**[ anisotropic Lifshitz scaling ]**

---

**BTZ black hole :** $\mathcal{L}_{\pm}$ **constants**

$$ds^2 = \ell^2 \left[ \frac{dr^2}{r^2} + \frac{\mathcal{L}_+}{4} \left(d\tilde{x}^+\right)^2 + \frac{\mathcal{L}_-}{4} \left(d\tilde{x}^-\right)^2 - \left( \frac{r^2}{\ell^2} + \frac{\ell^2 \mathcal{L}_+ \mathcal{L}_-}{16 r^2} \right) d\tilde{x}^+ d\tilde{x}^- \right]$$

**with** $d\tilde{x}^{\pm} = \mu^{\pm} dt \pm d\varphi$ **, and** $\mu^{\pm} = \mathcal{L}^k_{\pm}$

---

**For a generic choice of b.c.'s, lapse & shift obtained from** $\mu^{\pm} = \frac{\delta H^{\pm}}{\delta \mathcal{L}_{\pm}}$

**Euclidean metric becomes regular for :** $\mu^{\pm} = \frac{2\pi}{\sqrt{\mathcal{L}_{\pm}}}$

# Final remarks

- Dynamics of k-th KdV left and right movers can be fully geometrized

  - Parameters of k-th KdV acquire a gravitational meaning
  - Phenomena observed in KdV : interpreted in the context of gravitation and vice versa

# Final remarks

**Boundary conditions (k-th KdV) :**

---

**Solutions describe constant curvature spacetimes :**

**Locally AdS, but with anisotropic Lifshitz scaling ( $z = 2k + 1$ )**

**Interesting possibility : nonrelativistic holography**
**without the need of asymptotically Lifshitz spacetimes**

**[ Lifshitz scaling : not necessarily requires the use of Lifshitz spacetimes ]**

**Black hole entropy with our boundary conditions :**

**Successfully reproduced from the asymptotic growth of the number of states of a field theory with Lifshitz scaling (same z)**

# A different kind of asymptotic structure

$$A^{\pm} = b_{\pm}^{-1}\left(\mathrm{d} + \mathfrak{a}^{\pm}\right) b_{\pm}$$

radial dependence, fully captured by : $b_{\pm} = \exp\left(\pm \frac{1}{\ell \zeta^{\pm}} L_1\right) \exp\left(\pm \frac{\rho}{2} L_{-1}\right)$

$$\mathfrak{a}^{\pm} = L_0 \left(\pm \mathcal{J}^{\pm}\, \mathrm{d}\varphi + \zeta^{\pm}\, \mathrm{d}v\right)$$

: "diagonal gauge"

$\mathcal{J}^{\pm}, \zeta^{\pm}$ : arbitrary functions of $v$, $\varphi$

$\mathcal{J}^{\pm}$ : dynamical fields

$\zeta^{\pm}$ : Lagrange multipliers (held fixed at the boundary)

Afshar, Detournay, Grumiller, Merbis, Pérez, Tempo, Troncoso, arXiv:1603.04824 [hep-th]

**Connected with proposal about black holes with "soft hair" (soft gravitons) in the sense of :**

Hawking, Perry, Strominger, arXiv:1601.00921, 1611.09175 [hep-th]

# Remarks

- **Soft hairy black holes in 3D :**
  - **Stationary black holes, not necessarily spherically symmetric**
  - **"Black flowers" do not fulfill the Brown-Henneaux boundary conditions**
- **Asymptotic symmetries and global charges :**
  - **Remarkably simple set of asymptotic symmetries**
  - **Two independent affine u(1) currents (with very precise levels !)**
- **Soft hair**
  - **Hamiltonian commutes with the asymptotic symmetry generators**
- **Comparison with Brown-Henneaux**
  - **Virasoro currents turn out to be composite operators of the u(1)'s**

# Remarks

**Highest weight gauge :** $$a^{\pm} = a^{\pm}_{\varphi} d\varphi + a^{\pm}_{t} dt$$

$$a^{\pm}_{\varphi} = L_{\pm 1} - \frac{1}{4}\mathcal{L}_{\pm} L_{\mp 1} \; ; \; a^{\pm}_{t} = \pm \Lambda^{\pm}\left[\mu^{\pm}\right]$$

$$\Lambda^{\pm}\left[\mu^{\pm}\right] = \mu^{\pm}\left(L_{\pm 1} - \frac{1}{4}\mathcal{L}_{\pm} L_{\mp 1}\right) \mp \mu^{\pm\prime} L_0 + \frac{1}{2}\mu^{\pm\prime\prime} L_{\mp 1}$$

---

$\mu^{\pm}$ **depend nonlocally on** $\mathcal{L}_{\pm}$

---

**The choice of boundary conditions can be expressed through:**

$$\mu' - \mathcal{J}\mu = -\zeta$$

: **Nonlocal dependence on the dynamical fields**
$\zeta$ **held fixed at the boundary (without variation)**

$$\mathcal{L} = \frac{1}{2}\mathcal{J}^2 + \mathcal{J}'$$

: **Dynamical field is reexpressed**

# Extended KdV hierarchy

**Generalizations of KdV-type of boundary conditions :**

$$\mu' - \mathcal{J}\mu = -\zeta$$ : Nonlocal dependence on the dynamical fields

$$\mathcal{L} = \frac{1}{2}\mathcal{J}^2 + \mathcal{J}'$$ : Dynamical field is reexpressed

---

**Recovered through forcing the recursion relation (Gelfand-Dikii polynomials) to work backwards !**

$$\partial_\varphi R^{\pm}_{(j+1)} = \mathcal{D}^{\pm} R^{\pm}_{(j)} \qquad \mu^{\pm}_{(k)} = R^{\pm}_{(k)} = \frac{\delta H^{\pm}_{(k)}}{\delta \mathcal{L}_{\pm}}$$

---

**Brown-Henneaux:** $R^{\pm}_{(0)} = \mu^{\pm}_{(0)} = 1$ $\rightarrow$ **KdV:** $R^{\pm}_{(1)} = \mu^{\pm}_{(1)} = \mathcal{L}_{\pm}$ $\rightarrow$ **… (KdV hierarchy)**

---

**…** $\rightarrow$ **KdV:** $R^{\pm}_{(1)} = \mu^{\pm}_{(1)} = \mathcal{L}_{\pm}$ $\rightarrow$ **Brown-Henneaux** $R^{\pm}_{(0)} = \mu^{\pm}_{(0)} = 1$ $\rightarrow$ **( ? )** **Kernel of** $\mathcal{D}^{\pm}$

# Extended KdV hierarchy

**Generalizations of KdV-type of boundary conditions :**

$$\mu' - \mathcal{J}\mu = -\zeta$$

**: Nonlocal dependence on the dynamical fields**

$$\mathcal{L} = \frac{1}{2}\mathcal{J}^2 + \mathcal{J}'$$

**: Dynamical field is reexpressed**

---

**Precisely solve the kernel of $\mathcal{D}^{\pm}$ !**

**[ "Precursor" of the Brown-Henneaux boundary conditions ]**

**Anisotropic scaling with z = 0 ( z = 2k + 1)**

**Labelling of this set as a member of an extended hierarchy with k = - 1/2**

**[ Fractional extensions of the KdV hierarchy have been studied in the literature ]**