Wave Propagation & Shock formation in The most general scalar-tensor theory

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Wave Propagation and Shock formation in The most general scalar-tensor theory

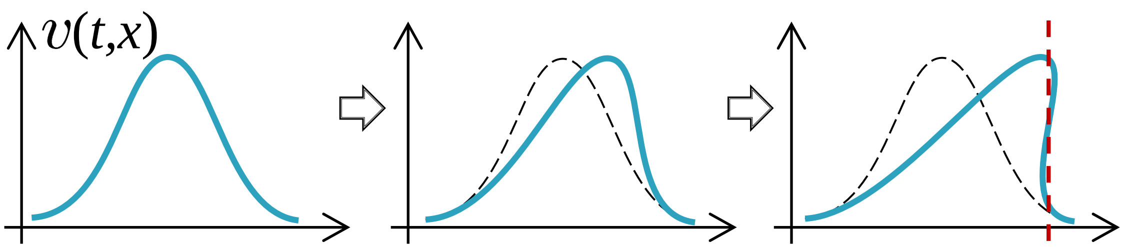
- Scalar & gravitational wave in modified gravity theories

General Relativity: GW propagates at the light speed c

Modified gravity : (GW speed) $\neq c$, environment dependent
Causal structure may be modified

- Waveform distortion & Shock formation

ex.) Burgers' eq. $$\partial_t v + v\,\partial_x v = 0$$



Q: Does this occur for scalar & gravitational waves in modified gravity?

If it occurs, it may be observationally important.

- Study these phenomena in the Horndeski theory.

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1. Introduction
   i. Horndeski theory
   ii. Characteristic surface
2. Wave propagation in Horndeski theory
3. Shock formation in Horndeski theory
4. Summary

Horndeski theory

- Horndeski theory [Horndeski 1974]
  - Scalar field ϕ & gravity in 4-dim. spacetime
  - The most general covariant theory with 2nd-order EoM

$$\mathcal{L} = K(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]$$
$$+ G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]$$

$$\left(X \equiv -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi\right)$$

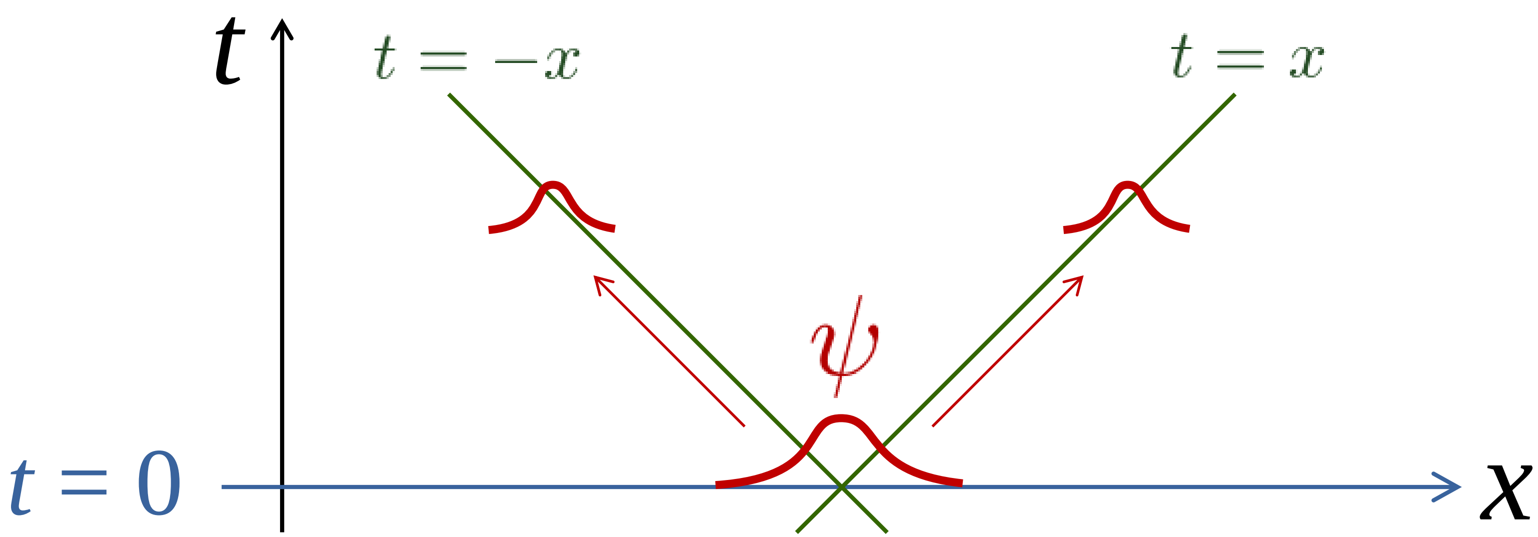
  - 4 arbitrary functions in Lagrangian
  - Applications to cosmology (e.g. inflation, DE, BH…)

Wave propagation surface = Characteristic surface

- Massless scalar in flat space

$$0 = g^{\mu\nu}\partial_\mu\partial_\nu\psi = \left(-\partial_t^2 + \partial_x^2\right)\psi$$

$$\Rightarrow \quad \psi = f_1(t-x) + f_2(t+x)$$

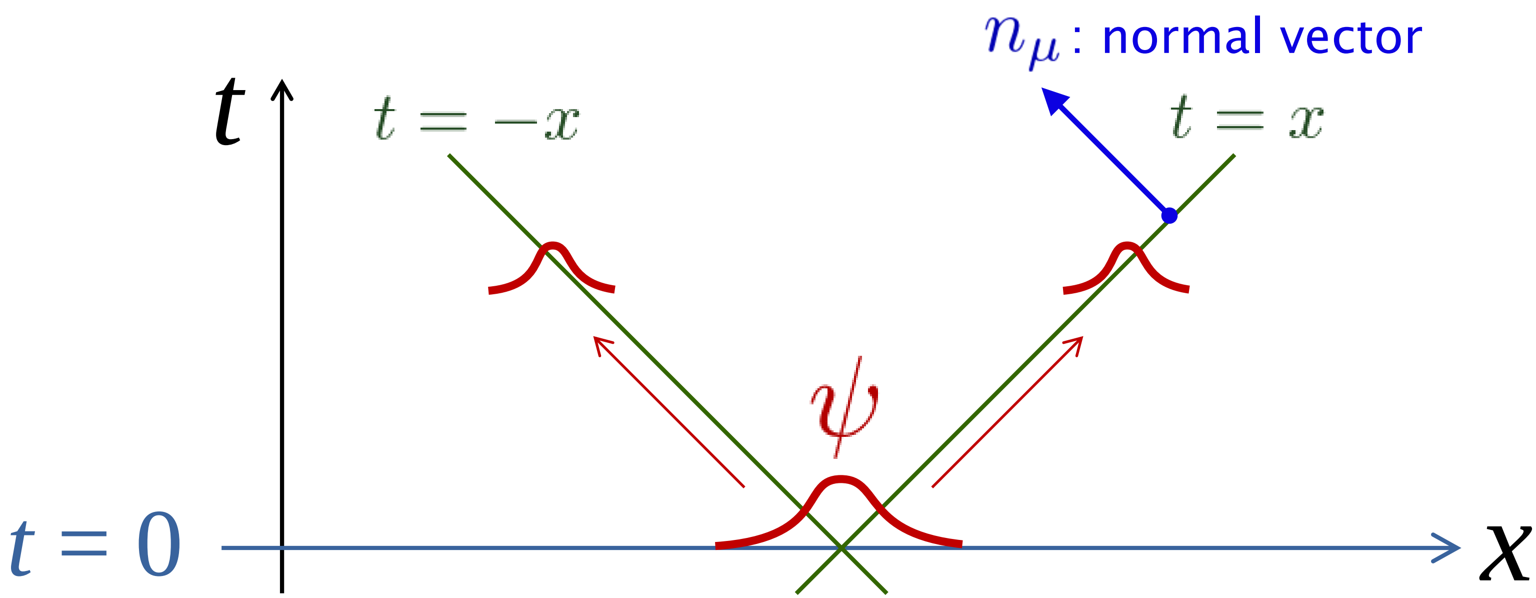


Wave propagation surface = Characteristic surface

- Massless scalar in flat space $\left(\psi = e^{i n_\mu x^\mu} \tilde{\psi}_k\right)$

$$0 = g^{\mu\nu}\partial_\mu\partial_\nu\psi = -g^{\mu\nu} n_\mu n_\nu e^{i n_\mu x^\mu} \tilde{\psi}_k$$

$$\Rightarrow \quad \psi = f_1(t-x) + f_2(t+x) \;\Leftrightarrow\; \boxed{g^{\mu\nu} n_\mu n_\nu = 0}$$



Wave propagation surface = Characteristic surface

- General EoM $E(v, \partial v, \partial^2 v) = 0$ for variable v :

$$0 = E\left(v, \partial v, \partial^2 v\right) = \frac{\partial E}{\partial\left(\partial_\mu \partial_\nu v\right)} \partial_\mu \partial_\nu v + \cdots$$
$$= \frac{\partial E}{\partial\left(\partial_\mu \partial_\nu v\right)} n_\mu n_\nu v + \cdots$$

- Characteristic equation for n^μ

$$\boxed{\frac{\partial E}{\partial\left(\partial_\mu \partial_\nu v\right)}} n_\mu n_\nu = 0$$

"effective metric"

- Surface w/ normal n_μ = Characteristic surface

  Physically, a characteristic surface is a wave propagation surface.

Wave propagation surface = Characteristic surface

- EoM of scalar-tensor theory

$$\begin{cases} 0 = E_{ab} = \dfrac{\partial E_{ab}}{\partial(\partial_\mu\partial_\nu g_{cd})}\partial_\mu\partial_\nu g_{cd} + \dfrac{\partial E_{ab}}{\partial(\partial_\mu\partial_\nu \phi_J)}\partial_\mu\partial_\nu \phi_J + \cdots \\ 0 = \ E_I \ = \dfrac{\partial E_I}{\partial(\partial_\mu\partial_\nu g_{cd})}\partial_\mu\partial_\nu g_{cd} + \dfrac{\partial E_I}{\partial(\partial_\mu\partial_\nu \phi_J)}\partial_\mu\partial_\nu \phi_J + \cdots \end{cases}$$

Write them collectively as

$$0 = E = \begin{pmatrix} \frac{\partial E_{ab}}{\partial(\partial_\mu\partial_\nu g_{cd})} & \frac{\partial E_{ab}}{\partial(\partial_\mu\partial_\nu \phi_J)} \\ \frac{\partial E_I}{\partial(\partial_\mu\partial_\nu g_{cd})} & \frac{\partial E_I}{\partial(\partial_\mu\partial_\nu \phi_J)} \end{pmatrix} n_\mu n_\nu \begin{pmatrix} \tilde{g}_{cd} \\ \tilde{\phi}_J \end{pmatrix} + \cdots$$

$$= P(n_\mu) \cdot r + \cdots$$

- Characteristic equation for n^μ

propagating mode

$$\det P(n_\mu) = 0 \iff P(n_\mu) \cdot r = 0 \quad \text{with} \quad r = \begin{pmatrix} \tilde{g}_{cd} \\ \tilde{\phi}_J \end{pmatrix} \neq 0$$

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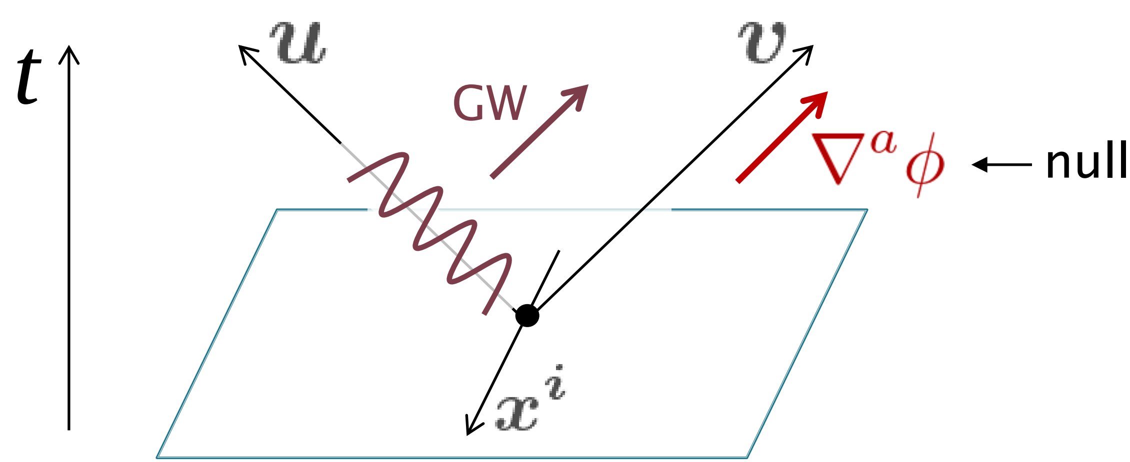
Wave Propagation on Plane wave background

- Background = Plane wave solution in Horndeski [Babichev 2012]

$$ds^2 = a_{ij}x^i x^j du^2 + 2dudv + \delta_{ij}dx^i dx^j, \quad \phi = \phi(u)$$



- Wave of perturbations on this background

Wave propagation surface with normal n_a $\Longleftrightarrow$ Find n_a satisfying $P(n_a) \cdot r = 0$ with $r = (r_{ab}, r_\phi) \neq 0$

Characteristic eq. on the plane wave background

$$P\cdot r = \begin{pmatrix} (P\cdot r)_{ij} \\ (P\cdot r)_{\phi} \end{pmatrix} = \begin{pmatrix} G^{ab}_{\mathrm{GW}} n_a n_b \, \hat{r}_{ij} \\ G^{ab}_{\phi} \, n_a n_b \, r_{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where G^{ab}_{GW}, G^{ab}_{ϕ} are

$$\begin{cases} G^{ab}_{\mathrm{GW}} = g^{ab} + \omega \, \nabla^a \phi \nabla^b \phi \quad \text{with} \quad \omega = \dfrac{G_{4X}}{G_4} \\[2ex] G^{ab}_{\phi} = g^{ab} + \omega \, \nabla^a \phi \nabla^b \phi \quad \text{with} \quad \omega = -\dfrac{K_{XX}}{K_X} - \dfrac{G_{4X}}{G_4} + \dfrac{2G_{3X}}{K_X} \dfrac{\phi''}{\phi'^2} \end{cases}$$

- ✓ Wave propagation surface is null w.r.t. "effective metric": $G^{ab} n_a n_b = 0$
- ✓ $G^{ab}_{\mathrm{GW}} n_a n_b = r_{\phi} = 0 \quad \Rightarrow \quad r = (\hat{r}_{ij}, 0)$: Gravitational wave
- ✓ $G^{ab}_{\phi} \, n_a n_b = \hat{r}_{ij} = 0 \quad \Rightarrow \quad r = (0, r_{\phi})$: Scalar field wave

- Characteristic eq. on the plane wave background

$$P\cdot r = \begin{pmatrix} (P\cdot r)_{ij} \\ (P\cdot r)_\phi \end{pmatrix} = \begin{pmatrix} G^{ab}_{\mathrm{GW}} n_a n_b \, \hat{r}_{ij} \\ G^{ab}_{\phi} \, n_a n_b \, r_\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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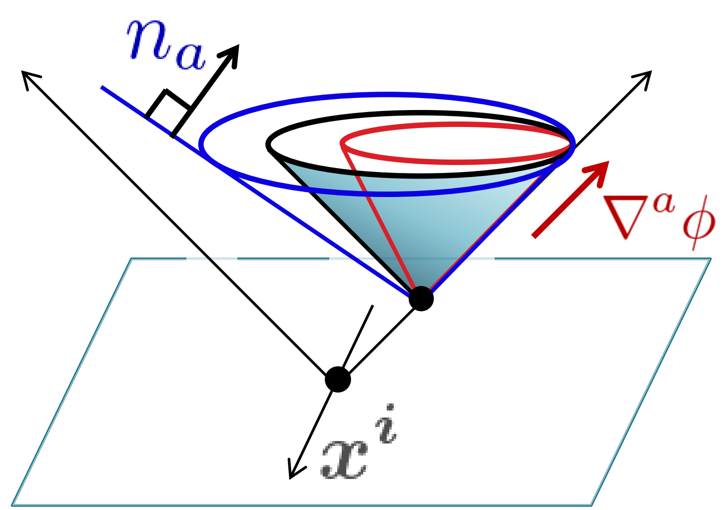
✓ Nested characteristic cones:
All cones // $\nabla^a\phi$

✓ Causality w.r.t. the largest cone:
larger ω $\Leftrightarrow$ larger cone



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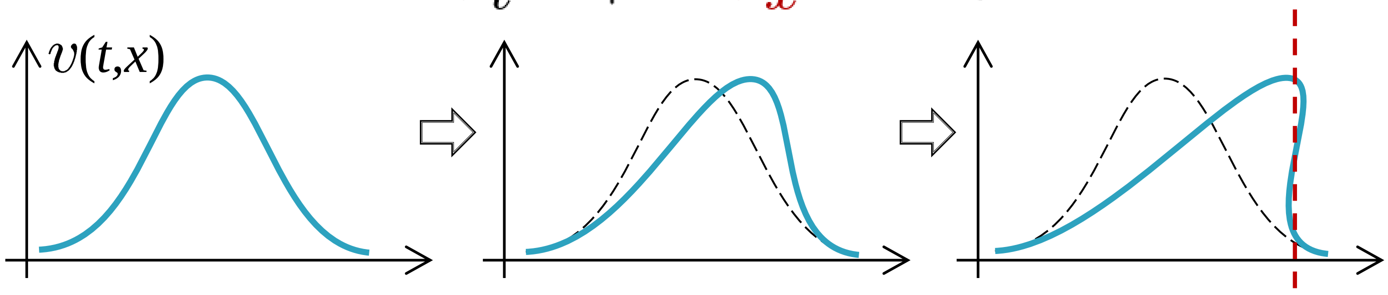
Shock formation in Horndeski theory

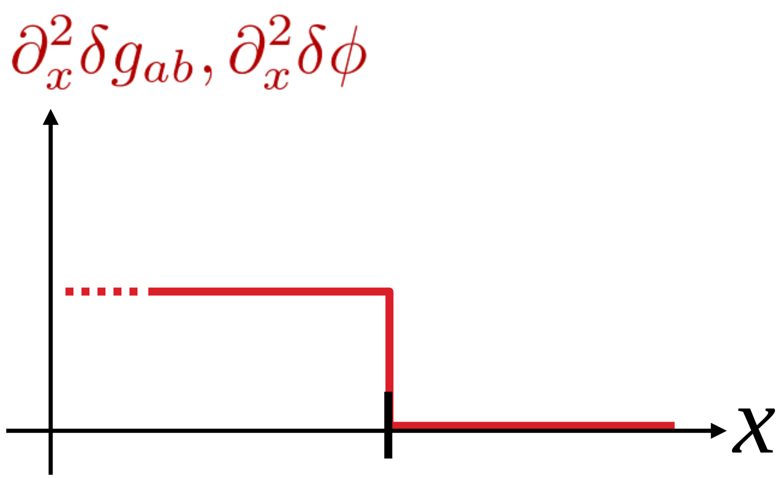
- Shock formation = Divergence in gradient of waveform

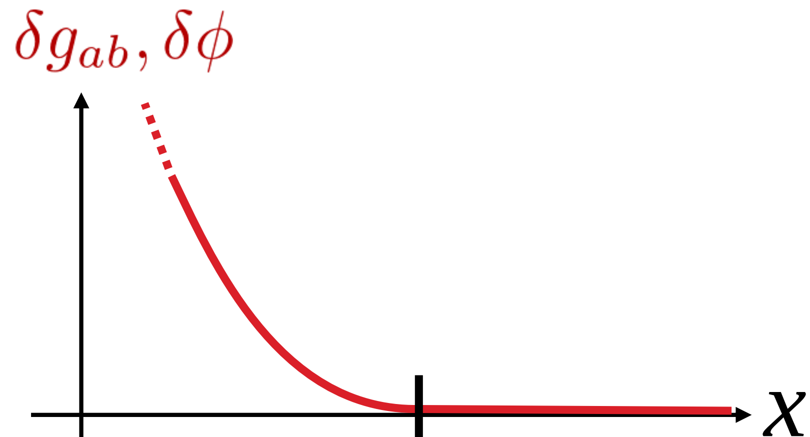
ex.) Burgers' eq. $\partial_t v + v\,\partial_x v = 0$



- For simplicity, we look at wave with discontinuity in second derivative:

(weak discontinuity)





Shock formation in Horndeski theory

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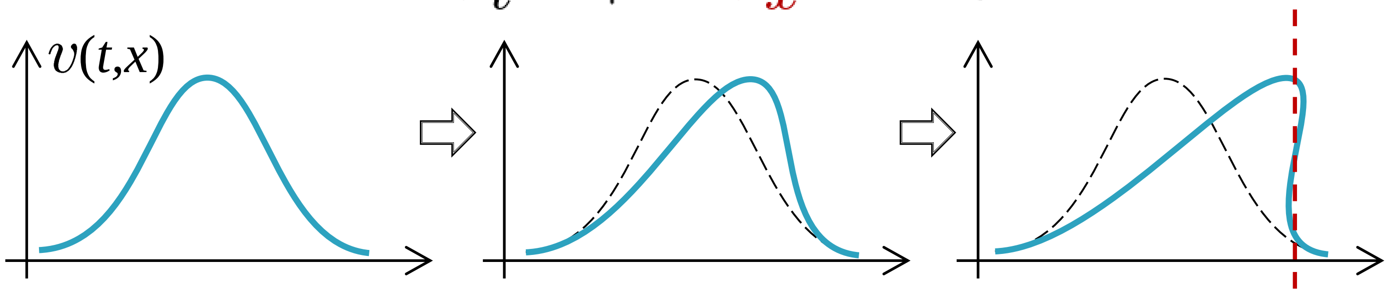

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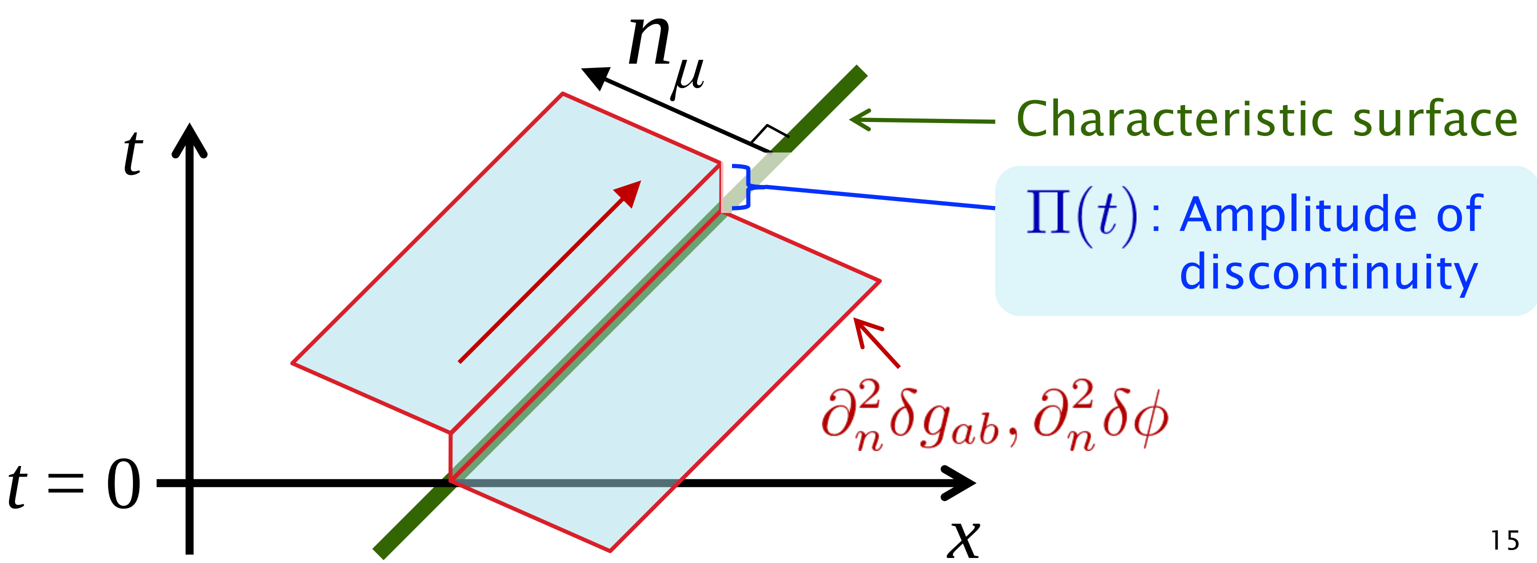


1. EoM: $\begin{cases} E_{ab} = \dfrac{\partial E_{ab}}{\partial(\partial_n^2 g_{cd})}\partial_n^2 g_{cd} + \dfrac{\partial E_{ab}}{\partial(\partial_n^2 \phi_J)}\partial_n^2 \phi_J + \cdots = 0 \\ E_I = \dfrac{\partial E_I}{\partial(\partial_n^2 g_{cd})}\partial_n^2 g_{cd} + \dfrac{\partial E_I}{\partial(\partial_n^2 \phi_J)}\partial_n^2 \phi_J + \cdots = 0 \end{cases}$

$$P \equiv \begin{pmatrix} \dfrac{\partial E_{ab}}{\partial g_{cd,nn}} & \dfrac{\partial E_{ab}}{\partial \phi_{J,n}} \\ \dfrac{\partial E_I}{\partial g_{cd,nn}} & \dfrac{\partial E_I}{\partial \phi_{J,n}} \end{pmatrix}$$

Write them collectively as

$$E_a = P_a{}^b \, \partial_n^2 v_b + \cdots = 0$$

2. Take discontinuous part

$$[E_a] = P_a{}^b \left[\partial_n^2 v_b\right] = 0 \quad \Rightarrow \quad \left[\partial_n^2 v_b\right] = \Pi(t)\, r_b$$

(discontinuous part)

For r_b s.t. $P \cdot r = 0$

3. Transport equation of amplitude $\Pi(t)$

$$[\,\partial_n E_a\,] = 0 \quad \Rightarrow \quad \boxed{\dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0}$$

where

$$N = \frac{\partial P^{ab}}{\partial\left(\partial_n v_c\right)} r_a r_b r_c$$

What happens when $N \neq 0$?

$$\dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$$

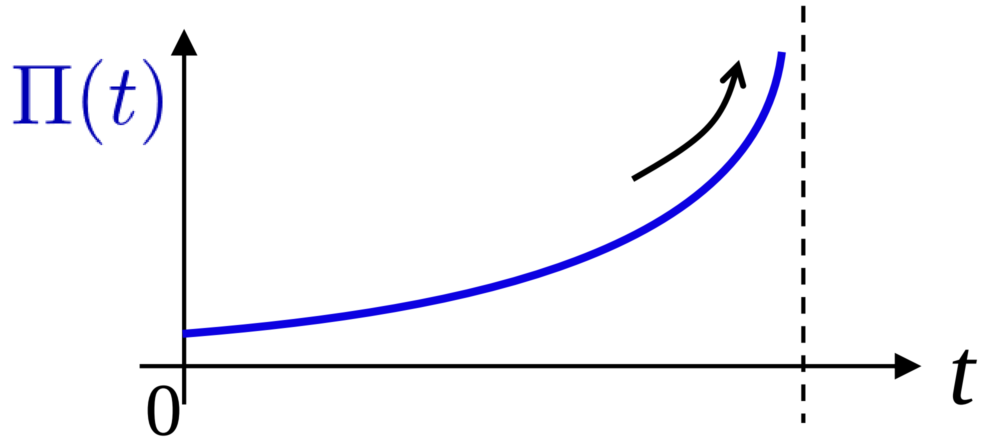
$$\left(\Phi(t) = \int_0^t M(t')dt'\right)$$

$$\Rightarrow \quad \Pi(t) = \frac{\Pi(0)e^{-\Phi(t)}}{1 + \Pi(0)\int_0^t N(t')e^{-\Phi(t')}dt'} \sim \frac{\Pi(0)}{1 + \Pi(0)\,N\,t}$$

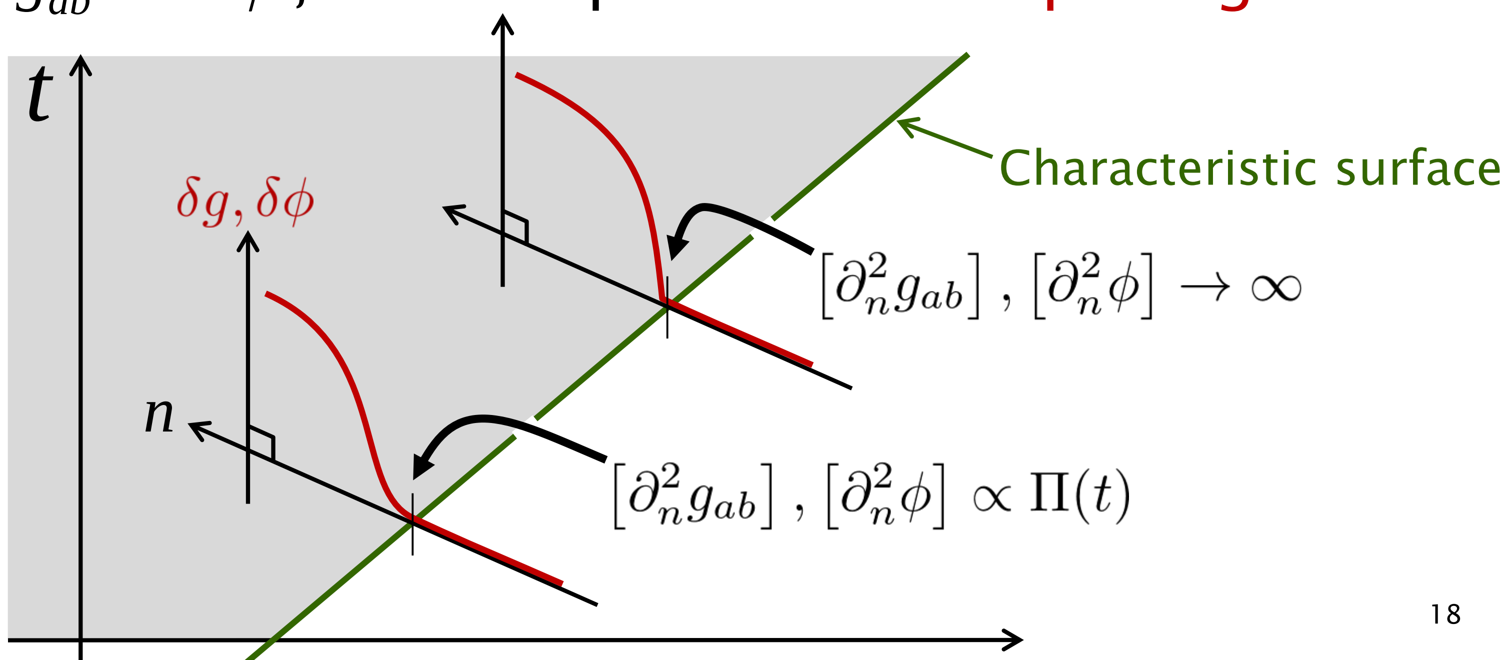
- GR : $N = 0 \quad \Rightarrow \quad \Pi(t)$ stays finite
- Modified grav : $N \neq 0 \quad \Rightarrow$ Denominator may vanish at $t \sim -1/\Pi(0)N$

  $\Rightarrow$ Amplitude $\Pi(t)$ diverges "Shock formation"

- Amplitude of discontinuity: $\Pi(t) = \left[\partial_n^2 g_{ab}\right], \left[\partial_n^2 \phi\right]$
- Our "shock formation" = 2nd derivative $\Pi(t) \to \infty$ at finite t

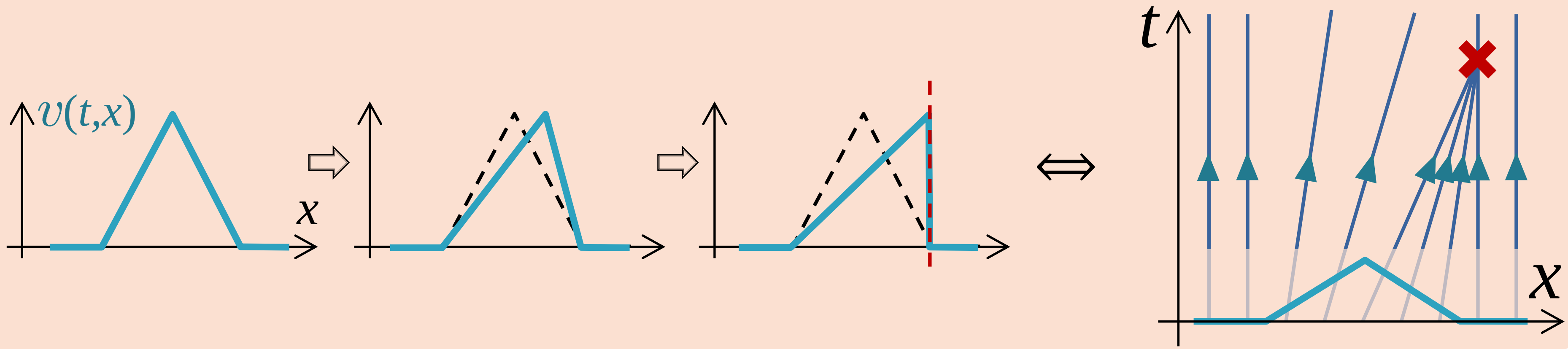


- In terms of g_{ab} and ϕ, it corresponds to "sharpening":



Shock formation caused by variable sound speed

Burgers' eq. $\partial_t v + v\,\partial_x v = 0 \;\Rightarrow$ propagation speed $c(v) \propto v$



In our case,

$\det P\left(n_\mu\right) = 0 \quad \Rightarrow$ propagation speed $c(v, \partial v)$

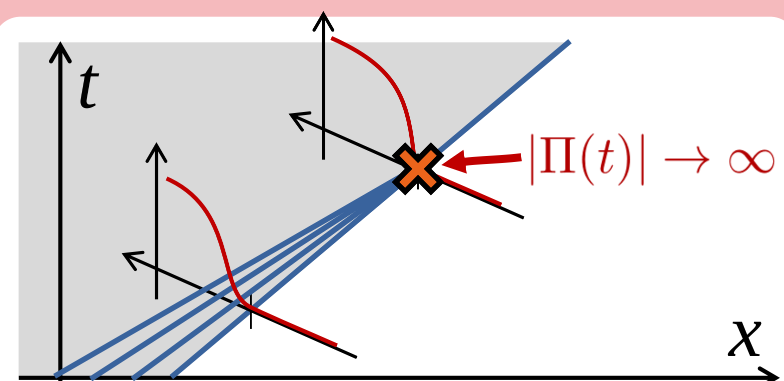
$\dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$ where $N = \dfrac{\partial P^{ab}}{\partial\left(\partial_n v_c\right)} r_a r_b r_c \sim \dfrac{\partial c(v,\partial v)}{\partial(\partial v)}$

$N \sim \dfrac{\partial c(v,\partial v)}{\partial(\partial v)} \neq 0$

$\Updownarrow$

$c(v, \partial v)$ depends on amplitude

$\Rightarrow$

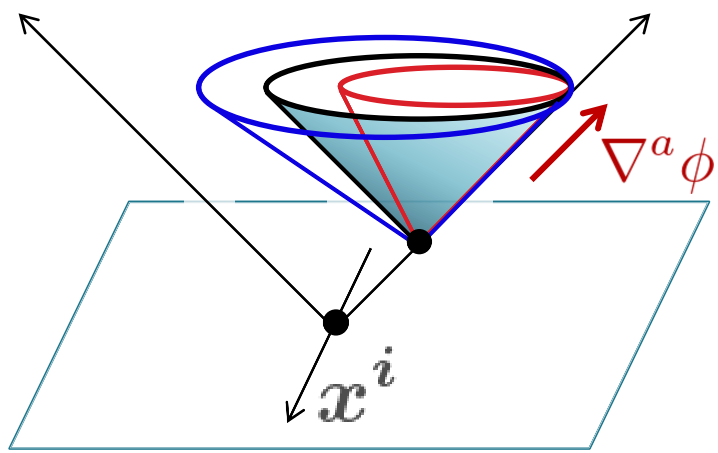


Shock formation on Plane wave background

- Example: Wave on Plane wave solution

$$ds^2 = a_{ij}x^i x^j du^2 + 2dudv + \delta_{ij}dx^i dx^j, \quad \phi = \phi(u)$$

[Babichev 2012]



✓ Scalar field & Gravitational wave propagate at different speeds

✓ Scalar : $N \neq 0$ → Shock formation

[Babichev 2016]
[Mukohyama, Namba, Watanabe 2016]
[de Rham, Motohashi 2016]

✓ GW : $N = 0$ → No shock formation

Shock formation on Plane wave background

$$N = C_+ f_+(t) + C_- f_-(t) + C_0$$

$$C_+ = \phi'^2 \left\{ -\frac{1}{2G_4}(2G_{3X}G_{4X} + K_X G_{5X}) + \frac{2K_{XX}G_{3X}}{K_X} - G_{3XX} \right\}$$

$$C_- = -2a\, G_{5X}$$

$$C_0 = \phi'^3 \left\{ \frac{3}{G_4}\left(-{G_{3X}}^2 - K_X G_{4XX} + K_{XX}G_{4X}\right) + \frac{3{K_{XX}}^2}{K_X} - K_{XXX} \right\}$$

$f_\pm(t)$: functions of background metric & ϕ

- ✓ $N \neq 0$ in general → shock formation for scalar mode
- ✓ $N = 0$ identically for scalar DBI model → shock formation suppressed

$$\left[K = \sqrt{1 + (\nabla\phi)^2} \right]$$

- ✓ Scalar field & Gravitational wave propagate at different speeds
- ✓ Scalar : $N \neq 0$ → Shock formation
- ✓ GW : $N = 0$ → No shock formation

[Babichev 2016]
[Mukohyama, Namba, Watanabe 2016]
[de Rham, Motohashi 2016]

Summary

- Wave propagation & Shock formation in Horndeski theory

  - Result 1:

    On the plane wave solution in Horndeski theory,

    wave propagation obeys the effective metric,

    and the causality can be defined from them.

  - Result 2:

    For perturbations on the plane wave background in Horndeski,

    - Shock formation occurs for scalar field wave
    - Shock formation does not occur for gravitational wave

✓ Shock formation DOES OCCUR for gravitational wave in Gauss-Bonnet gravity in higher dimensions.

[Tomimatsu-Ishihara 1987]
[Choquet-Bruhat 1989]
[Reall, NT, Way 2015]

$$\mathcal{L} = \mathcal{L}_{\mathrm{GR}} + \boxed{R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}}$$

(curvature)^2 term crucial for GW shock?

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    On the plane wave solution in Horndeski theory,

    wave propagation obeys the effective metric,

    and the causality can be defined from them.
  - Result 2:

    For perturbations on the plane wave background in Horndeski,
    - Shock formation occurs for scalar field wave
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- More complicated background in Horndeski
- More general modified gravity theories

Shock formation occurs even for gravitational wave?