Black Strings in Large Dimension Limit

Amruta Sadhu

IISER Pune

A Sadhu, V Suneeta, *Non-spherically symmetric black string perturbations in the large D limit*, Phys Rev D 93, 124002 (2016).

Motivation

- Black strings/ branes are objects with extended singularities. In our case, the black string metric is obtained by adding an extra dimension to the Schwarzschild-Tangherlini black holes.
- Perturbation equations of Schwarzschild black holes have been studied for years. But the addition of extra dimensions as in black strings/branes makes the perturbation equations considerably more complex.
- Spherically symmetric perturbations of black strings have been studied extensively. Particularly the Gregory-Laflamme instability and its end point.
- However so far the the non-spherically symmetric perturbations of black strings were intractable. In this talk I will present these perturbations in simplest form to date.
- We will analyse them in large dimension limit of General Relativity.

Obtaining The Equations-I

- The black string metric is $D = n+3$ dimensional, obtained by adding a flat extra dimension to an $n+2$ dimensional Schwarzschild-Tangherlini metric.

$$g_{MN}dx^M dx^N = -\left(1 - \frac{b^{n-1}}{r^{n-1}}\right) dt^2 + \left(1 - \frac{b^{n-1}}{r^{n-1}}\right)^{-1} dr^2 + r^2 d\Omega_n^2 + dz^2; \tag{1}$$

- Where $d\Omega_n^2 = \gamma_{ij} d\tilde{y}^i d\tilde{y}^j$ is the metric of a n-dimensional sphere of unit radius. We shall denote $f(r) = \left(1 - \frac{b^{n-1}}{r^{n-1}}\right)$
- The perturbed metric is $\bar{g}_{AB} = g_{AB} + \bar{h}_{AB}$. The linearised Einstein equation for the black string perturbations is

$$\delta R_{MN} = 0 \tag{2}$$

Obtaining The Equations-II

- We make the gauge choice $\bar{h}_{Mz} = 0$ and make the ansatz $\bar{h}_{\mu\nu}(y,z) = e^{i\lambda z} h_{\mu\nu}(y)$. The perturbation $h_{\mu\nu}$ is only on the black hole part of the metric.
- The gauge choice we made earlier implies that $h_{\mu\nu}$ is transverse and traceless. Putting this in (2), we reduce this equation to

$$\delta G_{\mu\nu} = -\frac{1}{2}\lambda^2 h_{\mu\nu}; \tag{3}$$

  where $\delta G_{\mu\nu}$ is the first variation of the Einstein tensor on the Schwarzschild-Tangherlini background.
- We use the variables introduced by Ishibashi and Kodama for studying black hole perturbations to analyse our black string perturbation equations.

Vector Perturbation variables

- The indices a, b denote the (r, t) part of the metric and i, j are indices on the n-sphere.
- We choose

$$h_{ab} = 0 \qquad h_{ai} = r f_a^{vector} V_i \qquad h_{ij} = 2r^2 H_T^{vector} V_{ij}.$$

- Here $f_a^{vector}, H_T^{vector}$ are functions of r, t. Vector harmonics V_i and V_{ij} are defined by

$$(\hat{\Delta} + k_v^2) V_i = 0, \hat{D}_i V^i = 0 \tag{4}$$

$$V_{ij} = -\frac{1}{2k_v}(\hat{D}_i V_j + \hat{D}_j V_i). \tag{5}$$

$k_v^2 = \ell(\ell + n - 1) - 1$ and $\ell = 2, \ldots.$ We denote $\hat{\Delta} = \gamma^{ij} \hat{D}_i \hat{D}_j$.

- We now form a new variable

$$F_a = f_a^{vector} + \frac{r}{k_v} D_a H_T^{vector} \tag{6}$$

Scalar Perturbation variables

- We choose

$$h_{ab} = f_{ab} S \qquad h_{ai} = r f_a S_i \qquad h_{ij} = 2r^2 (H_L \gamma_{ij} S + H_T S_{ij})$$

- S, S_i and S_{ij} are scalar harmonics satisfying

$$(\hat{\Delta} + k^2) S = 0 \quad S_i = -\frac{1}{k} \hat{D}_i S \quad S_{ij} = \frac{1}{k^2} \hat{D}_i \hat{D}_j S + \frac{1}{n} \gamma_{ij} S \quad S_i^i = 0$$

$k^2 = \ell(\ell + n - 1)$ and $\ell = 0, 1, 2....$

- To construct gauge invariant variables for scalar perturbations (not defined for $\ell = 0$ and $\ell = 1$),first we define

$$X_a = \frac{r}{k} \left(f_a + \frac{r}{k} D_a H_T \right) \tag{7}$$

- In terms of X_a, the new variables are

$$F_{ab} = f_{ab} + D_a X_b + D_b X_a \tag{8}$$

$$F = H_L + \frac{1}{n} H_T + \frac{1}{r} D^a X_a \tag{9}$$

Vector Perturbation Equations

- We write the equations $\delta G_{ai} = -\frac{1}{2}\lambda^2 h_{ai}$ and $\delta G_{ij} = -\frac{1}{2}\lambda^2 h_{ij}$ in terms of the IK variables. Combining the two equations, we get

$$\Box F_a - D^b D_a F_b + D_a D^b F_b + n\frac{D^b r D_b F_a}{r} - 2\frac{D^b r D_a F_b}{r}$$
$$- \frac{\Box r}{r} F_a - n\frac{(Dr)^2}{r^2} F_a - (n-2)\frac{D^b r D_a r}{r^2} F_b + \frac{D^b D_a r}{r} F_b$$
$$+ (n-1)\frac{D_a D^b r}{r} F_b - \frac{\alpha}{r^2} F_a = \lambda^2 F_a \tag{10}$$

  where $\alpha = k_v^2 - (n-1)$.

- Explicitly evaluating the covariant derivatives, we get a system of coupled partial differential equations for F_r and F_t. We then do a modal decomposition of F_t and F_r.

$$F_t = A(r)e^{i\omega t} = A(r)e^{\Omega t} \qquad F_r = \frac{B(r)}{f(r)}e^{i\omega t} = \frac{B(r)}{f(r)}e^{\Omega t} \tag{11}$$

Vector Perturbation Equations II

- The resulting equations for $A(r)$ and $B(r)$ are:

$$\frac{d^2A}{dr^2}+\frac{n}{r}\frac{dA}{dr}+\left(-\frac{n}{r^2}-\frac{\alpha}{fr^2}-\frac{\lambda^2}{f}+\frac{\omega^2}{f^2}\right)A = \left(\frac{2}{rf}-\frac{(n-1)b^{n-1}}{f^2r^n}\right)i\omega B \qquad (12)$$

$$\frac{d^2B}{dr^2}+\frac{(n-2)}{r}\frac{dB}{dr}+\left(-\frac{2(n-1)}{r^2}-\frac{\alpha}{fr^2}-\frac{\lambda^2}{f}+\frac{\omega^2}{f^2}\right)B = -\frac{(n-1)b^{n-1}}{f^2r^n}i\omega A \qquad (13)$$

- For discussions on stability, we need to investigate if there are normalizable solutions to the set of coupled equations that are regular at the horizon with Ω positive.
- In order to analyse these equations, we have to resort to large n limit, where n is the number of dimensions the sphere part of the metric (1). This limit was employed to study higher dimensional black holes by Emparan et. al, Kol et. al.

The large n limit

- In the large n limit, the function $f(r) = \left(1 - \frac{b^{n-1}}{r^{n-1}}\right)$ increases steeply from zero in the interval $b < r < b + \frac{b}{n}$ and is almost constant for $r > \frac{b}{n}$.
- This step-like behaviour of $f(r)$ divides the spacetime in two different regions.
- We define a near-horizon region and far region as:

$$\text{Near region} \quad r - b \ll b, \qquad \text{Far region} \quad r - b \gg \frac{b}{n-1}$$

- The two regions overlap in $\frac{b}{n-1} \ll r - b \ll b$.
- We define a new coordinate $R = (\frac{r}{b})^{n-1}$. In the near-region approximation, r can be written in terms of R as

$$r \sim b\left[1 + \frac{\ln R}{n-1}\right] \tag{14}$$

The large *n* limit

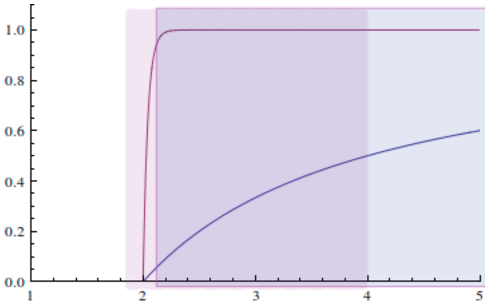

Figure: f(r) for n=2 and n=50

Equations in Large n

- We use the notation $k_v^2/n^2 = \hat{k_v}^2$, $\lambda^2/n^2 = \hat{\lambda}^2$, $i\omega = \Omega$ and $\Omega^2/n^2 = \hat{\Omega}^2$.
- We expand A and B as,

$$A = \sum_{i \geq 0} \frac{A_i}{n^i} \qquad B = \sum_{i \geq 0} \frac{B_i}{n^i} \tag{15}$$

- The leading order near region equations then become,

$$\frac{d^2A}{dR^2} + \frac{2}{R}\frac{dA}{dR} - \left[\frac{\hat{k}_v^2}{R(R-1)} + \frac{\hat{\lambda}^2 b^2}{R(R-1)} + \frac{\hat{\Omega}^2 b^2}{(R-1)^2}\right] A = -\frac{\hat{\Omega} b B}{R(R-1)^2} \tag{16}$$

$$\frac{d^2B}{dR^2} + \frac{2}{R}\frac{dB}{dR} - \left[\frac{\hat{k}_v^2}{R(R-1)} + \frac{\hat{\lambda}^2 b^2}{R(R-1)} + \frac{\hat{\Omega}^2 b^2}{(R-1)^2}\right] B = -\frac{\hat{\Omega} b A}{R(R-1)^2} \tag{17}$$

Far Region Equations

- The far region is defined by $r \gg b + \frac{b}{n}$. Therefore, in this limit $f \to 1$ as $(b^{n-1}/r^{n-1}) \sim e^{-n \ln r}$ is a small quantity for large n and large r.
- The equations in the far region hence can be obtained by setting the terms with $f'(r)$ to zero.
- We shall assume A and B to be of the similar r behaviour. The far region equations in this case are,

$$\frac{d^2A}{dr^2} + \frac{n}{r}\frac{dA}{dr} + \left(-\frac{k_v^2}{r^2} - \lambda^2 - \Omega^2\right) A = \left(\frac{2}{r}\right) \Omega B \tag{18}$$

$$\frac{d^2B}{dr^2} + \frac{n}{r}\frac{dB}{dr} + \left(-\frac{k_v^2}{r^2} - \lambda^2 - \Omega^2\right) B = 0 \tag{19}$$

Stability of Vector Perturbations

- For the desired range of parameters $\Omega \geq 0$ and $\ell > 2$, the solutions from near and far region do not match.
- This implies that there are no normalisable solutions for A and B that are growing in time.
- Here Ω, λ and ℓ are at least $\mathcal{O}(n)$, but this result is true when the parameters are of $\mathcal{O}(1)$.
- The analysis of scalar perturbations also shows that there are no normalisable solutions for $\Omega \geq 0$.
- In particular, in the static limit, with $\Omega = 0$, we have no instability. This is proof that the Gross-Perry-Yaffe mode for semiclassical black hole perturbations is the unique unstable mode even in the large dimension limit.

Scalar Perturbation Equations

- From the components of the IK variables F_{ab} and F, we construct new variables

$$W = r^{n-2}(F^t_t - 2F) \qquad Y = r^{n-2}(F^r_r - 2F) \qquad Z = r^{n-2}F^r_t. \tag{20}$$

- Traceless part of the eigenvalue equation $2\delta G_{ij} = -\lambda^2 h_{ij}$ give us relation between these variables.

$$W + Y + 2nF = 2\lambda^2 \frac{r^2}{k^2} H_T \tag{21}$$

- Using this relation we write the four IK variables $(F_{rr}, F_{rt}, F_{tt}, F)$ in terms of (W, Y, Z, H_T).
- Substituting our new variables in the eigenvalue equations $2\delta G_{\mu\nu} = -\lambda^2 h_{\mu\nu}$, we obtain six equations. But these equations have terms containing H_T.
- We can eliminate the unwanted H_T factors by further combining the six equations. The three final equations are written completely in terms of W, Y and Z.

Scalar Perturbation Equations II

- Our later computations are made simpler by the further change of variables:

$$\hat{\psi} = \frac{f^{1/2}}{r^{(n-4)/2}} W \qquad \hat{\phi} = \frac{f^{1/2}}{r^{n/2}} Y \qquad \hat{\eta} = \frac{1}{r^{(n-2)/2} f^{1/2}} Z. \tag{22}$$

- We can assume a time dependence of the form $\hat{\psi}(r,t) = e^{i\omega t}\psi(r)$ for all three variables. Finally, the three coupled perturbation equations are:

$$-\frac{d^2\psi}{dr^2} + \left[\frac{n^3 - 2n^2 + 8n - 8}{4nr^2} + \frac{f'^2}{4f^2} - \frac{(n^2+2n-4)}{2n}\frac{f'}{fr} - \frac{f''}{2f} - \frac{2(n-1)}{nr^2 f} + \frac{k^2}{fr^2} + \frac{\lambda^2}{f} - \frac{\omega^2}{f^2}\right]\psi = \left[\frac{4}{f} - \frac{2f'r}{f^2}\right](i\omega)\eta + \left[\frac{2(n-1)}{nf} + \frac{2}{n} - \frac{n+2}{n}\frac{rf'}{f} - \frac{r^2 f''}{f} + \frac{f'^2 r^2}{2f^2}\right]\phi \tag{23}$$

Scalar Perturbation Equations III

$$
\begin{aligned}
&-\frac{d^2\phi}{dr^2}+\Bigg[\frac{n^3-2n^2+8n-8}{4nr^2}+\frac{f'^2}{4f^2}-\frac{(n^2+2n-4)}{2n}\frac{f'}{fr}-\frac{f''}{2f}\\
&-\frac{2(n-1)}{nr^2f}+\frac{k^2}{fr^2}+\frac{\lambda^2}{f}-\frac{\omega^2}{f^2}\Bigg]\phi=\\
&\frac{2f'}{f^2r}\eta(i\omega)+\left[\frac{2(n-1)}{nr^4f}-\frac{2(n-1)}{nr^4}-\frac{2-n}{nr^3}\frac{f'}{f}-\frac{f''}{r^2f}+\frac{f'^2}{2f^2r^2}\right]\psi \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{d^2\eta}{dr^2}+\left[\frac{n^2-2n}{4r^2}-\frac{(n+2)f'}{2rf}+\frac{3f'^2}{4f^2}-\frac{3f''}{2f}+\frac{k^2}{fr^2}-\frac{\omega^2}{f^2}+\frac{\lambda^2}{f}\right]\eta\\
=\left[\frac{f'}{f}-\frac{2}{r}\right]\frac{r(i\omega)}{f}\phi-\frac{f'}{f^2}\frac{(i\omega)}{r}\psi \qquad (25)
\end{aligned}
$$

Quasinormal Modes in Large n

- We can find the quasinormal modes of flat black strings/branes using the perturbation equation obtained for vector and scalar case. We can systematically add corrections of the order $1/n$ to the find the modes at all orders.
- In the large n limit, depending on the n-dependence of λ, ℓ and Ω we get different classes of quasinormal modes. Most notable among them are the modes having all the parameters of $\mathcal{O}(1)$ called the decoupled modes and the modes for which all the parameters are of $\mathcal{O}(n)$ called the non-decoupled modes.
- Decoupled modes are localized in the near region where $f(r)$ is steeply increasing. In the far region these modes decay rapidly.
- The non-decoupled modes obey quasinormal mode boundary conditions at both horizon and infinity.
- Analysis of the decoupled modes for black holes in large n has been done in many papers by various authors.
- For $\lambda = 0$, our results so far have reproduced the old results in decoupled sector and we have managed to get the leading order answer in the non-decoupled case.

Thank you

References

- A Sadhu, V Suneeta, *Non-spherically symmetric black string perturbations in the large D limit*, Phys Rev D 93, 124002 (2016).
- R Gregory, R Laflamme, *Black strings and p-branes are unstable*, Phys Rev Lett 70:2837-2840 (1993).
- A Ishibashi and H Kodama, *Perturbations and stability of black holes in higher dimensions* Prog. Theor. Phys. Supplement (2011) 189 165-209.
- R Emparan, R Suzuki, K Tanabe, *The large D limit of General Relativity*, JHEP06 009 (2013).

Near Region Solutions

- The expansion of A and B in (15) implies that the leading order n behaviour of the two variables has to be equal.
- We define

$$\xi = (R-1)^{-\hat{\Omega} b}(A+B) \qquad \zeta = (R-1)^{\hat{\Omega} b}(A-B)$$

- The equations for ξ and ζ are decoupled and can be written as hypergeometric equations.
- The solution for ξ is,

$$\xi = C_1 F(p, q, 2\hat{\Omega} b; 1-R) + C_2 (R-1)^{1-2\hat{\Omega} b} F(2-p, 2-q, 2-2\hat{\Omega} b; 1-R); \tag{26}$$

where

$$p = \frac{1}{2}\left[1 + 2\hat{\Omega} b + \sqrt{1 + 4\hat{\Omega}^2 b^2 + 4(\hat{k}_v^2 + \hat{\lambda}^2 b^2)}\right]$$
$$q = \frac{1}{2}\left[1 + 2\hat{\Omega} b - \sqrt{1 + 4\hat{\Omega}^2 b^2 + 4(\hat{k}_v^2 + \hat{\lambda}^2 b^2)}\right]$$

Near Region Solutions II

- We want A, B to be, at the very least, finite at the horizon. This implies $C_2 = 0$.
- Hence the solution for $(A + B)$ is,

$$(A + B) = (R - 1)^{\hat{\Omega} b} C_1 F(p, q, 2\hat{\Omega} b; 1 - R) \tag{27}$$

- We now need to extend the near region solution to overlap region. This is done by using the transformation properties of hypergeometric functions. In overlap region,

$$A + B = C_1 \frac{\Gamma(p + q - 1)\Gamma(q - p)}{\Gamma(q)\Gamma(q - 1)} R^{-\frac{1}{2} - \frac{\sqrt{1 + 4\hat{\Omega}^2 b^2 + 4(\hat{k}_v^2 + \hat{\lambda}^2 b^2)}}{2}} + C_1 \frac{\Gamma(p + q - 1)\Gamma(p - q)}{\Gamma(p)\Gamma(p - 1)} R^{-\frac{1}{2} + \frac{\sqrt{1 + 4\hat{\Omega}^2 b^2 + 4(\hat{k}_v^2 + \hat{\lambda}^2 b^2)}}{2}} \tag{28}$$

Far Region Solutions

- The general solution for B is given in terms of modified Bessel functions of order $\nu = \sqrt{\frac{(n-1)^2}{4} + k_v^2}$ as

$$B = r^{\frac{1-n}{2}}[D_1 I_\nu(\sqrt{\lambda^2 + \Omega^2} r) + D_2 K_\nu(\sqrt{\lambda^2 + \Omega^2} r)] \quad (29)$$

- The modified functions have large order and large argument in large n limit. Let us define a new coordinate $z = \frac{\sqrt{\lambda^2+\Omega^2}}{\nu} r$.
- In this limit, as $r \to \infty$, $I_\nu(\nu z) \sim e^{\nu z} \to \infty$ and $K_\nu(\nu z) \sim e^{-\nu z} \to 0$. Normalizability thus implies in (29) that $D_1 = 0$.
- Rewriting the solution in terms of R, we get,

$$B = D_1 R^{-\frac{1}{2} - \frac{\sqrt{1+4\hat{\Omega}^2 b^2 + 4(\hat{k}_v^2 + \hat{\lambda}^2 b^2)}}{2}} \quad (30)$$