Dynamical analysis and growth of matter perturbations in Finsler-Randers Cosmology

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Plan of the talk

- **Finsler Randers Cosmology**
- Finsler Randers Cosmology with a scalar field
- Field equations FR
- **Dynamical Analysis**
- **System of equations** expressed in dimensionless variables
- Exponential Potential & Hyperbolic Potential
- Interaction of Scalar Field and matter

Plan of the talk

- **Growth of matter perturbations**
- FR with dark matter
- FR with collisional matter
- FR with collisional matter and scalar field
- **Conclusions**

- This work is a collaboration of G.Papagiannopoulos, S.Basilakos, A.Paliathanasis, S.Savvidou, P.C.Stavrinos
- This work has been accepted for publication in Classical and Quantum Gravity (http://arxiv.org/abs/1709.03748)

Finslerian Geometry

- The Finsler-Randers (FR) cosmological model is based on the Finslerian Geometry: a natural generalization of the standard Riemannian. In the FR space time the metric is

$$f_{\mu\nu} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^\mu \partial y^\nu},$$

where

$$F(x,y) = \sigma(x,y) + u_\mu(x) y^\mu, \quad \sigma(x,y) = \sqrt{a_{\mu\nu} y^\mu y^\nu},$$

and $u_\mu = (u_0, 0, 0, 0)$ is a weak primordial vector field with $\|u_\mu\| \ll 1$ and $a_{\mu\nu}$ the metric of the Riemannian space.

- The Finslerian contribution is provided by the vector field u_μ which introduces a preferred direction in space time and causes a differentiation of geodesics from a Riemannian spacetime.
- The field equations in the FR cosmology are written as

$$L_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}), \qquad (1)$$

where $L_{\mu\nu}$ is the Finslerian Ricci Tensor which can be written

as the sum of the nominal Ricci tensor plus a small tensor perturbation.(Stavrinos & Diakogiannis)

Finsler Randers Cosmology

We consider that the expanding universe can be described by a Finslerian perfect fluid with velocity 4-vector field u_μ.With ρ being the total energy density and p the corresponding pressure. Considering a flat FLRW metric we obtain the modified Friedmann's Equations.

$$\dot{H} + H^2 + \frac{3}{4}HZ_t = -\frac{4\pi G}{3}(\rho + 3P), \tag{2}$$

$$\dot{H} + 3H^2 + \frac{11}{4}HZ_t = 4\pi G(\rho - P), \tag{3}$$

where $H = \dot{a}/a$, $Z_t = \dot{u}_0 < 0$.Combining them we arrive at

$$H^2 + HZ_t = \frac{8\pi G}{3}\rho \tag{4}$$

Finsler Randers Cosmology

The Bianchi identities (ensure the covariance of the theory) impose:

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (5)$$

If we assume a mixture of two barotropic fluids, matter and scalar field, then the total density and pressure are given by

$$\rho = \rho_m + \rho_\phi, \quad p = p_m + p_\phi. \quad (6)$$

and using a standard scalar field language one can write ρ_ϕ,p_ϕ

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\ ,\ p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi),\ w_\phi = \frac{(\dot{\phi}^2/2) - V(\phi)}{(\dot{\phi}^2/2) + V(\phi)}$$

The conservation law now becomes

$$\dot{\rho}_m + 3H(1 + w_m)\rho_m = Q \quad , \quad \dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = -Q, \quad (7)$$

where Q is the rate of interaction between the scalar field and the matter source. Here we consider only one of the most simple expressions: $Q = a_m \rho_m H$. If $a_m > 0$ then the scalar field decays into matter, while the opposite holds for $a_m < 0$.

Dynamical Analysis - introduction

If we consider a non conservative system

$$\dot{x} = f(x, y), \dot{y} = g(x, y) \tag{8}$$

To get an image of the general beahvior of the system → spot the critical points $f(x_0, y_0) = g(x_0, y_0) = 0$ and consider a small perturbations around them. Keeping only the linear terms in terms of $\delta x, \delta y$ (this can be generalized for n equations)

$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0)\\ g_x(x_0, y_0) & g_y(x_0, y_0)\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \tag{9}$$

By calculating the eigenvalues of the above, its general solution is

$$\delta Q = c_1 D_1 e^{\lambda_1 t} + c_2 D_2 e^{\lambda_2 t} \tag{10}$$

Dynamical Analysis - introduction

$$\delta Q = c_1 D_1 e^{\lambda_1 t} + c_2 D_2 e^{\lambda_2 t}$$

If the eigenvalues only have an imaginary part then the perturbation will be a closed ellipsis and thus the point is stable.
Case· $\lambda_1 < \lambda_2 < 0$: the point is stable
Case· $\lambda_1 > \lambda_2 > 0$: the point is unstable
Case· $\lambda_1 < 0 < \lambda_2$: the point is hyperbolic,
Case· $\lambda_1 = -a + ib, \lambda_2 = -a - ib, a, b > 0$: the point is stable
Case· $\lambda_1 = a + ib, \lambda_2 = a - ib, a, b > 0$: the point is unstable
Case· $\lambda_1 = +ib, \lambda_2 = -ib, b > 0$: the point is elliptically stable

Dynamical Analysis - Minimally coupled fluids

We introduce a new set of dimensionless variables

$$x = \frac{\dot{\phi}}{\sqrt{6}H} \; , \; y = \frac{\sqrt{V}}{\sqrt{3}H} \; , \; \Omega_z = \frac{Z_t}{H} \; , \; \Omega_m = \frac{\rho_m}{3H^2}, \tag{11}$$

we have set $8\pi G = 1$ and $c = 1$. Therefore, the first Friedmann's equation (2) becomes

$$(1 + \Omega_z) - x^2 - y^2 = (1 + \Omega_z) - \Omega_\phi = \Omega_m, \tag{12}$$

Dynamical Analysis - System of Equations

The field equations are given by the following system of first-order ordinary differential equations

$$\frac{dx}{dN} = -3x + x(1 + \frac{3}{4}\Omega_Z + \frac{1}{2}\Omega_m(1 + 3w_m) + 2x^2 - y^2) + \sqrt{\frac{3}{2}}\lambda y^2, \tag{13}$$

$$\frac{dy}{dN} = y(1 + \frac{3}{4}\Omega_Z + \frac{1}{2}\Omega_m(1 + 3w_m) + 2x^2 - y^2 - \sqrt{\frac{3}{2}\lambda x}), \tag{14}$$

$$\frac{d\Omega_z}{dN} = \Omega_Z\left(1 + \frac{3}{4}\Omega_Z + \frac{1}{2}\Omega_m(1 + 3w_m) + 2x^2 - y^2\right), \tag{15}$$

$$\frac{d\lambda}{dN} = \sqrt{6}x\lambda^2[1 - \Gamma(\lambda)], \text{with } \lambda = -\frac{V_{,\phi}}{V}\ ,\ \Gamma = \frac{V_{,\phi\phi}V}{V_{,\phi}^2}. \tag{16}$$

where $N = \ln(a)$ is the new lapse function, $V_{,\phi} = dV/d\phi$ and $V_{,\phi\phi} = d^2V/d\phi^2$.

Dynamical Analysis - System of Equations

Moreover we calculate that

$$\frac{\dot{H}}{H^2} = -1 - \frac{3}{4}\Omega_z - 2x^2 + y^2 - \frac{1}{2}\Omega_m(1+3w_m), \tag{17}$$

hence the total equation of the state parameter $w_{\rm tot}$ can be explicitly derived in terms of the new variables x, y and Ω_z.

$$w_{tot} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = -\frac{1}{3} + \frac{1}{2}\Omega_z + \frac{4}{3}x^2 - \frac{2}{3}y^2 + \frac{1}{3}\Omega_m(1+3w_m), \tag{18}$$

The above dynamical system is an algebraic-differential system which is valid for a general potential $V(\phi)$. Notice, that the Friedmann Equation is a constraint equation.

Dynamical Analysis - Eponential Potential- Critical Points

We study the exponential potential case. **The free variables are** $\{x, y, \Omega_z\}$**.**The dynamical system contains six critical points, among which five points are similar to those of GR and only one is a new critical point accommodated by FR gravity model.

Table: Critical points and cosmological parameters for scalar field cosmology in the Finsler-Randers theory for exponential potential.

Point	Ω_m	Ω_z	w_{tot}	**Acceleration**	**Existence**
$P_1^{\pm}$	0	0	1	No	Always
P_2	1	0	w_m	Yes for $w_m < -1/3$	Always
P_3	0	0	$-1 + \frac{\lambda^2}{3}$	$-\sqrt{2} \leq \lambda \leq \sqrt{2}$	$\lambda^2 < 6$
P_4	$\frac{\lambda^2 - 3(1+w_m)}{\lambda^2}$	0	w_m	Yes for $w_m < -1/3$	$\frac{\lambda^2}{3} - 1 \geqq w_m > -1$
F_1	$1 - \frac{6(w_m+1)}{6w_m+5}$	$-\frac{6(w_m+1)}{6w_m+5}$	-1	Always	$w_m \neq -\frac{5}{6}$, ρ_m violates S.E.C.

Dynamical Analysis - Eponential Potential- Eigenvalues

Table: Eigenvalues and stability for the critical points of scalar field cosmology in the Finsler-Randers theory for exponential potential.

Point/Eigenvalues	m_1	m_2	m_3	Stable?
P_1	3	$\frac{1}{2}(6+\sqrt{6}\lambda)$	$3(1-w_m)$	No
P_2	$-\frac{3}{2}(1-w_m)$	$\frac{3}{2}(1+w_m)$	$\frac{3}{2}(1+w_m)$	No
P_3	$\frac{\lambda^2}{2}$	$\frac{1}{2}(-6+\lambda^2)$	$-3+\lambda^2-3w_m$	No
P_4	$\frac{3}{2}(1+w_m)$	$-\frac{3}{4}(1-w_m)-\frac{A}{4\lambda^2}$	$-\frac{3}{4}(1-w_m)+\frac{A}{4\lambda^2}$	No
F_1	-3	0	$-\frac{3}{2}(1+w_m)$	CMT/ Yes for $w_m > -1$

where $P_1^{\pm}:(\pm 1,0,0)$, $P_2:(0,0,0)$, $P_3:(\frac{\lambda}{\sqrt{6}},\pm\sqrt{1-\frac{\lambda^2}{6}},0)$, $P_4:(\sqrt{\frac{3}{2}}\frac{1+w_m}{\lambda},\pm\sqrt{\frac{3}{2}}\ \frac{\sqrt{1-w_m^2}}{\lambda},0)$, $F_1:(0,0,-\frac{6(w_m+1)}{6w_m+5})$

Dynamical Analysis - Hyperbolic Potential- Critical Points

For $V = V_0 \cosh^q(p\phi)$ equation (16) becomes

$$\frac{d\lambda}{dN} = \frac{\sqrt{6}}{q}(qp - \lambda)(qp + \lambda)x \; . \qquad (19)$$

Table: Critical points and cosmological parameters for scalar field cosmology in the Finsler-Randers theory for hyperbolic potential.

Point	$(x, y, \lambda, \Omega_z)$	Ω_m	Ω_ϕ	w_{tot}	Existence
$\bar{P}_1^{\pm}$	$(\pm 1, 0, \pm pq, 0)$	0	1	1	Always
$\bar{P}_2$	$(0, 0, \lambda_c, 0)$	1	0	$-\frac{1}{3} + \frac{1}{2}(1 + 3w_m)$	Always
$\bar{P}_3^{\pm}$	$\left(\pm \frac{pq}{\sqrt{6}}, \sqrt{1 - \frac{(pq)^2}{6}}, \pm pq, 0\right)$	0	1	$-1 + \frac{(pq)^2}{3}$	$(pq)^2 \leq 6$
$\bar{P}_4^{\pm}$	$\left(\pm \sqrt{\frac{3}{2}} \frac{1+w_m}{pq}, \sqrt{\frac{3}{2}} \frac{\sqrt{1-w_m^2}}{pq}, \pm pq, 0\right)$	$\frac{(pq)^2 - 3(1+w_m)}{(pq)^2}$	$1 - \Omega_m$	w_m	$\frac{(pq)^2}{3} - 1 \geqq w_m > -1$
$\bar{P}_5$	$(0, 1, 0, 0)$	0	1	-1	Always
$\bar{F}_1$	$(0, 0, \lambda_c, -\frac{6(w_m+1)}{6w_m+5})$	$1 - \frac{6(w_m+1)}{6w_m+5}$	0	-1	$w_m \neq -\frac{5}{6}$, ρ_m can violate S.E.C.
$\bar{F}_2$	$(0, y_c, 0, -\frac{6(w_m+1-y^2-w_m y^2)}{6w_m+5})$	$1 - \Omega_z - y_c^2$	y_c^2	-1	$w_m \neq -\frac{5}{6}$, ρ_m can violate S.E.C.

Various points describe de Sitter phases of the universe exist unlike the GR case (if the matter source violates the strong energy condition)

Dynamical Analysis - Hyperbolic Potential- Eigenvalues

Table: Eigenvalues and stability for the critical points of scalar field cosmology in the Finsler-Randers theory for hyperbolic potential.

Point	m_1	m_2	m_3	m_4	Stable
$\bar{P}_1^{\pm}$	3	$\pm 2\sqrt{6}p$	$\frac{1}{2}(6-\sqrt{6}pq)$	$\pm 3(1-w_m)$	No
$\bar{P}_2$	0	$-\frac{3}{2}(1-w_m)$	$\frac{3}{2}(1+w_m)$	$\frac{3}{2}(1+w_m)$	No
$\bar{P}_3^{\pm}$	$-2p^2q$	$\frac{p^2q^2}{2}$	$\frac{1}{2}(-1+p^2q^2)$	$-3+p^2q^2-3w_m$	No
$\bar{P}_4^{\pm}$	$\frac{-6(1+w_m)}{q}$	$\frac{3(1+w_m)}{2}$	$-\frac{3}{4}(1-w_m)-\frac{B}{4p^2q^2}$	$-\frac{3}{4}(1-w_m)+\frac{B}{4p^2q^2}$	No
$\bar{P}_5$	0	$-\frac{1}{2}(3+\sqrt{9+12p^2q})$	$-\frac{1}{2}(3-\sqrt{9+12p^2q})$	$-3(1+w_m)$	For $p^2q<0$
$\bar{F}_1$	0	0	-3	$-\frac{3}{2}(1+w_m)$	No
$\bar{F}_2$	0	$-\frac{3}{2}(1+w_m)(1+y^2)$	$-\frac{3}{2}+\frac{C}{2(5+6w_m)}$	$-\frac{3}{2}-\frac{C}{2(5+6w_m)}$	No

Dynamical Analysis - Interaction of Scalar Field and matter

Here we allow interactions between scalar field and matter $Q = a_m \rho_m H$. From our initial system only the first equation is altered

$$\frac{dx}{dN} = -3x + x\left(1 + \frac{3}{4}\Omega_Z + \frac{1}{2}\Omega_m(1 + 3w_m) + 2x^2 - y^2\right) + \sqrt{\frac{3}{2}}\lambda y^2 - \bar{Q} \quad (20)$$

where $\bar{Q} = \frac{\alpha_m}{2}\Omega_m$. For $\lambda^2 = pq$ the two potentials share the same critical points. Therefore, we focus our analysis on the exponential case.

Table: Critical points and cosmological parameters for scalar field cosmology in the Finsler-Randers theory with interaction for exponential potential.

Point	Ω_m	Ω_z	w_{tot}
$A_1^{\pm}$	0	0	1
$A_2^{\pm}$	$1-\frac{a_m}{3(1-w_m)}$	0	$w_m+\frac{a_m}{3}$
A_3	0	0	$-1+\frac{\lambda^2}{3}$
A_4	$\frac{(3(1+w_m)+a_m)^2\left(\frac{\lambda^2}{(3(1+w_m)+a_m)}-1\right)}{3\lambda^2(1+w_m)}$	0	$w_m+\frac{a_m}{3}$
R_1	$-\frac{6}{30+11a_m+36w_m}$	$-\frac{12(3w_m+3+a_m)}{30+11a_m+36w_m}$	-1

where $A_1^{\pm}:(\pm1,0,0)$, $A_2^{\pm}:\left(\pm\sqrt{\frac{a_m}{3(1-w_m)}},0,0\right)$, $A_3:$ $\left(\frac{\lambda}{\sqrt{6}},\pm\sqrt{1-\frac{\lambda^2}{6}},0\right)$, $A_4:\left(\frac{3(1+w_m)+a_m}{\sqrt{6}\lambda},y\left(A_4\right),0\right)$, $R_1:$ $\left(-\sqrt{\frac{-a_m}{30+11a_m+36w_m}},0,-\frac{12(3w_m+3+a_m)}{30+11a_m+36w_m}\right)$

Table: Critical and stability for scalar field cosmology with interaction in the Finsler-Randers theory for exponential potential.

Point	Acceleration	Existence	Stability
A_1	No	yes	No
A_2	Yes for $w_m + \frac{a_m}{3} < -1/3$	$0 < \frac{a_m}{3(1-w_m)} < 1$	No
A_3	Yes $-\sqrt{2} \leq \lambda \leq \sqrt{2}$	$\lambda^2 \leq 6$	No
A_4	Yes for $w_m + \frac{a_m}{3} < -1/3$	$\lambda^2 \geq 3(1+w_m) + a_m$	No
R_1	Yes	Yes for $\frac{-a_m}{30+11a_m+36w_m} > 0$	Yes for specific a_m, w_m

Growth of matter Perturbations

We study the linear growth of matter fluctuations for FR cosmology and compare our results with those of the DGP and ΛCDM models. Owing to the fact that we are well inside in the matter epoch we can neglect the radiation component from the Hubble expansion.The basic differential equation which provides the evolution of linear matter perturbations

$$\ddot{\delta}_m + 2\nu H\dot{\delta}_m - 4\pi G\mu\rho_m\delta_m = 0 \ . \qquad (21)$$

where $\nu \equiv 1 + Q/H$ and $\mu \equiv G_{\rm eff}/G_N$ are related with the physics of dark energy.We consider no interactions in the Dark Sector so $Q = 0$ and $\nu = 1$. Also, $\mu \equiv G_{\rm eff}/G_N = 1$.

Growth of matter Perturbations

Introducing the growth rate of clustering

$$f(a) = \frac{d\ln\delta_m}{d\ln a} \simeq \Omega_m^{\gamma}(a) \ , \tag{22}$$

and a change of variables we can rewrite the intial equation as

$$\frac{d\omega}{d\ln a}(\gamma + \omega\frac{d\gamma}{d\omega}) + e^{\omega\gamma} + 2\nu + \frac{d\ln E}{d\ln a} = \frac{3}{2}\mu e^{\omega(1-\gamma)}, \tag{23}$$

where $\omega = \ln\Omega_m(a)$ At $a \to 0$ then $\Omega_m(a) \to 1$ [or $\omega \to 0$].
Regarding the evolution of growth index we use the methodology of Steigerwald et al., who proposed the parametrization

$$\gamma(a) = \gamma_0 + \gamma_1\omega(a). \tag{24}$$

Obviously $\gamma_\infty \approx \gamma_0$.

Growth of matter Perturbations

The coefficients γ_0 and γ_1 are given by

$$\gamma_0 = \frac{3(M_0 + M_1) - 2(\mathcal{H}_1 + N_1)}{2 + 2X_1 + 3M_0} \tag{25}$$

and

$$\gamma_1 = 3\frac{M_2 + 2M_1B_1(1 - y_1) + M_0B_2(1 - y_1, -y_2)}{2(2 + 4X_1 + 3M_0)} - 2\frac{B_2(y_1, y_2) + X_2\gamma_0 + \mathcal{H}_2 + N_2}{2(2 + 4X_1 + 3M_0)}$$

The following quantities have been defined as:

$$\begin{aligned} X_n &= \left.\frac{d^n(d\omega/d\ln a)}{d\omega^n}\right|_{\omega=0} \quad M_n = \left.\frac{d^n\mu}{d\omega^n}\right|_{\omega=0}, \\ N_n &= \left.\frac{d^n\nu}{d\omega^n}\right|_{\omega=0}, H_n = -\frac{1}{2}X_n = \left.\frac{d^n(d\ln E/d\ln a)}{d\omega^n}\right|_{\omega=0} \end{aligned} \tag{26}$$

Finsler Randers with cold dark matter

$p_m = 0$ ($w_m = 0$), $\rho_\phi(a) = 0$

- **FR model**. In this case we find

  $$\{M_0, M_1, M_2, N_1, N_2\} = \{1, 0, 0, 0, 0\}, \{\mathcal{H}_1, \mathcal{H}_2, X_1, X_2\} = \{-\frac{3}{4}, 0, \frac{3}{2}, 0\}$$

  and thus

  $$\gamma_\infty^{(FR)} \approx \gamma_0^{(FR)} = \frac{9}{16}, \qquad \gamma_1^{(FR)} = -\frac{15}{5632} \approx -0.0027.$$

- **DGP model**

  $$\{M_0, M_1, M_2, N_1, N_2\} = \{1, \frac{1}{3}, 0, 0, 0\}, \{\mathcal{H}_1, \mathcal{H}_2, X_1, X_2\} = \{-\frac{3}{4}, 0, \frac{3}{2}, 0\}$$

  and thus

  $$\gamma_\infty^{(DGP)} \approx \gamma_0^{(DGP)} = \frac{11}{16}, \qquad \gamma_1^{(DGP)} = -\frac{7}{5632} \approx -0.0012.$$

- **ΛCDM model**

  $$\{M_0, M_1, M_2, N_1, N_2\} = \{1, 0, 0, 0, 0\}, \{\mathcal{H}_1, \mathcal{H}_2, X_1, X_2\} = \{-\frac{3}{2}, -\frac{3}{2}, 3, 3\}$$

  and thus

  $$\gamma_\infty^{(\Lambda)} \approx \gamma_0^{(\Lambda)} = \frac{6}{11}, \qquad \gamma_1^{(\Lambda)} = -\frac{15}{2057} \approx -0.0073.$$

Finsler Randers with collissional matter

$w_m = \frac{p_m}{\rho_m} \neq 0$,
$\rho_m \propto a^{-3(1+w_m)}$, $\mu = (1+3w_m)(1+w_m)$, $\rho_\phi(a) = 0$

- **FR model**. Here we obtain

$$\{M_0, M_1, M_2, N_1, N_2\} = \{(1+3w_m)(1+w_m), 0, 0, 0, 0\}$$

$$\{\mathcal{H}_1, \mathcal{H}_2, X_1, X_2\} = \{-\frac{3}{4}(1+w_m), 0, \frac{3}{2}(1+w_m), 0\},$$

hence

$$\gamma_\infty^{(FR)} \approx \gamma_0^{(FR\phi)} = \frac{9(1+w_m)(1+2w_m)}{2(8+3w_m(5+3w_m))}\,,\ \gamma_1^{(FR)} = -\frac{3(1+w_m)(5+21w_m)(1+3w_m(4+3w_m))}{8(11+9w_m(2+w_m))(8+3w_m(5+3w_m))^2}.$$

- Λ**CDM model**. In this case we find

$$\{M_0, M_1, M_2, N_1, N_2\} = \{(1+3w_m)(1+w_m), 0, 0, 0, 0\}$$

$$\{\mathcal{H}_1, \mathcal{H}_2, X_1, X_2\} = \{-\frac{3}{2}(1+w_m), -\frac{3}{2}(1+3w_m), 3(1+w_m), 3(1+w_m)\}$$

and thus

$$\gamma_\infty^{(\Lambda)} \approx \gamma_0^{(\Lambda)} = \frac{3(1+w_m)(2+3w_m)}{11+9w_m(2+w_m)}, \gamma_1^{(\Lambda)} = -\frac{3(1+w_m)(2+3w_m)(5+3w_m)(1+3w_m(4+3w_m))}{2(11+9w_m(2+w_m))^2(17+3w_m(8+3w_m))}.$$

Finsler Randers with collissional matter and Scalar Field

$w_\phi = \frac{p_\phi}{\rho_\phi} = const, \rho_\phi = \rho_{\phi 0} a^{-3(1+w_\phi)}$

In this case our main coefficients become

$$M_0 = (1+3w_m)(1+w_m), \ M_{1,2} = 0, \ N_{1,2} = 0, \mathcal{H}_2 = X_2 = 0,$$

and

$$\mathcal{H}_1 = \frac{1}{2}X_1 = -\frac{3}{4}(1+w_m) + \frac{9c_\phi}{4}(1-w_m+2w_\phi)(2+w_m+w_\phi),$$

where $c_\phi = lim_{a\to 0}\left[\frac{\Omega_\phi(a)}{(d\ln\Omega_m/d\ln a)}\right]$.

Inserting the above coefficients into Eqs.(25-26) we can trivially calculate $\{\gamma_0, \gamma_1\}$

$$\gamma_0^{(FR\phi)} = \frac{9[c_\phi(1-w_m+2w_\phi)(2+w_m+w_\phi) - (1+w_m)(1+2w_m)]}{2[c_\phi(1-w_m+2w_\phi)(2+w_m+w_\phi) - 8 - 3w_m(5+3w_m)]}$$

and

$$\gamma_1^{(FR\phi)} = -\frac{3[(1+w_m)(1+3w_m)(7+3w_m-9C)^2 - 54(C-(1+w_m)(1+2w_m))^2]}{8(11+9w_m(2+w_m)-18C)(8-9C+3w_m(5+3w_m))^2}.$$

where we have considered $C = c_\phi(1-w_m+2w_\phi)(2+w_m+w_\phi)$.

We studied the cosmological behavior of the FR cosmological model at the background and at the perturbation level.

- For different types of FR models we **performed a critical point analysis** in order to study the various phases of the FR gravity theory in order to be able to compare the current results with those of General Relativity.
- For all FR models we found solutions which accommodate cosmic acceleration and under specific conditions they provide **de-Sitter points as stable late-time attractors**.

We analytically studied the growth of perturbations in FR cosmologies.

- **In the context of CDM** we found that $\gamma_{\infty}^{(FR)} \approx \frac{9}{16}$ which is somewhat greater ($\sim 3\%$) than that of ΛCDM model $\gamma_{\infty}^{(\Lambda)} \approx \frac{6}{11}$.
- We confirmed that **the FR model mimics the DGP gravity model** as far as the cosmic expansion is concerned. They can be distinguished at the perturbation level, since $\Delta\gamma_{\infty}(\%) = 100 \times [\gamma_{\infty}^{(FR)} - \gamma_{\infty}^{(DGP)}]/\gamma_{\infty}^{(DGP)}$ is $\sim -18.2\%$, where $\gamma_{\infty}^{(DGP)} \approx \frac{11}{16}$.
- If we allow pressure in the matter fluid then $\gamma_{\infty}^{(FR)} \approx \frac{9(1+w_m)(1+2w_m)}{2[8+3w_m(5+3w_m)]}$ and $\gamma_{\infty}^{(\Lambda)} \approx \frac{3(1+w_m)(2+3w_m)}{11+9w_m(2+w_m)}$, where $w_m = p_m/\rho_m$.
- We generalized the growth index analysis by including the effects of the scalar field. The evolution of the growth index in FR cosmologies **is affected by the scalar field.**