Beyond Einstein Gravity: Decoupling gravitational sources

by

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Motivation

- In most cases solving Einstein field equations is a difficult task. Indeed, it is hard to obtain analytical solutions having some physical relevance, except for some specific situations.
- Particular cases: (i) Vacuum ($\rho = p = Q = ... = 0$); (ii) The spherically symmetric space-time with a perfect fluid $\hat{T}_{\mu\nu}$ as a gravitational source.
- As soon as the perfect fluid is coupled to complex forms of matter-energy to describe more realistic scenarios, namely,

$$T_{\mu\nu} = \hat{T}_{\mu\nu} + \alpha\,\theta_{\mu\nu}\ , \tag{1}$$

  with $\theta_{\mu\nu}$ any other form of gravitational source, then the situation changes radically, making it almost impossible to obtain analytical results that can be easily interpreted.
- In this talk: the first simple, systematic and direct approach to decoupling gravitational sources in general relativity.

A naive question

Would it not be ideal to solve Einstein field equations by solving the field equations for each gravitational source individually? That is, we could find the metric $g_{\mu\nu}$, and both energy-momentum tensors $\hat{T}_{\mu\nu}$ and $\theta_{\mu\nu}$, not by solving

$$G_{\mu\nu} = -k^2 \left(\hat{T}_{\mu\nu} + \alpha\, \theta_{\mu\nu} \right) \;\; ; \quad k^2 = 8\pi \;, \tag{2}$$

but

$$\hat{G}_{\mu\nu} = -k^2\, \hat{T}_{\mu\nu} \;, \quad \text{to find} \;\; \{\hat{g}_{\mu\nu}, \hat{T}_{\mu\nu}\} \tag{3}$$

and then

$$G^*_{\mu\nu} = -k^2\, \theta^*_{\mu\nu} \;, \quad \text{to find} \;\; \{g^*_{\mu\nu}, \theta^*_{\mu\nu}\} \tag{4}$$

and finally, we could obtain the metric $g_{\mu\nu}$ in Eq. (2) by a simple combination of the two metrics found by Eqs. (3) and (4), namely, $\hat{g}_{\mu\nu}$ and $g^*_{\mu\nu}$.

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and finally, we could obtain the metric $g_{\mu\nu}$ in Eq. (2) by a simple combination of the two metrics found by Eqs. (3) and (4), namely, $\hat{g}_{\mu\nu}$ and $g^*_{\mu\nu}$.

Consensus: No way!!! (given the highly non-linear and complex structure of Einstein field equations)

Einstein Equations

Let us start from Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -k^2\, T^{(\mathrm{tot})}_{\mu\nu} \;, \qquad (5)$$

with

$$T^{(\mathrm{tot})}_{\mu\nu} = T^{(\mathrm{m})}_{\mu\nu} + \alpha\, \theta_{\mu\nu} \qquad (6)$$

where

$$T^{(\mathrm{m})}_{\mu\nu} = (\rho + p)\, u_\mu\, u_\nu - p\, g_{\mu\nu} \qquad (7)$$

is the four-dimensional energy-momentum tensor of ordinary matter, described by a perfect fluid.

$\theta_{\mu\nu} \rightarrow$ any additional gravitational source coupled with the perfect fluid by the constant α

$$\nabla_\nu\, T^{(\mathrm{tot})\mu\nu} = 0 \;. \qquad (8)$$

Einstein Equations

$$ds^2 = e^{\nu(r)}\, dt^2 - e^{\lambda(r)}\, dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \ . \qquad (9)$$

Einstein equations

$$-k^2 \left(\rho + \alpha\, \theta_0^0 \right) = -\frac{1}{r^2} + e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) \ , \qquad (10)$$

$$-\, k^2 \left(-p + \alpha\, \theta_1^1 \right) = -\frac{1}{r^2} + e^{-\lambda} \left(\frac{1}{r^2} + \frac{\nu'}{r} \right) \ , \qquad (11)$$

$$-\, k^2 \left(-p + \alpha\, \theta_2^2 \right) = \frac{1}{4} e^{-\lambda} \left[2\,\nu'' + \nu'^2 - \lambda'\,\nu' + 2\,\frac{\nu' - \lambda'}{r} \right] \ , \qquad (12)$$

while the conservation equation

$$-\, p' - \frac{\nu'}{2}(\rho + p) + \alpha({\theta_1}^1)' - \frac{\nu'}{2}\alpha({\theta_0}^0 - {\theta_1}^1) - \frac{2\alpha}{r}({\theta_2}^2 - {\theta_1}^1) = 0 \ , \qquad (13)$$

where $f' \equiv \partial_r f$.

Minimal Geometric Deformation

Let us start by considering a solution with $\alpha = 0$, namely, a perfect fluid solution $\{\xi, \mu, \rho, p\}$, where ξ and μ are the metric functions

$$ds^2 = e^{\xi(r)}\, dt^2 - \mu(r)^{-1}\, dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \ . \tag{14}$$

Now let us turn on the parameter α to consider the effects of the source $\theta_{\mu\nu}$ on the perfect fluid solution $\{\xi, \mu, \rho, p\}$.

$$\xi \ \rightarrow \ \nu = \xi + \alpha\, g \ , \tag{15}$$

$$\mu \ \rightarrow \ e^{-\lambda} = \mu + \alpha\, f . \tag{16}$$

where f and g are, respectively, the **geometric deformations** undergone by the perfect fluid geometry $\{\xi, \mu\}$. Of all the possibilities contained in Eqs. (15) and (16), there is a specific one, the so-called minimal geometric deformation (MGD), for which

$$g \ \rightarrow \ 0 \tag{17}$$

$$f \ \rightarrow \ f^* \ . \tag{18}$$

The metric $\{\xi, \mu\}$ is thus *minimally deformed* by $\theta_{\mu\nu}$

Decoupling: Set (A)

$$\mu(r) \to \; e^{-\lambda(r)} = \mu(r) + \alpha \, f^*(r) \; , \tag{19}$$

Now let us plug the decomposition in Eq. (19) in the Einstein equations. The system is thus *separated in two sets*: (A) one having the standard Einstein field equations for a perfect fluid ($\alpha = 0$),

$$-k^2\rho = -\frac{1}{r^2} + \frac{\mu}{r^2} + \frac{\mu'}{r} \; , \tag{20}$$

$$-\,k^2\,(-p) = -\frac{1}{r^2} + \mu\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) \; , \tag{21}$$

$$-\,k^2\,(-p) = \frac{\mu}{4}\left(2\nu'' + \nu'^2 + \frac{2\nu'}{r}\right) + \frac{\mu'}{4}\left(\nu' + \frac{2}{r}\right) \; , \tag{22}$$

$$\nabla_\nu \, T^{(\mathrm{m})\mu\nu} = 0 \; \to \; p' + \frac{\nu'}{2}(\rho + p) = 0 \; . \tag{23}$$

$$\boxed{\hat{g}_{\mu\nu} \to \; ds^2 = e^{\nu(r)}\, dt^2 - \mu(r)^{-1}\, dr^2 - r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \; .} \tag{24}$$

and (B) one for the source $\theta_{\mu\nu}$, which reads

$$-k^2\,\theta_0^0 = \frac{f^*}{r^2} + \frac{{f^*}'}{r}\ , \tag{25}$$

$$-\,k^2\,\theta_1^1 = f^*\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)\ , \tag{26}$$

$$-\,k^2\,\theta_2^2 = \frac{f^*}{4}\left(2\nu'' + \nu'^2 + 2\frac{\nu'}{r}\right) + \frac{{f^*}'}{4}\left(\nu' + \frac{2}{r}\right)\ . \tag{27}$$

$$\nabla_\nu\,\theta^{\mu\nu} = 0 \to ({\theta_1}^1)' - \frac{\nu'}{2}({\theta_0}^0 - {\theta_1}^1) - \frac{2}{r}({\theta_2}^2 - {\theta_1}^1) = 0\ . \tag{28}$$

Under these conditions, there is no exchange of energy-momentum between $\hat{T}_{\mu\nu}$ and $\theta_{\mu\nu}$; their interaction is **purely gravitational**. The Set (B) looks very similar to Einstein field equations.

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Under these conditions, there is no exchange of energy-momentum between $\hat{T}_{\mu\nu}$ and $\theta_{\mu\nu}$; their interaction is **purely gravitational**. The Set (B) looks very similar to Einstein field equations. But it is not: *missed*. It represents a "*quasi − Einstein system*".

Decoupling: Set (B): Quasi-Einstein or Einstein system

The Set (B) may be formally identified as Einstein equations for an anisotropic system with energy-momentum tensor $\theta^*_{\mu\nu}$ defined as

$$k^2\,\theta^{*\,\nu}_{\mu} = k^2\,\theta^{\nu}_{\mu} + \frac{1}{r^2}\left(\delta_{\mu}^{\ 0}\,\delta_{0}^{\ \nu} + \delta_{\mu}^{\ 1}\,\delta_{1}^{\ \nu}\right)\ , \tag{29}$$

with conservation equation

$$(\theta^{*\,1}_{1})' - \frac{\nu'}{2}(\theta^{*\,0}_{0} - \theta^{*\,1}_{1}) - \frac{2}{r}(\theta^{*\,2}_{2} - \theta^{*\,1}_{1}) = 0\ , \tag{30}$$

and metric

$$\boxed{g^*_{\mu\nu} \to\ ds^2 = e^{\nu(r)}\,dt^2 - \frac{dr^2}{f^*(r)} - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\ .} \tag{31}$$

MGD: linear scheme for gravitational decoupling

What happens if we consider an additional source $\Psi_{\mu\nu}$?, namely,

$$G_{\mu\nu} = -k^2 \left(\hat{T}_{\mu\nu} + \alpha\,\theta_{\mu\nu} + \beta\,\Psi_{\mu\nu} \right) \ . \qquad (32)$$

We can easily follow the same scheme to find a successful decoupling by

$$e^{-\lambda(r)} = \mu(r) + \alpha\,f^*(r) + \beta\,h^*(r) \ , \qquad (33)$$

$$\text{where} \qquad \tilde{G}_{\mu\nu} = -k^2\,\tilde{\Psi}_{\mu\nu} \ , \quad \text{to find } \{\tilde{g}_{\mu\nu}, \tilde{\Psi}_{\mu\nu}\} \ , \qquad (34)$$

with the metric $\tilde{g}_{\mu\nu}$ given by

$$ds^2 = e^{\nu(r)}\,dt^2 - \frac{dr^2}{h^*(r)} - r^2 \left(d\theta^2 + \sin^2\theta\,d\phi^2 \right) \ . \qquad (35)$$

We can see that this approach represents a *linear scheme* for decoupling gravitational sources.

$$T_{\mu\nu} = \sum_{i=0}^{n} \alpha_i \, T_{\mu\nu}^{(i)} \; ; \;\; \alpha_0 = 1 \; ; \;\; T_{\mu\nu}^{0} = \hat{T}_{\mu\nu} \; . \tag{36}$$

The solution to $G_{\mu\nu} = -k^2 \, T_{\mu\nu}$, will be given by

$$g_{\mu\nu} = \hat{g}_{\mu\nu} = g_{\mu\nu}^{(i)} \; ; \; \mu = \nu \neq 1 \; , \tag{37}$$

$$g^{11} = \hat{g}^{11} + \alpha_1 \, g^{11(1)} + \ldots + \alpha_n \, g^{11(n)} \; . \tag{38}$$

$g_{\mu\nu}$ is found by first solving Einstein field equations $\hat{T}_{\mu\nu}$

$$\hat{G}_{\mu\nu} = -k^2 \, \hat{T}_{\mu\nu} \; ; \qquad \nabla_\nu \, \hat{T}^{\mu\nu} = 0 \; , \tag{39}$$

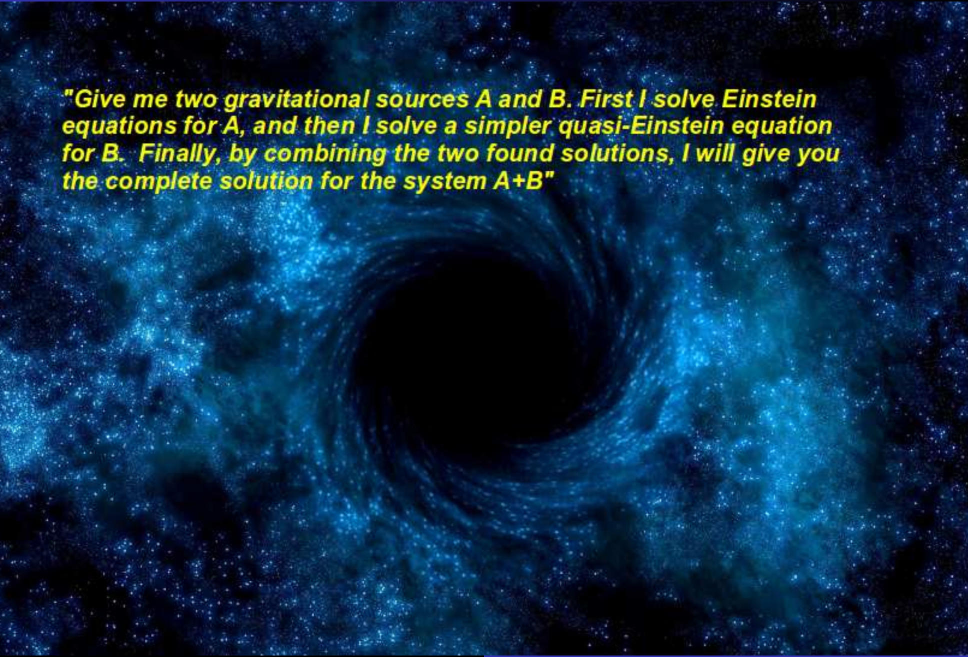
and then by solving the remaining n "quasi-Einstein equations"

$$\begin{aligned} \tilde{G}_{\mu\nu}^{(1)} &= -k^2 \, T_{\mu\nu}^{(1)} \; ; \quad \nabla_\nu \, T^{(1)\mu\nu} = 0 \; , \\ &\vdots \\ \tilde{G}_{\mu\nu}^{(n)} &= -k^2 \, T_{\mu\nu}^{(n)} \; ; \quad \nabla_\nu \, T^{(n)\mu\nu} = 0 \; , \end{aligned} \tag{40}$$

where the "quasi-Einstein" tensor $\tilde{G}_{\mu\nu}$ and $G_{\mu\nu}$ are related by

$$\tilde{G}_\mu{}^\nu = G_\mu{}^\nu + \Gamma_\mu{}^\nu(g) \; . \tag{41}$$

What does MGD tell us?



Applications: From perfect to anisotropic fluids

$$ds^2 = e^{\nu(r)}\, dt^2 - e^{\lambda(r)}\, dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\ . \qquad (43)$$

Einstein equations for an anisotropic self-gravitating system

$$-k^2 \underbrace{\left(\rho + \alpha\, \theta_0^0\right)}_{\tilde{\rho}} = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)\ , \qquad (44)$$

$$-\, k^2 \underbrace{\left(-p + \alpha\, \theta_1^1\right)}_{-\tilde{p}_r} = - = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)\ , \qquad (45)$$

$$-\, k^2 \underbrace{\left(-p + \alpha\, \theta_2^2\right)}_{-\tilde{p}_t} = \frac{1}{4}e^{-\lambda}\left[2\,\nu'' + \nu'^2 - \lambda'\,\nu' + 2\,\frac{\nu' - \lambda'}{r}\right]\ , \qquad (46)$$

$$-\, p' - \frac{\nu'}{2}(\rho + p) + \alpha({\theta_1}^1)' - \frac{\nu'}{2}\alpha({\theta_0}^0 - {\theta_1}^1) - \frac{2\alpha}{r}({\theta_2}^2 - {\theta_1}^1) = 0\ , \qquad (47)$$

Let us find $\{\nu, \lambda, \tilde{\rho}, \tilde{p}_r, \tilde{p}_t\}$

Applications: From perfect to anisotropic fluids

To solve the set (A), we simply choose an already-known solution with physical relevance, for instance, the well-known Tolman IV solution (ν, μ, ρ, p) for perfect fluids, namely,

$$e^{\nu} = B^2 \left(1 + \frac{r^2}{A^2}\right) , \tag{48}$$

$$\mu = \frac{\left(1 - \frac{r^2}{C^2}\right)\left(1 + \frac{r^2}{A^2}\right)}{1 + \frac{2\,r^2}{A^2}} , \tag{49}$$

$$\rho(r) = \frac{3A^4 + A^2\left(3C^2 + 7r^2\right) + 2r^2\left(C^2 + 3r^2\right)}{k^2\, C^2\left(A^2 + 2r^2\right)^2} , \tag{50}$$

and

$$p(r) = \frac{C^2 - A^2 - 3r^2}{k^2\, C^2\left(A^2 + 2r^2\right)} . \tag{51}$$

The constants A, B and C are found by matching conditions, yielding $A^2/R^2 = \frac{1-3\,c}{c}$; $B^2 = 1 - 3\,c$ and $C^2/R^2 = c^{-1}$, with $c \equiv M_0/R < 4/9$ and M_0 the total mass $m(R)$.

We need to solve the Sector (B), namely, the $\theta_{\mu\nu}$-sector,

$$-k^2\,\theta_0^0 = \frac{f^*}{r^2} + \frac{f^{*'}}{r}\ , \tag{52}$$

$$-\,k^2\,\theta_1^1 = f^*\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)\ , \tag{53}$$

$$-\,k^2\,\theta_2^2 = \frac{f^*}{4}\left(2\nu'' + \nu'^2 + 2\frac{\nu'}{r}\right) + \frac{f^{*'}}{4}\left(\nu' + \frac{2}{r}\right)\ . \tag{54}$$

$$\nabla_\nu\,\theta^{\mu\nu} = 0 \rightarrow\ ({\theta_1}^1)' - \frac{\nu'}{2}({\theta_0}^0 - {\theta_1}^1) - \frac{2}{r}({\theta_2}^2 - {\theta_1}^1) = 0\ . \tag{55}$$

We have four unknown $\{{\theta_0}^0, {\theta_1}^1, {\theta_2}^2, f^*\}$. Hence we need to provide a a constraint.
We must be careful in keeping the physical acceptability of our solution, which **is not a trivial matter**.

Applications: From perfect to anisotropic fluids

We impose the "mimic" constraint $\theta_1^{\ 1}(r) = p(r)$. Hence the radial metric component reads

$$e^{-\lambda(r)} = \mu(r)(1-\alpha) + \alpha\left(\frac{A^2+r^2}{A^2+3\,r^2}\right)\ , \tag{56}$$

and

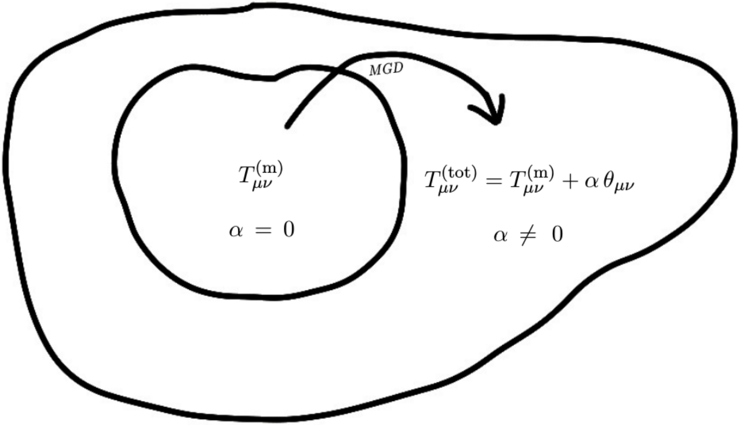
$$\tilde{p}_r(r,\alpha) = \frac{3\,(1-\alpha)(R^2-r^2)}{k^2\,(A^2+3\,R^2)(A^2+2\,r^2)}\ . \tag{57}$$

$$\tilde{\rho}(r,\alpha) = \rho(r)\,(1-\alpha) + 6\frac{\alpha}{k^2}\frac{A^2+r^2}{(A^2+3\,r^2)^2} \tag{58}$$

$$\tilde{p}_t(r,\alpha) = \tilde{p}_r(r,\alpha) + 3\frac{\alpha}{k^2}\frac{r^2}{(A^2+3\,r^2)^2}\ . \tag{59}$$

The expression for e^{ν} along with Eqs. (56)-(59) represent an exact analytic solution to the system Eqs. (44)-(46). This solution represent the **Tolman IV anisotropic solution**, or what is the same, the Tolman IV solution minimally deformed by the anisotropic source $\theta_{\mu\nu}$.

Applications: From perfect to anisotropic fluids



Under the MGD approach, each perfect fluid solution can be consistently extended to the anisotropic domain.

Applications: Einstein-Klein-Gordon

$$-k^2\left[\rho+\alpha\,\theta_0^0\right]=-\frac{1}{r^2}+e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)\ , \tag{60}$$

$$-\,k^2\left[-p+\alpha\,\theta_1^1\right]=-\frac{1}{r^2}+e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right)\ , \tag{61}$$

$$-\,k^2\left[-p+\alpha\,\theta_2^2\right]=\frac{1}{4}e^{-\lambda}\left[2\,\nu''+\nu'^2-\lambda'\,\nu'+2\,\frac{\nu'-\lambda'}{r}\right]\ , \tag{62}$$

$$\theta_0^0=\frac{1}{2}e^{-\lambda}\psi'^2+V\ , \tag{63}$$

$$\theta_1^1=-\frac{1}{2}e^{-\lambda}\psi'^2+V\ , \tag{64}$$

$$\theta_2^2=\frac{1}{2}e^{-\lambda}\psi'^2+V\ . \tag{65}$$

with $\psi''+\left[\frac{2}{r}+\frac{1}{2}(\nu'-\lambda')\right]\psi'=e^{\lambda}\frac{d\,V}{d\,\psi}\ .$ (66)

Applications: Einstein-Klein-Gordon: Vacuum

Set (A): $\rho = p = 0 \rightarrow$ Schwarzschild. Set (B): $\{f^*, \psi, V\}$

$$-k^2\left(\frac{1}{2}e^{-\lambda}\psi'^2 + V\right) = \frac{f^*}{r^2} + \frac{f^{*'}}{r}\ , \tag{67}$$

$$-\, k^2\left(-\frac{1}{2}e^{-\lambda}\psi'^2 + V\right) = f^*\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)\ , \tag{68}$$

$$-\, k^2\left(\frac{1}{2}e^{-\lambda}\psi'^2 + V\right) = \frac{f^*}{4}\left(2\nu'' + \nu'^2 + 2\frac{\nu'}{r}\right) + \frac{f^{*'}}{4}\left(\nu' + \frac{2}{r}\right)\ .$$

$$e^{-\lambda} = \left(1 - \frac{2\,M}{r}\right)\left(1 + \frac{\alpha\,M^2}{r^2\left(1 - \frac{3\,M}{r}\right)^2}\right)\ , \tag{69}$$

$$V = \frac{\alpha\,M}{k^2\,r^5\left(1 - \frac{3\,M}{r}\right)^3}\ , \tag{70}$$

$$\psi'^2 = \frac{2\alpha}{k^2\,r^2\left(1 - \frac{3\,M}{r}\right)\left[\alpha + r^2\left(1 - \frac{2\,M}{r}\right)\right]}\ . \tag{71}$$

Horizon $h = 2M$. If $\alpha \rightarrow$-α, $h = (3+\alpha)M$, but unphysical ψ

MGD extended (MGDe-Decoupling)

Let us start by considering a solution with $\alpha = 0$, namely, a perfect fluid solution $\{\xi, \mu, \rho, p\}$, where ξ and μ are the metric functions

$$ds^2 = e^{\xi(r)}\, dt^2 - \mu(r)^{-1}\, dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \ . \tag{72}$$

Now let us turn on the parameter α to consider the effects of the source $\theta_{\mu\nu}$ on the perfect fluid solution $\{\xi, \mu, \rho, p\}$.

$$\xi \rightarrow \nu = \xi + \alpha\, g \ , \tag{73}$$

$$\mu \rightarrow e^{-\lambda} = \mu + \alpha\, f . \tag{74}$$

where f and g are, respectively, the **geometric deformations** undergone by the perfect fluid geometry $\{\xi, \mu\}$.

The Set (B) may be formally identified as Einstein equations for an anisotropic system with energy-momentum tensor $\theta^*_{\mu\nu}$ defined as

$$\begin{aligned} k^2\,\theta^{*\ \nu}_{\mu} &= k^2\,\theta_{\mu}^{\nu} + \frac{1}{r^2}\delta_{\mu}^{\ 0}\,\delta_{0}^{\ \nu} + \left(Z_1 + \frac{1}{r^2}\right)\delta_{\mu}^{\ 1}\,\delta_{1}^{\ \nu} \\ &\quad + Z_2\left(\delta_{\mu}^{\ 2}\,\delta_{2}^{\ \nu} + \delta_{\mu}^{\ 3}\,\delta_{3}^{\ \nu}\right)\ , \end{aligned} \tag{75}$$

with conservation equation

$$(\theta^{*\ 1}_{1})' - \frac{\nu'}{2}(\theta^{*\ 0}_{0} - \theta^{*\ 1}_{1}) - \frac{2}{r}(\theta^{*\ 2}_{2} - \theta^{*\ 1}_{1}) = 0\ , \tag{76}$$

and metric

$$g^*_{\mu\nu} \to ds^2 = e^{\alpha\, g(r)}\,dt^2 - \frac{dr^2}{f(r)} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\ . \tag{77}$$

Decoupling Einstein-Maxwell

$\theta_{\mu\nu}$ represents the Maxwell energy-momentum tensor

$$\theta_{\mu\nu} = F_{\mu\alpha} F^{\alpha}_{\ \nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{78}$$

where $F_{\mu\nu}$ in Eq. (78) satisfies Maxwell's equations

$$\nabla_\nu \left[(-g)^{1/2} F^{\mu\nu} \right] = (-g)^{1/2} j^\mu \tag{79}$$

and its respective Bianchi identity. Only $F_{01} = E_r$ is non-vanishing.

$$-\mathcal{E}^2 = \frac{\alpha f}{r^2} + \frac{\alpha f'}{r} \ , \tag{80}$$

$$-\mathcal{E}^2 - \alpha Z_1 = \alpha f \left(\frac{1}{r^2} + \frac{\alpha g'}{r} \right) \ , \tag{81}$$

$$\mathcal{E}^2 - \alpha Z_2 = \frac{\alpha f}{4} \left(2 \alpha g'' + (\alpha g')^2 + 2 \frac{\alpha g'}{r} \right) + \frac{\alpha f'}{4} \left(\alpha g' + \frac{2}{r} \right) , \tag{82}$$

$$\frac{\alpha g'}{2 r} (\mu' - \mu \xi') + (\mathcal{E}^2)' + \frac{4 \mathcal{E}^2}{r} = 0 \ , \tag{83}$$

where $\mathcal{E} = |\vec{E}|$ is the electric field intensity.

Decoupling Einstein-Maxwell: Electrovacuum

If we already have a perfect fluid solution $\{\rho, p, \xi, \mu\}$ for the Set (A), then we end up with three unknown functions $\{\mathcal{E}, f, g\}$ to be determined by Set (B).
The simplest situation: the electrovacuum region defined by $r > R$; $\rho = p = 0$.
Set (A): the Schwarzschild solution; Set (B): $\{\mathcal{E}, f, g\}$.
Since for the Schwarzschild solution $e^{\xi} = \mu$, then

$$\underbrace{\frac{\alpha\, g'}{2\, r}\left(\mu' - \mu\, \xi'\right)}_{0} + (\mathcal{E}^2)' + \frac{4\mathcal{E}^2}{r} = 0 \;, \tag{84}$$

$$\rightarrow \mathcal{E} = \frac{Q}{r^2} \;, \tag{85}$$

with Q a constant which eventually is identified as the total electric charge.

By using Eq. (85) in Eq. (80) we obtain $f(r)$ as

$$\alpha\, f(r) = \frac{c_1}{r} + \frac{Q^2}{r^2}\ , \tag{86}$$

with c_1 a constant. Combining Eqs. (80) and (81) we obtain

$$\alpha\, g'(r) = \frac{\alpha\, f' - \alpha\, f\, \xi'}{\mu + \alpha\, f} \tag{87}$$

which can easily be integrated, yielding to

$$e^{\alpha\, g} = c_2 \left(1 - \frac{2\,M}{r}\right)^{-1} \left(1 - \frac{2\,M}{r} + \frac{c_1}{r} + \frac{Q^2}{r^2}\right)\ , \tag{88}$$

with c_2 an integration constant, which can be taken as 1 without loss of generality.

The "deformed" Schwarzschild, according to Eqs. (73) and (74), is given by

$$ds^2 = \left(1 - \frac{2\,M}{r}\right)\, e^{\alpha\, g(r)}\, dt^2 - \left(1 - \frac{2\,M}{r} + \alpha\, f(r)\right)^{-1} dr^2 - r^2 d\Omega^2\,. \tag{89}$$

Using the expressions in Eq. (86) and Eq. (88) in the metric shown in Eq. (89) we obtain the well known Reissner-Nordstrom solution, where we can take $c_1 = 0$ or just to consider $\mathcal{M} \equiv M - \frac{c_1}{2}$.

$$ds^2 = \left(1 - \frac{2\,\mathcal{M}}{r} + \frac{Q}{r^2}\right)\, dt^2 - \left(1 - \frac{2\,\mathcal{M}}{r} + \frac{Q}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2\,. \tag{90}$$

Conclusions and outlook

- The first simple, systematic and direct approach to decoupling gravitational sources in general relativity.
- Every GR perfect fluid solution can be consistently extended to the anisotropic domain.
- We can solve Einstein + KG in an easy way. For instance we can find the scalar field ψ consistent with a physically acceptable interior solution.
- The MGD-decoupling can be consistently extended when both metric functions are deformed: MGDe-decoupling.
- By using the MGDe-decoupling, we find a new way to try the Einstein-Maxwell system. This is particularly useful to find physically acceptable interior solutions. The electrovacuum solution is reproduced,
- Extension beyond the spherical symmetry? Time-dependent? Extra-dimensional theories? Lovelock, Horndeski, etc.

Thanks!