Gravitational Collapse
of a Homogeneous Scalar Field
Coupled Kinematically to Einstein Tensor

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Overview

1 Introduction
   - Why generalize General Relativity?
   - Horndeski Theory - The NMDC case

2 Gravitational Collapse of a Scalar Field with NMDC
   - Gravitational collapse of a scalar field (*how to...*)
   - A simple example
   - The NMDC case
   - Boundary surface formation

3 Conclusions
1 Introduction
   - Why generalize General Relativity?
   - Horndeski Theory - The NMDC case

2 Gravitational Collapse of a Scalar Field with NMDC
   - Gravitational collapse of a scalar field \((how\ to...)\)
   - A simple example
   - The NMDC case
   - Boundary surface formation

3 Conclusions
GR is, itself, a generalization of *Newton’s Theory of Gravity*.

- **Hot Big Bang problems:**
  - **Flatness Problem:** $\Omega - 1 = \frac{K}{a^2 H^2}$ ← decreasing function
  - **Horizon Problem:** $a \lambda, a \sim t^p, 0 < p < 1 \Rightarrow \frac{dH}{dt} \sim t^{-1/3} \approx 10^{-2}$.
  - **Unwanted Relics:** Energy density of *forbidden* particles $\rho_m \sim a^{-3}$ while we observe $\rho_r \sim a^{-4}$ radiation.

- Accelerating expansion is driven by a new energy density component with negative pressure, termed *Dark Energy* ($\Omega E$).
  - Possible origin: *positive cosmological constant* $\Lambda \Rightarrow \Lambda CDM$ model
  - Unknown origin
  - Fine Tuning Problem: $\rho_\Lambda = \frac{\Lambda}{8\pi G} = 6.72 \times 10^{-24} \text{ g m}^{-3}$ for $\Omega E$
    - while at early epochs (Plank scale) $\rho_{Pl} = 5.16 \times 10^{99} \text{ g m}^{-3}$
    - $\frac{\rho_\Lambda}{\rho_{Pl}} \sim 10^{-123}$!

- The energy component giving rise to the distribution of *large scale structure* is assumed to be pressureless and termed *Dark Matter* ($DM$).
  - Introduced “*by hand*” in order to fit observational data.

- 1971 (Hawking): *Primordial black holes* $\sim 10^{-8} \text{ kg}$ (Planck relics)
GR is, itself, a generalization of Newton’s Theory of Gravity.

Hot Big Bang problems:

\[ \text{Flatness Problem: } \Omega - 1 = \frac{K}{a^2 H^2} \leftarrow \text{decreasing function} \]

\[ \text{Horizon Problem: } a \ll a \sim t^p, 0 < p < 1 \implies \frac{dH(t_{\text{dec}})}{dH(t_0)} \approx \left( \frac{t_{\text{dec}}}{t_0} \right)^{1/3} \approx 10^{-2}. \]

\[ \text{Unwanted Relics: } \text{Energy density of forbidden particles } \rho_m \sim a^{-3} \text{ while we observe } \rho_r \sim a^{-4} \text{ radiation.} \]

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Simplest generalization: introduction of a **scalar field**.

- Inflaton, with a *strange* equation of state $\rho + 3P < 0 \equiv\text{INFLATION.}$
  

- **Inflation** ⇔ $\ddot{a} > 0$ ⇔ $\frac{d}{dt} \left(\frac{1}{aH}\right) < 0$ as a solution to:
  
  - ✓ **Flatness Problem**: $\Omega - 1 = \frac{K}{a^2 H^2}$ \leftarrow \text{increasing function}
  
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  - ✓ **Unwanted Relics**: Energy density of forbidden particles $\rho_m \sim a^{-3}$ while the energy density of the Universe decreases slowly $\rho \sim a^{-2/p}$.

Theories of inflation and present accelerating expansion:

- ▶ Quintessence: $\mathcal{L} = X - V(\phi)$  ▶ k-essence: $\mathcal{L} = K(\phi, X)$

- ▶ $f(R)$ Gravity: $\mathcal{L} = \frac{M_{Pl}^2}{2} f(R)$  ▶ Scalar-Tensor Gravity: $\mathcal{L} = F(\phi) R + K(\phi, X)$

- ▶ Galileon Gravity: EoM with *shift symmetry* $\phi \rightarrow \phi + b_\mu x^\mu$. *Dropping* the shift symmetry $\rightarrow$ *rediscovery* of the **Horndeski Theory** (Nicolis et al. 2009, Deffayet et al. 2009)

- ▶ **Horndeski Theory**: The most general *Second-Order Scalar Field Equations in 4d.*
  
  - **naked singularity** ×
  
  (Choptuik 1993, Christodoulou 1994)

- **Gravitational Collapse of a scalar field**
  
  - **black hole formation** ✓
  
  (Goswami, Joshi 2008, Baier et al. 2015)
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- **Gravitational Collapse of a scalar field**: *redumbly singular* $\times$

  (Choptuik 1993, Christodoulou 1994)

- **black hole formation** $\checkmark$

  (Goswami, Joshi 2006, Baier et al. 2015)
Introduction

Gravitational Collapse of a Scalar Field with NMDC

Conclusions

Why generalize General Relativity?

Horndeski Theory - The NMDC case

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• Gravitational Collapse of a scalar field $\nrightarrow$ naked singularity $\times$
  (Choptuik 1993, Christodoulou 1994)

• black hole formation $\checkmark$
  (Goswami, Joshi 2006, Baier et al. 2015)
By defining $X = -\nabla_\mu \phi \nabla^\mu \phi / 2$ the Horndeski Lagrangian reads $\mathcal{L} = \sum_{i=2}^{5} \mathcal{L}_i$, with

$\mathcal{L}_2 = K(\phi, X),$ \\
$\mathcal{L}_3 = -G_3(\phi, X) \Box \phi,$ \\
$\mathcal{L}_4 = G_4(\phi, X) R + G_{4,X} [(\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi) (\nabla^\mu \nabla^\nu \phi)],$ \\
$\mathcal{L}_5 = G_5(\phi, X) G_{\mu\nu} (\nabla^\mu \nabla^\nu \phi) - \frac{1}{6} G_{5,X} [(\Box \phi)^3 - 3(\Box \phi) (\nabla_\mu \nabla_\nu \phi) (\nabla^\mu \nabla^\nu \phi) + 2(\nabla^\mu \nabla_\alpha \phi) (\nabla^\alpha \nabla^\beta \phi) (\nabla^\beta \nabla_\mu \phi)].$

**Non-Minimal Derivative Coupling:** $G_5(\phi, X) = -\lambda_5 \phi \rightarrow \mathcal{L}_5 = \lambda_5 G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi$

- *Cosmological scenarios with NMDC* (Saridakis & Sushkov 2010)
- *Successful Higgs inflation scenario* (Germani & Kehagias 2010)
- *Observational tests of inflation with NMDC* (Tsujikawa 2012)
- *Perturbation analysis and observational constraints in cosmology with NMDC* (Dent, Dutta, Saridakis, Xia 2013)
- *Black holes with NMDC* (Rinaldi 2012)
- *Which do not have hair* (Hui & Nicolis 2012)
- *Unless they do* (Kolyvaris, Koutsoumbas, Papantonopoulos 2012 / Babichev & Charmousis 2013)
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Section 2

Gravitational Collapse of a Scalar Field with NMDC

- Gravitational collapse of a scalar field (how to...)
- A simple example
- The NMDC case
- Boundary surface formation

Conclusions
Friedmann equation of a *homogeneous scalar field* in an FRW metric

\[
\dot{a}^2 = \frac{8\pi}{3} a^2 \rho
\]

Initial data: \(a(0) = a_0, \phi(0) = \phi_0, \dot{\phi}(0) = \dot{\phi}_0\) \(\rightarrow\) *singular* state at \(a = 0\)

*If* \(a(t)\) is a monotonically decreasing function of \(t\).

\[
\rho = \frac{3}{8\pi} \left( \frac{\psi(a)}{a} \right)^2, \quad \psi(a) > 0 \quad \Rightarrow \quad \dot{a} = -\psi(a)
\]

\[
\psi(a) = a^{-\nu} \quad \text{(Goswami, Joshi 2004)}
\]

An *apparent horizon* is defined by:

\[
R(r, t_h) = 2m(r, t_h)
\]

and it is the *boundary* of the *trapped* region

\[
\mathcal{T} = \{(r, t) : R(r, t) \leq 2m(r, t)\}
\]

*\(R\): *areal radius* of the metric

*\(m\): Misner-Sharp mass defined as

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m \equiv \frac{R}{2} \left(1 - \nabla_\mu R \nabla^\mu R\right).
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(Giambo 2005):

**Proposition**

If \( \psi(a) \) is bounded in \((0, a_0)\), there exists \( r_b > 0 \) such that, for any shell of matter \( r \leq r_b \), no apparent horizon forms during the evolution.

**Proposition**

If \( \lim_{a \to 0^+} \psi(a) = +\infty \), for any \( r > 0 \) such that the initial data are taken outside the trapped region \( \mathcal{T} \), the shell labeled \( r \) becomes trapped strictly before it becomes singular, and so a black hole forms.

- **Aim** is to match the scalar field spacetime with an exterior solution at \( r = r_b \)
- **Israel-Darmois junction conditions**: Continuity across a hypersurface \( \Sigma \) of the first and second fundamental forms induced on \( \Sigma \) by the two spacetimes

\[
I_\Sigma = g_{\alpha\beta} \frac{dx^\alpha}{dy^a} \frac{dx^\beta}{dy^b} \, dy^a dy^b, \quad II_\Sigma = -n_\alpha \left( \frac{\partial^2 x^\alpha}{\partial y^a \partial y^b} + \Gamma^\alpha_{\rho\sigma} \frac{\partial x^\rho}{\partial y^a} \frac{\partial x^\sigma}{\partial y^b} \right)
\]

**Theorem**

If \( \psi(a) \) is bounded and there is \( r_b \) such that \( 1 - \psi^2(a) r^2 \) is bounded away from 0, then the boundary \( \Sigma = \{ r = r_b \} \) of the scalar field collapses to a naked singularity.
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Giambò 2005:

\[ \psi(a) = \sqrt{\frac{8\pi}{3}} a^\beta, \quad -2 < \beta < 1 \]

\[
    a(t) = \left( a_0^{1-\beta} - \sqrt{\frac{8\pi}{3}} (1 - \beta) t \right)^{\frac{1}{1-\beta}} \Rightarrow \dot{a} = -\sqrt{\frac{8\pi}{3}} \left( a_0^{1-\beta} - \sqrt{\frac{8\pi}{3}} (1 - \beta) t \right)^{\frac{\beta}{1-\beta}}
\]

i) \quad 0 \leq \beta < 1: \psi(a) \text{ bounded. No apparent horizon forms}

ii) \quad -2 \leq \beta < 0: \lim_{a \to 0} \psi(a) = +\infty. \text{ apparent horizon existance}

\[
    S = \int d^4x \sqrt{-g} \left[ \frac{R - 2\Lambda}{16\pi G} - \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi \right]
\]

\[
    a(t) = \left( \frac{\cosh \left( \sqrt{3|\Lambda|}(t - C) \right)}{\cosh \left( \sqrt{3|\Lambda|}C \right)} \right)^{1/3} > 0 \quad \text{but for} \quad \Lambda < 0 \quad a(t) = \left( \frac{\cos \left( \sqrt{3|\Lambda|}(t - C) \right)}{\cos \left( \sqrt{3|\Lambda|}C \right)} \right)^{1/3}
\]

\[
    \dot{a} \left( t_s = C + \frac{\pi}{2\sqrt{3|\Lambda|}} \right) = \left. \frac{\sqrt{|\Lambda|} \sin \left( \sqrt{3|\Lambda|}(t - C) \right)}{\cos^{2/3} \left( \sqrt{3|\Lambda|}(t - C) \right)} \right|_{t_s = C + \frac{\pi}{2\sqrt{3|\Lambda|}}} = -\infty
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**Giambo 2005:**

\[ \psi(a) = \sqrt{\frac{8\pi}{3}} a^\beta, \quad -2 < \beta < 1 \]

\[ a(t) = \left( a_0^{1-\beta} - \sqrt{\frac{8\pi}{3}} (1 - \beta) t \right) \frac{1}{1 - \beta} \Rightarrow \dot{a} = -\sqrt{\frac{8\pi}{3}} \left( a_0^{1-\beta} - \sqrt{\frac{8\pi}{3}} (1 - \beta) t \right) \frac{\beta}{1 - \beta} \]

i) \( 0 \leq \beta < 1 \): \( \psi(a) \) **bounded.** No apparent horizon forms

ii) \( -2 \leq \beta < 0 \): \( \lim_{a \to 0} \psi(a) = +\infty \). apparent horizon exisance

\[ S = \int d^4x \sqrt{-g} \left[ \frac{R - 2\Lambda}{16\pi G} - \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi \right] \]

\[ a(t) = \left( \frac{\cosh \left( \sqrt{3|\Lambda|} (t - C) \right)}{\cosh \left( \sqrt{3|\Lambda|} C \right)} \right)^{1/3} \quad \text{but for } \Lambda < 0 \]

\[ \dot{a} \left( t_s = C + \frac{\pi}{2\sqrt{3|\Lambda|}} \right) = \frac{\sqrt{|\Lambda|}}{\sqrt{3} \cos \left( \sqrt{3\Lambda} C \right)} \left. \frac{\sin \left( \sqrt{3|\Lambda|} (t - C) \right)}{\cos^{2/3} \left( \sqrt{3|\Lambda|} (t - C) \right)} \right|_{t_s=C+\frac{\pi}{2\sqrt{3|\Lambda|}}} = -\infty \]
Gravitational Collapse of a Scalar Field with NMDC

Conclusions

Gravitational collapse of a scalar field \((\text{how to...})\)

A simple example

The NMDC case

Boundary surface formation

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{R - 2\Lambda}{16\pi G} - \left[ \frac{1}{2} g^{\mu\nu} - \frac{1}{2} \lambda \mathcal{G}^{\mu\nu} \right] \partial_\mu \phi \partial_\nu \phi \right\} \]

Klein-Gordon:
\[ \left( 1 + \frac{3\lambda \dot{a}(t)}{a^2(t)} \right) \ddot{\phi}(t) + \left( 3 \frac{\dot{a}(t)}{a(t)} + 3\lambda \left( \frac{\dot{a}^3(t)}{a^3(t)} + 2 \frac{\dot{a}(t)\ddot{a}(t)}{a^2(t)} \right) \right) \dot{\phi}(t) = 0 \]

Friedmann:
\[ \frac{3\dot{a}^2(t)}{a^2(t)} - \Lambda = 4\pi \left( 1 + 9\lambda \frac{\dot{a}^2(t)}{a^2(t)} \right) \phi^2(t) \]

\[ \Lambda = -0.0015 \]

\[ \lambda_1 = 0 \]

\[ \lambda_2 = 32 \]

\[ \lambda_3 = 70 \]

or approximately near the singularity:
\[ a(t) \sim \left( t_s - t \right)^{\frac{2}{3}} \quad \Rightarrow \quad \dot{a}(t) \sim -\left( t_s - t \right)^{-\frac{1}{3}} \]

- Stronger coupling \(\lambda\) \textit{increases} the collapsing time.
- End state of the collapse \textit{does not} depend on \(\lambda\).
- An apparent horizon \textit{always} forms \(\rightarrow\) \textit{Black hole} formation.

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FRW metric

\[ ds_{\text{int}}^2 = -dt^2 + a^2(t)dr^2 + r^2a^2(t)d\Omega^2 \]

matched with a Schwarzschild-AdS\(_4\) metric

\[ ds_{\text{ext}}^2 = -\chi(Y)dT^2 + \chi(Y)^{-1}dY^2 + Y^2d\Omega^2, \quad \chi(Y) = 1 - \frac{2M}{Y} + \frac{Y^2}{l^2} \]

First fundamental forms

- \( I_{\Sigma}^{\text{int}} = -d\tau^2 + r_b^2a^2(\tau)d\Omega^2 \)
- \( I_{\Sigma}^{\text{ext}} = \left( -\chi(Y)\dot{T}^2 + \frac{1}{\chi(Y)}\dot{Y}^2 \right)d\tau^2 + Y^2(\tau)d\Omega^2 \)

Second fundamental forms

- \( II_{\Sigma}^{\text{int}} = 0 \cdot d\tau^2 + r_ba(\tau)d\Omega^2 \)
- \( II_{\Sigma}^{\text{ext}} = -\frac{\chi^2(Y)\chi(Y),Y\dot{T}^3 + 3\chi(Y),Y\dot{Y}^2 + 2\chi(Y)\left(\dot{Y}\dot{T} - \dot{T}\dot{Y}\right)}{2\chi(Y)}d\tau^2 + Y\chi(Y)\dot{T}d\Omega^2 \)
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**First fundamental forms**

- $I^\text{int}_\Sigma = -d\tau^2 + r_b^2 a^2(\tau) d\Omega^2$
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**Second fundamental forms**

- $II^\text{int}_\Sigma = 0 \cdot d\tau^2 + r_b a(\tau) d\Omega^2$
- $II^\text{ext}_\Sigma = \frac{-\chi^2(Y)\chi(Y),Y\dot{T}^3 + 3\chi(Y),Y\dot{T}\dot{Y}^2 + 2\chi(Y) (\dot{Y}\ddot{T} - \dot{T}\ddot{Y})}{2\chi(Y)} d\tau^2 + Y\chi(Y)\dot{T} d\Omega^2$

**Matched first fundamental forms**

$\left(-\chi(Y)\dot{T}^2 + \frac{1}{\chi(Y)} \dot{Y}^2\right) = -1 \quad \Rightarrow \quad M(\tau) = \frac{1}{2} r_b^3 a(\tau) \left(\dot{a}^2(\tau) + \frac{a^2(\tau)}{l^2}\right)$

- $Y(\tau) = r_b a(\tau) \quad \Rightarrow \quad \text{boundary of the collapsing shell}$

**Matched second fundamental forms**

\[-\frac{\chi^2(Y)\chi(Y),Y\dot{T}^3 + 3\chi(Y),Y\dot{T}\dot{Y}^2 + 2\chi(Y) (\dot{Y}\ddot{T} - \dot{T}\ddot{Y})}{2\chi(Y)} = 0 \quad \rightarrow \quad \text{holds identically}\]

- $Y\chi(Y)\dot{T} = r_b a(\tau) \quad \Rightarrow \quad T_s = \int_{\tau_c}^{\tau_s} \frac{d\tau}{1 - r_b^2 \dot{a}^2(\tau)} \quad \Rightarrow \quad \text{no naked singularity}$
Introduction

Why generalize General Relativity?
- Horndeski Theory - The NMDC case

Gravitational Collapse of a Scalar Field with NMDC
- Gravitational collapse of a scalar field (how to...)
- A simple example
- The NMDC case
- Boundary surface formation

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Gravitational Collapse of a Scalar Field with NMDC

Conclusions

- $Y(\tau) = r_b a(\tau)$, $M(\tau) = \frac{1}{2} r_b^3 a(\tau) \left( \dot{a}^2(\tau) + \frac{a^2(\tau)}{l^2} \right)$
- collapsing shell becomes trapped $\Rightarrow$ **black hole** forms
- Black hole mass depending on $\lambda$ ($M \sim 10^{-8}$ kg)
- Black hole “size” depending on $\lambda$
Conclusions

- **NMDC** acts as a friction term on the collapsing process.
- **Black hole** formation in the absence of a potential \((M \sim M_{pl})\)
- The presence of **NMDC** allows the formation of “heavier” and “larger” black holes.
- Studying additional terms of the Horndeski Lagrangian opens the road for new cosmological models.

*open question*

- Which combination of Horndeski terms could possibly drive the collapse of a scalar field?