Gravitational Collapse of a Homogeneous Scalar Field Coupled Kinematically to Einstein Tensor

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Overview

Section 1

- *GR* is, itself, a generalization of *Newton's* Theory of Gravity.
- Hot Big Bang problems:
  - ✗ *Flatness Problem*: $\Omega - 1 = \frac{K}{a^2 H^2}$ ← decreasing function
  - ✗ *Horizon Problem*: $\begin{array}{c} a\lambda, a \sim t^p, 0 < p < 1 \\ H^{-1} \sim t \end{array} \Rightarrow \frac{d_H(t_{dec})}{d_H(t_0)} \approx \left(\frac{t_{dec}}{t_0}\right)^{1/3} \approx 10^{-2}$.
  - ✗ *Unwanted Relics*: Energy density of *forbidden* particles $\rho_m \sim a^{-3}$ while we observe $\rho_r \sim a^{-4}$ radiation.
- *Accelerating expansion* is driven by a new energy density component with negative pressure, termed *Dark Energy* (*DE*).
  - ✓ Possible origin: positive cosmological constant $\Lambda \Rightarrow$ **ΛCDM** model
    - ✗ Uknown origin
    - ✗ Fine Tuning Problem: $\rho_\Lambda = \frac{\Lambda}{8\pi G} = 6.72 \times 10^{-24} g\, m^{-3}$ for *DE* while at early epochs (Plank scale) $\rho_{Pl} = 5.16 \times 10^{99} g\, m^{-3} \Rightarrow \frac{\rho_\Lambda}{\rho_{Pl}} \sim 10^{-123}$!
- The energy component giving rise to the distribution of *large scale structure* is assumed to be pressureless and termed *Dark Matter* (*DM*).
  - ✗ Introduced "*by hand*" in order to fit observational data.
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- Inflaton, with a *strange* equation of state $\rho + 3P < 0 \equiv$ INFLATION.

(Guth 1981, Albrecht and Steinhardt 1982, Linde 1982,1983)

- *Inflation* $\Leftrightarrow \ddot{a} > 0 \Leftrightarrow \frac{d}{dt}\left(\frac{1}{aH}\right) < 0$ as a *solution* to:
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- Theories of inflation and present accelerating expansion:
  - ▷ Quintessence: $\mathcal{L} = X - V(\phi)$ ▷ k-essence: $\mathcal{L} = K(\phi, X)$
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  - ▷ Galileon Gravity: EoM with *shift symmetry* $\phi \to \phi + b_\mu x^\mu$. *Dropping* the shift symmetry → *rediscovery* of the Horndeski Theory (Nicolis *et al.* 2009, Deffayet *et al.* 2009)
  - ▷ *Horndeski Theory*: The most general *Second-Order Scalar Field Equations in 4d*.
- Gravitational Collapse of a *scalar field*
  - ↗ naked singularity ✗ (Choptuik 1993, Christodoulou 1994)
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By defining $X = -\nabla_\mu\phi\nabla^\mu\phi/2$ the Horndeski Lagrangian reads $\mathcal{L} = \sum_{i=2}^{5}\mathcal{L}_i$, with

$$\mathcal{L}_2 = K(\phi, X),$$
$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi,$$
$$\mathcal{L}_4 = G_4(\phi, X)\,R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)\,(\nabla^\mu\nabla^\nu\phi)\right],$$
$$\mathcal{L}_5 = G_5(\phi, X)\,G_{\mu\nu}\,(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}\,G_{5,X}\left[(\Box\phi)^3 - 3(\Box\phi)\,(\nabla_\mu\nabla_\nu\phi)\,(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)\,(\nabla^\alpha\nabla_\beta\phi)\,(\nabla^\beta\nabla_\mu\phi)\right].$$

Non-Minimal Derivative Coupling: $G_5(\phi, X) = -\lambda_5\phi \rightarrow \mathcal{L}_5 = \lambda_5 G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi$

- *Cosmological scenarios with NMDC* (Saridakis & Sushkov 2010)
- *Successful Higgs inflation scenario* (Germani & Kehagias 2010)
- *Observational tests of inflation with NMDC* (Tsujikawa 2012)
- *Perturbation analysis and observational constraints in cosmology with NMDC* (Dent,Dutta,Saridakis,Xia 2013)
- *Black holes with NMDC* (Rinaldi 2012)
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Section 2

Friedmann equation of a *homogeneous scalar field* in an *FRW* metric

$$\dot{a}^2 = \frac{8\pi}{3} a^2 \rho$$

Initial data : $a(0) = a_0,\ \phi(0) = \phi_0,\ \dot{\phi}(0) = \dot{\phi}_0 \quad \rightarrow \quad$ *singular* state at $a = 0$

If $a(t)$ is a *monotonically decreasing* function of t.

$$\rho = \frac{3}{8\pi}\left(\frac{\psi(a)}{a}\right)^2, \quad \psi(a) > 0 \qquad \Rightarrow \qquad \dot{a} = -\psi(a)$$

$\psi(a) = a^{-\nu}$ (Goswami, Joshi 2004)

An *apparent horizon* is defined by :

$$R(r, t_h) = 2m(r, t_h)$$

and it is the *boundary* of the *trapped* region

$$\mathcal{T} = \{(r, t) : R(r, t) \leq 2m(r, t)\}$$

R: *areal radius* of the metric
m: *Misner-Sharp mass* defined as $m \equiv \frac{R}{2}(1 - \nabla_\mu R \nabla^\mu R)$.

Friedmann equation of a *homogeneous scalar field* in an *FRW* metric

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(Giambo 2005):

Proposition

If $\psi(a)$ is bounded in $(0, a_0)$, there exists $r_b > 0$ such that, for any shell of matter $r \leq r_b$, no apparent horizon forms during the evolution.

Proposition

If $\lim_{a\to 0^+} \psi(a) = +\infty$, for any $r > 0$ such that the initial data are taken outside the trapped region $\mathcal{T}$, the shell labeled r becomes trapped strictly before it becomes singular, and so a black hole forms.

- Aim is to ***match*** the scalar field spacetime with an exterior solution at $r = r_b$
- *Israel-Darmois junction conditions*: Continuity across a *hypersurface* Σ of the **first** and **second fundamental forms** induced on Σ by the two spacetimes

$$I_\Sigma = g_{\alpha\beta}\frac{dx^\alpha}{dy^a}\frac{dx^\beta}{dy^b}dy^a dy^b\ , \quad II_\Sigma = -n_\alpha\left(\frac{\partial^2 x^\alpha}{\partial y^a \partial y^b} + \Gamma^\alpha_{\rho\sigma}\frac{\partial x^\rho}{\partial y^a}\frac{\partial x^\sigma}{\partial y^b}\right)$$

Theorem

If $\psi(a)$ is bounded and there is r_b such that $1 - \psi^2(a)r^2$ is bounded away from 0, then the boundary $\Sigma = \{r = r_b\}$ of the scalar field collapses to a naked singularity

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Giambo 2005: $$\psi(a) = \sqrt{\frac{8\pi}{3}}a^{\beta}, \quad -2 < \beta < 1$$

$$a(t) = \left(a_0^{1-\beta} - \sqrt{\frac{8\pi}{3}}(1-\beta)t\right)^{\frac{1}{1-\beta}} \Rightarrow \dot{a} = -\sqrt{\frac{8\pi}{3}}\left(a_0^{1-\beta} - \sqrt{\frac{8\pi}{3}}(1-\beta)t\right)^{\frac{\beta}{1-\beta}}$$

- *i*) $0 \leq \beta < 1 : \psi(a)$ *bounded*. No apparent horizon forms
- *ii*) $-2 \leq \beta < 0 : \lim_{a\to 0}\psi(a) = +\infty$. apparent horizon existance

$$S = \int d^4x\,\sqrt{-g}\left[\frac{R-2\Lambda}{16\pi G} - \frac{1}{2}g_{\mu\nu}\partial^{\mu}\phi\partial^{\nu}\phi\right]$$

$$a(t) = \left(\frac{\cosh\left(\sqrt{3\Lambda}(t-C)\right)}{\cosh\left(\sqrt{3\Lambda}C\right)}\right)^{1/3} > 0 \quad \text{but for } \Lambda < 0 \quad a(t) = \left(\frac{\cos\left(\sqrt{3|\Lambda|}(t-C)\right)}{\cos\left(\sqrt{3|\Lambda|}C\right)}\right)^{1/3}$$

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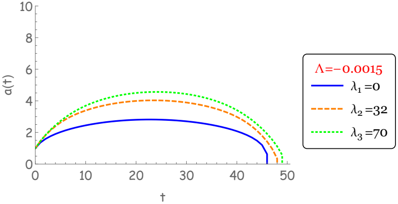
$$a(t) = \left(\frac{\cosh\left(\sqrt{3\Lambda}(t-C)\right)}{\cosh\left(\sqrt{3\Lambda}C\right)}\right)^{1/3} > 0 \quad \text{but for } \Lambda < 0 \quad a(t) = \left(\frac{\cos\left(\sqrt{3|\Lambda|}(t-C)\right)}{\cos\left(\sqrt{3|\Lambda|}C\right)}\right)^{1/3}$$

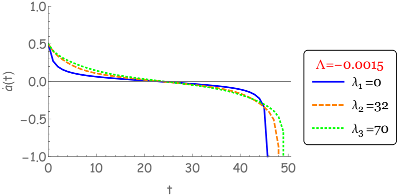
$$\dot{a}\left(t_s = C + \frac{\pi}{2\sqrt{3|\Lambda|}}\right) = \frac{\sqrt{|\Lambda|}}{\sqrt{3}\cos\left(\sqrt{3\Lambda}(C)\right)} \frac{\sin\left(\sqrt{3|\Lambda|}(t-C)\right)}{\cos^{2/3}\left(\sqrt{3|\Lambda|}(t-C)\right)}\Bigg|_{t_s = C + \frac{\pi}{2\sqrt{3|\Lambda|}}} = -\infty$$

$$S=\int d^4x\sqrt{-g}\left\{\frac{R-2\Lambda}{16\pi G}-\left[\frac{1}{2}g^{\mu\nu}-\frac{1}{2}\lambda G^{\mu\nu}\right]\partial_\mu\phi\partial_\nu\phi\right\}$$

Klein-Gordon: $$\left(1+\frac{3\lambda\dot{a}^2(t)}{a^2(t)}\right)\ddot{\phi}(t)+\left(3\frac{\dot{a}(t)}{a(t)}+3\lambda\left(\frac{\dot{a}^3(t)}{a^3(t)}+2\frac{\dot{a}(t)\ddot{a}(t)}{a^2(t)}\right)\right)\dot{\phi}(t)=0$$

Friedmann: $$\frac{3\dot{a}^2(t)}{a^2(t)}-\Lambda=4\pi\left(1+9\lambda\frac{\dot{a}^2(t)}{a^2(t)}\right)\dot{\phi}^2(t)$$





or approximately *near the singularity*:

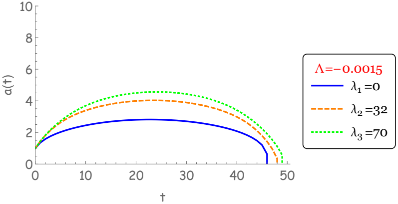
$$a(t)\sim(t_s-t)^{\frac{2}{3}}\quad\Rightarrow\quad\dot{a}(t)\sim-(t_s-t)^{-\frac{1}{3}}$$

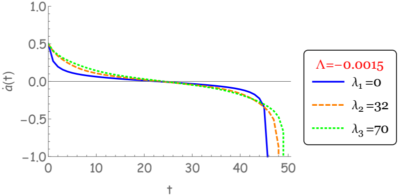
- Stronger coupling λ ***increases*** the collapsing time.
- End state of the collapse ***does not*** depend on λ.
- An *apparent horizon* always forms $\rightarrow$ ***Black hole*** formation.

$$S = \int d^4x\sqrt{-g}\left\{\frac{R-2\Lambda}{16\pi G} - \left[\frac{1}{2}g^{\mu\nu} - \frac{1}{2}\lambda G^{\mu\nu}\right]\partial_\mu\phi\partial_\nu\phi\right\}$$

Klein-Gordon: $$\left(1+\frac{3\lambda\dot{a}^2(t)}{a^2(t)}\right)\ddot{\phi}(t) + \left(3\frac{\dot{a}(t)}{a(t)} + 3\lambda\left(\frac{\dot{a}^3(t)}{a^3(t)} + 2\frac{\dot{a}(t)\ddot{a}(t)}{a^2(t)}\right)\right)\dot{\phi}(t) = 0$$

Friedmann: $$\frac{3\dot{a}^2(t)}{a^2(t)} - \Lambda = 4\pi\left(1+9\lambda\frac{\dot{a}^2(t)}{a^2(t)}\right)\dot{\phi}^2(t)$$





or approximately *near the singularity*:

$$a(t) \sim (t_s - t)^{\frac{2}{3}} \quad \Rightarrow \quad \dot{a}(t) \sim -(t_s - t)^{-\frac{1}{3}}$$

- Stronger coupling λ ***increases*** the collapsing time.
- End state of the collapse ***does not*** depend on λ.
- An *apparent horizon* always forms → ***Black hole*** formation.

FRW metric

$$ds^2_{int} = -dt^2 + a^2(t)dr^2 + r^2 a^2(t)d\Omega^2$$

matched with a Schwarzschild-AdS_4 metric

$$ds^2_{ext} = -\chi(Y)dT^2 + \chi(Y)^{-1}dY^2 + Y^2 d\Omega^2, \qquad \chi(Y) = 1 - \frac{2M}{Y} + \frac{Y^2}{l^2}$$

***First** fundamental forms*

- $I^{int}_{\Sigma} = -d\tau^2 + {r_b}^2 a^2(\tau)d\Omega^2$
- $I^{ext}_{\Sigma} = \left(-\chi(Y)\dot{T}^2 + \frac{1}{\chi(Y)}\dot{Y}^2\right)d\tau^2 + Y^2(\tau)d\Omega^2$

***Second** fundamental forms*

- $II^{int}_{\Sigma} = 0 \cdot d\tau^2 + r_b a(\tau)d\Omega^2$
- $II^{ext}_{\Sigma} = -\frac{\chi^2(Y)\chi(Y)_{,Y}\dot{T}^3 + 3\chi(Y)_{,Y}\dot{T}\dot{Y}^2 + 2\chi(Y)\left(\dot{Y}\ddot{T} - \dot{T}\ddot{Y}\right)}{2\chi(Y)}d\tau^2 + Y\chi(Y)\dot{T}d\Omega^2$

First *fundamental forms*

- $I_{\Sigma}^{int} = -d\tau^2 + {r_b}^2 a^2(\tau) d\Omega^2$
- $I_{\Sigma}^{ext} = \left(-\chi(Y)\dot{T}^2 + \frac{1}{\chi(Y)}\dot{Y}^2 \right) d\tau^2 + Y^2(\tau) d\Omega^2$

Second *fundamental forms*

- $II_{\Sigma}^{int} = 0 \cdot d\tau^2 + r_b a(\tau) d\Omega^2$
- $II_{\Sigma}^{ext} = -\frac{\chi^2(Y)\chi(Y)_{,Y}\dot{T}^3 + 3\chi(Y)_{,Y}\dot{T}\dot{Y}^2 + 2\chi(Y)\left(\dot{Y}\ddot{T} - \dot{T}\ddot{Y}\right)}{2\chi(Y)} d\tau^2 + Y\chi(Y)\dot{T} d\Omega^2$

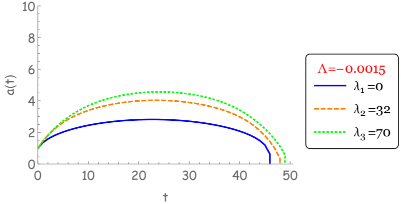
Matched ***first*** *fundamental forms*

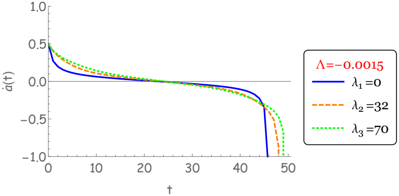
- $\left(-\chi(Y)\dot{T}^2 + \frac{1}{\chi(Y)}\dot{Y}^2 \right) = -1 \quad \Rightarrow \quad \boldsymbol{M}(\tau) = \frac{1}{2}{r_b}^3 a(\tau)\left(\dot{a}^2(\tau) + \frac{a^2(\tau)}{l^2} \right)$
- $Y(\tau) = r_b a(\tau) \quad \Rightarrow$ *boundary of the collapsing shell*

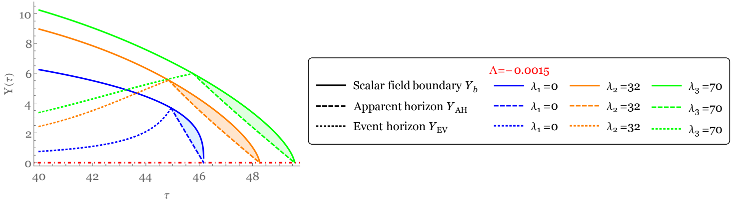
Matched ***second*** *fundamental forms*

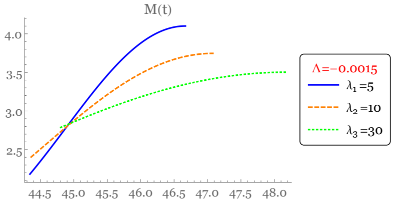
- $-\frac{\chi^2(Y)\chi(Y)_{,Y}\dot{T}^3 + 3\chi(Y)_{,Y}\dot{T}\dot{Y}^2 + 2\chi(Y)\left(\dot{Y}\ddot{T} - \dot{T}\ddot{Y}\right)}{2\chi(Y)} = 0 \ \rightarrow$ *holds identically*
- $Y\chi(Y)\dot{T} = r_b a(\tau) \ \Rightarrow \ T_s = \int_{\tau_c}^{\tau_s} \frac{d\tau}{1 - {r_b}^2 \dot{a}^2(\tau)} \quad \Rightarrow$ ***no*** *naked singularity*

Section 3









- $Y(\tau) = r_b a(\tau), \quad \boldsymbol{M}(\tau) = \frac{1}{2} {r_b}^3 a(\tau) \left(\dot{a}^2(\tau) + \frac{a^2(\tau)}{l^2} \right)$
- *collapsing shell becomes trapped* ⇒ ***black hole*** *forms*
- *Black hole mass depending on* λ $(\mathrm{M} \sim 10^{-8}\ \mathrm{kg})$
- *Black hole "size" depending on* λ

Conclusions

- NMDC acts as a friction term on the collapsing process.
- **Black hole** formation in the absence of a potential $(M \sim M_{pl})$
- The presence of NMDC allows the formation of "heavier" and "larger" *black holes*.
- Studying additional terms of the Horndeski Lagrangian opens the road for new cosmological models.

open question

- Which combination of Horndeski terms could possibly *drive* the collapse of a scalar field ?