Phase transitions of extremal black holes coupled to the scalar field

Olivera Mišković, Paula Quezada

Pontificia Universidad Católica de Valparaíso, Chile

Alessio Marrani

Centro Studi e Ricerche Enrico Fermi, Roma & Universitá di Padova, Italy

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Outline

• Motivation - Why to expect a phase transition of extremal black holes
• Method - Entropy function formalism
• System - Charged black hole coupled to a scalar field
• Critical behavior of the extremal black hole
• Conclusions
Motivation

- **Free energy** of the black hole carries an important information about its thermodynamic stability.

- **Thermal phase transitions** arise due to fluctuations of the temperature of black hole, when a new phase has smaller free energy. For example, BH can develop hair below some $T_c$.

- Using the **AdS/CFT correspondence**, a dual theory can describe a phase transition in condensed matter physics, such as holographic superconductors [e.g., Hartnoll, Herzog Horowitz 2008]

- An **equilibrium state** of a thermodynamic system corresponds to a minimum of the internal energy in the energy representation of states, or a maximum of the entropy in the entropy representation of states.
Motivation

- The extremal black hole is the smallest mass black hole for a given \((Q, J)\) (in the flat space) and it has \(T = 0\).

- **Entropy** of extremal black hole arises due to a degenerate quantum ground state; it is a suitable quantity for studying its equilibria.

- **Quantum phase transition** arises due to quantum fluctuations, which produce instabilities of the system around a critical point. This phenomenon is known in the physics of condensed matter (*spin glasses)*.

- **Macroscopic entropy** of extremal black hole can be calculated using the entropy function formalism [Sen 2005].

- **Isometries of a near-horizon geometry of an extremal BH in 4D:**
  - spherically symmetric \(\rightarrow SO(2,1) \otimes SO(3)\);
  - rotating \(\rightarrow SO(2,1) \otimes U(1)\);
  - topological \(\rightarrow SO(2,1) \otimes SO(2,1), SO(2,1) \otimes \mathbb{R}^2\);
  - can be generalized to other extremal geometries, such as warped ones [Astefanesei, Miskovic, Olea 2012]
Motivation

- **Our interest – gravity with the cosmological constant**
  - Horizon geometry of an extremal BH is $\text{AdS}_2 \otimes \Sigma_k$
  - 2D transversal section $\Sigma_k$ can be a \[ \begin{cases} 
  2\text{-sphere} & k = 1 \\
  2\text{D plane} & k = 0 \\
  2\text{-hyperboloid} & k = -1 
\end{cases} \]

- Entropy function formalism is based on a **variational principle** applied to a generic class of functions of the charges, scalar fields and the parameters of the near-horizon geometry.

- Extremization of the entropy function determines **all the near-horizon parameters** without knowledge of a particular solution.
Motivation

- **Recent results about the horizon instabilities:**
  - A massless scalar field produces an instability at the horizon of an extreme RN BH [Aretakis 2013]
  - The axisymmetric extremal horizons are unstable under linear scalar perturbations [Aretakis 2015], [Lucietti, Murata, Reall 2013]
  - Also Kerr horizons are unstable in presence of a scalar [Zimmerman 2016]

- **Non-extremal BH:** St"uckelberg scalar has been known to describe both first and second order thermal phase transitions. A question is whether a similar change would also occur at $T = 0$.

- **We study phase transitions of a 4D extremal charged black hole** in General Relativity, when it is coupled to a St"uckelberg scalar field.
Near-horizon geometry of the extremal black hole in 4D has topology $\text{AdS}_2 \otimes \Sigma_k$

$$ds^2 = g_{\mu\nu}(x) \, dx^\mu \, dx^\nu = v_1 \left( -r^2 \, dt^2 + \frac{dr^2}{r^2} \right) + v_2 \, d\Omega^2_{(k)}$$

- $r$ = radial distance from the horizon
- $v_1$ = radius of 2D anti-de Sitter space $\text{AdS}_2$
- $v_2$ = radius of the transversal section $\Sigma_k$ with the metric $\gamma_{nm}(y)$

$$d\Omega^2_{(k)} = \begin{cases} 
    d\theta^2 + \sin^2 \theta \, d\phi^2 , & k = 1 \\
    dx^2 + dy^2 , & k = 0 \\
    d\chi^2 + \sinh^2 \chi \, d\phi^2 , & k = -1
\end{cases}$$
• Action for gravity coupled to the EM and scalar fields

\[ I = \int d^4x \sqrt{-g} \, L(g, A, \phi) \]

• Extremal BH near the horizon

\[ H: \quad g_{\mu\nu} \rightarrow (v_1, v_2), \quad A_\mu \rightarrow (e, p), \quad \phi \rightarrow u \]

The scalar field, due to the attractor mechanism, depends only on its value on the horizon, \( u \).

The electromagnetic field on the horizon is \( F_{rt} = e, \; F_{34} = \sqrt{\gamma} \frac{p}{4\pi} \)

• Action evaluated on the horizon

\[ f(v, e, p, u) = \int \limits_H d^2y \sqrt{-g} \, L(v, e, p, u) \]
The function $f(v, e, p, u)$ satisfies the action principle – it has an extremum on the equations of motion, for given boundary conditions.

**Boundary conditions:** Asymptotic electric charge $q$ and magnetic charge $p$ are kept fixed.

**Entropy function:** Legendre transformation of the function $f$ with respect to the electric field

$$E(v, e, p, u) = 2\pi [eq - f(v, e, p, u)]$$

**Parameters near the horizon:** calculated as an extremum of the entropy function [Sen 2005]

$$\frac{\partial E}{\partial v_i} = 0, \quad \frac{\partial E}{\partial u} = 0, \quad \frac{\partial E}{\partial e} = 0, \quad \frac{\partial E}{\partial p} = 0$$

**Black hole entropy:** extremum of the entropy function

$$S = E_{\text{extr}}$$

Therefore, finding the entropy function $E(v, e, p, u)$ and its maximum, one can calculate the entropy, electric field and AdS$_2$ and $\Sigma_k$ radii of the extremal black hole, independently on a particular solution considered.
Charged black hole coupled to a scalar field

- **Action for GR\(_\Lambda\) + Maxwell field + complex scalar field**

\[
l = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (R - 2\Lambda) - \frac{1}{4} F^2 + L_S(\phi e^{i\sigma}) \right]
\]

- **Stückelberg complex scalar**

\[
L_S = -\frac{1}{2} \left[ (\partial\phi)^2 + m^2\phi^2 + P(\phi)(\partial\sigma - A)^2 \right]
\]

\[
P(\phi) = \phi^2 - \frac{a}{4} \phi^4 \geq 0 \text{ nonminimal coupling } a \neq 0
\]

\[
a = \text{coupling constant}
\]

- When \( P(\phi) = \phi^2 \), the above action describes minimally coupled scalar field \( \hat{\phi} = \phi e^{i\sigma} \) of the form \( L_S = |(\partial - iA) \hat{\phi}|^2 - m^2 |\hat{\phi}|^2 \).

- When \( P(\phi) = P(|\hat{\phi}|) \) is not minimal, the action still possesses a \( U(1) \) symmetry.
Charged black hole coupled to a scalar field

- **Equations of motion**

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 8\pi G T_{\mu\nu}
\]
\[
\nabla_\mu F^{\mu\nu} = P(\phi) (\nabla^\mu \sigma - A^\mu)
\]
\[
(\Box - m^2)\phi = \frac{1}{2} P(\phi) (\nabla \sigma - A)^2
\]
\[
\nabla_\mu [P(\phi) (\nabla^\mu \sigma - A^\mu)] = 0 \quad (\text{not independent})
\]

- **Near-horizon parameters**

- Field equation for \(\sigma(x)\) is not independent due to the \(U(1)\) symmetry and it can be gauge-fixed to \(\sigma = 0\).

\[\Rightarrow\] Extremal BH configurations are replaced by five parameters \((v_1, v_2, e, p, u)\)
Charged black hole coupled to a scalar field

- **Lagrangian evaluated on the horizon**

\[
L = \frac{1}{8\pi G} \left( \frac{k}{v_2} - \frac{1}{v_1} - \Lambda \right) + \frac{e^2}{2v_1^2} - \frac{p^2}{32\pi^2 v_2^2} - \frac{1}{2} m^2 u^2 + \frac{1}{2} P(u) \left( \frac{e^2}{v_1} - \frac{p^2 z_k(y)}{16\pi^2 v_2} \right)
\]

- **Transversal section volume element**

\[
\text{Vol}(\Sigma_k) = \int d^2 y \sqrt{\gamma}
\]

- **Auxiliary function**

\[
f = \int d^2 y \sqrt{\gamma} v_1 v_2 L
\]
Charged black hole coupled to a scalar field

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- **Explicit dependence on** \( y^m \) **in the function** \( z_k(y) \) **makes** \( f \) **divergent, unless the magnetic charge vanishes,** \( p = 0 \).

- **Magnetic field** \( F_{mn} \neq 0 \) **breaks a spherical symmetry of the black hole solution.**
Charged black hole coupled to a scalar field

- **Charge density** \( Q = \frac{q}{\text{Vol}(\Sigma_k)} \)

- **Entropy function**
  \[
  E = 2\pi \text{Vol}(\Sigma_k) \left[ eQ - \frac{k v_1 - \Lambda v_1 v_2}{8\pi G} - \frac{e^2 v_2}{2v_1} + \frac{v_1 v_2}{2} \left( m^2 u^2 - \frac{e^2}{v_1} P \right) \right]
  \]

- **Equations of motion** (extremum of \( E \))

\[
\begin{align*}
0 &= \frac{\partial E}{\partial v_1} \quad \Rightarrow \quad k - \Lambda v_2 = v_2 \left( \frac{e^2}{v_1^2} + m^2 u^2 \right) \\
0 &= \frac{\partial E}{\partial v_2} \quad \Rightarrow \quad 1 + \Lambda v_1 = \frac{e^2}{v_1} - v_1 m^2 u^2 + e^2 P \\
0 &= \frac{\partial E}{\partial e} \quad \Rightarrow \quad Q = v_2 e \left( \frac{1}{v_1} + P \right) \\
0 &= \frac{\partial E}{\partial u} \quad \Rightarrow \quad 0 = 2v_1 m^2 u - e^2 P'(u)
\end{align*}
\]

- **Convention** \( 4\pi G = 1; \ Choice \) \( e, Q > 0 \) (no loss of generality)
NORMAL PHASE \( RN_\Lambda \) black hole

- \( u = 0 \) it is always a particular solution of the scalar equation
- \( k = \pm 1 \) otherwise we have undetermined geometry

- **General solution for fixed** \( Q \)
  
  \[
  \nu_1^{(k)}(Q) = \frac{2Q^2}{1-4\Lambda Q^2 + k\sqrt{1-4\Lambda Q^2}} ,
  \]
  
  \[
  \nu_2^{(k)}(Q) = \frac{2Q^2 k}{1 + k\sqrt{1-4\Lambda Q^2}}
  \]
  
  \[
  e^{(k)}(Q) = Q \frac{1 + k\sqrt{1-4\Lambda Q^2}}{1-4\Lambda Q^2 + k\sqrt{1-4\Lambda Q^2}}
  \]

- **Existence of the solution**

  \( \Lambda < 0 \) There are 2 solutions with \( k = \pm 1 \)
  
  \( 0 < \Lambda < \frac{1}{4Q^2} \) There is 1 solution with \( k = +1 \)

  \( \Lambda = 0 \) Can be reproduced from the limit \( \Lambda \to 0 \) of the branch ‘+’

  \( \Lambda = \frac{1}{4Q^2} \) There is no finite solution for \( \nu_1 \)
Critical behavior

- Extremum of the entropy function

\[
S_k(Q) = 2\pi Q^2 \text{Vol}(\Sigma_k) \frac{1 - k + k\sqrt{1 - 4\Lambda Q^2}}{1 - 4\Lambda Q^2 + k\sqrt{1 - 4\Lambda Q^2}}
\]

- When \( \Lambda = 0 \), one gets the known result \( S = q^2 / 4 \).

**HAIRY PHASE**

- Solution with hair exists only if \( a \neq 0 \) (nonlinear interaction), for scalar masses \( m \neq 0, 1, \frac{1}{2} \) and the cosmological constant \( \Lambda \neq 0 \)

- Three solutions for the scalar field \( u = 0, \quad u = \pm \sqrt{\frac{2}{a}} \left(1 - \frac{v_1 m^2}{e^2}\right)\)

- When \( u \neq 0 \), the equations are invariant under the replacement \( u \rightarrow -u \), so we can choose \( u > 0 \).
Critical behavior

Critical point

- Two solutions $u = 0$ and $u \geq 0$ co-exist, there can happen a phase transition from one configuration to another.

- **Critical point** exists if the hairy phase limit $u \to 0$ exists

  $$v_{1c} = \frac{m^2 - 1}{\Lambda}, \quad Q_c = \frac{k}{2m^2 - 1} \sqrt{\frac{m^2(m^2 - 1)}{\Lambda}}$$

  $$v_{2c} = \frac{k(m^2 - 1)}{\Lambda(2m^2 - 1)}, \quad e_c = \sqrt{\frac{m^2(m^2 - 1)}{\Lambda}}$$

- The following inequalities must be fulfilled:

  $$m^2 > 0, \quad k(2m^2 - 1) > 0, \quad \frac{m^2 - 1}{\Lambda} > 0$$

- **Positive branch** ‘+’ reproduces the known critical results of the parameters,

  $$v_i^+(Q_c) = v_{ic}, \quad e^+(Q_c) = e_c$$

- **Critical entropy** is continuous only for $k = +1$

  $$S_c = \frac{4\pi^2(m^2 - 1)}{\Lambda(2m^2 - 1)} = S_+(Q_c) > 0$$
Critical behavior

Near-critical behaviour of the entropy

- We focus to spherical horizons
- We introduce a small parameter $\epsilon = Q - Q_c$
- **Critical exponent** $\beta$: Describes how $u = A\epsilon^\beta + \cdots \to 0$ when $Q \to Q_c$
- Equations of motion give the critical exponent $\beta = 1/2$ (*mean field theory*)

- Solution of the fields equations
  
  $u = A\epsilon^{1/2} + \tilde{A}\epsilon^{3/2} + \cdots$ \quad $v_1 = v_{1c} + B\epsilon + \tilde{B}\epsilon^2 + \cdots$  
  
  $e = e_c + D\epsilon + \tilde{D}\epsilon^2 + \cdots$ \quad $v_2 = v_{2c} + C\epsilon + \tilde{C}\epsilon^2 + \cdots$  

- **First coefficients**

  $A = \left( \sqrt{\frac{m^2 - 1}{\Lambda m^2}} \frac{4\Lambda^2(2m^2-1)^2}{\Lambda a + 4(m^2-1)^3} \right)^{1/2}$
  
  $B = \sqrt{\frac{m^2(m^2-1)}{\Lambda}} \frac{2\Lambda a (2m^2-1)^2}{\Lambda a + 4(m^2-1)^3}$
  
  $C = \sqrt{\frac{m^2(m^2-1)}{\Lambda}} \frac{2\Lambda a - 4(m^2-1)^2}{\Lambda a + 4(m^2-1)^3}$
  
  $D = \frac{\Lambda a (2m^2-1)^3}{\Lambda a + 4(m^2-1)^3}$
Critical behavior

- **Two different solutions that extremize the entropy**
  
  \[ u = 0 \] for any \( Q \) and \[ u = A\sqrt{Q - Q_c} + \cdots \geq 0 \] for \( Q > Q_c \)

- **Finding the entropy near the critical point:**
  - When \( Q > Q_c \), non-linear equations can be solved approximatively in \( \epsilon \).
  - When \( Q < Q_c \), then \( u = 0 \) and \( S_+(Q) \) is known exactly. This result can be compared with a previous one via the Taylor expansion, \( Q = Q_c + \epsilon \).

- **Entropy**

  \[
  S_{u \neq 0} = S_c + 8\pi^2 e_c \epsilon + 4\pi^2 (2m^2 - 1)^3 \omega \epsilon^2 + O(\epsilon^3) \\
  S_{u = 0} = S_c + 8\pi^2 e_c \epsilon + 4\pi^2 (2m^2 - 1)^3 \epsilon^2 + O(\epsilon^3)
  \]

  where \( \omega = \frac{\Lambda a}{\Lambda a + 4(m^2 - 1)^3} \)

- **Second Law of Thermodynamics:** \( \Delta S = S_{u \neq 0} - S_{u = 0} > 0 \)
  
  - Gives that a favorable phase satisfies \( \omega > 1 \) or \( m^2 < 1 \)

- **Quantum phase transition:** allowed values of \( m \) and \( \Lambda \) are

  \[
  \frac{1}{2} < m^2 < 1, \quad \Lambda < 0
  \]

- **Discontinuity typical for phase transitions**

  \( S''(+Q_c) \neq S''(-Q_c) \)
Conclusions

- **Extremal AdS$_4$ black holes** with spherical horizons can develop hair above some $Q_c$, due to variations of electric charge.

- The mass of the Stückelberg scalar has to be in the interval $\frac{1}{2} < m^2 < 1$ and the Stückelberg interaction $a \neq 0$ non-linear.

- The phase transition does not occur when $\Lambda = 0$ and $\Lambda > 0$.

- All calculations are done **analytically using the entropy function formalism**.

- The extreme hairy AdS$_4$ solution should be studied in the **whole spacetime**.

- One should also study the AdS black holes with **hyperbolic horizons**.

- Describing this process (naturally) in the **SUGRA context**?

- Quantum phase transition in a dual theory via the **AdS/CFT correspondence**?
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**Thank you!**