Phase transitions of extremal black holes coupled to the scalar field

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Einstein's theory of gravity and its modifications: from theory to observations

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Outline

- Motivation - Why to expect a phase transition of extremal black holes
- Method - Entropy function formalism
- System - Charged black hole coupled to a scalar field
- Critical behavior of the extremal black hole
- Conclusions

Motivation

- **Free energy** of the black hole carries an important information about its thermodynamic stability.
- **Thermal phase transitions** arise due to fluctuations of the temperature of black hole, when a new phase has smaller free energy. For example, BH can develop hair below some T_c.
- Using the **AdS/CFT correspondence**, a dual theory can describe a phase transition in condensed matter physics, such as holographic superconductors [*e.g.*, Hartnoll, Herzog Horowitz 2008]
- An **equilibrium state** of a thermodynamic system corresponds to a minimum of the internal energy in the energy representation of states, or a maximum of the entropy in the entropy representation of states.

Motivation

- The extremal black hole is the smallest mass black hole for a given (Q, J) (in the flat space) and it has $T = 0$
- **Entropy** of extremal black hole arises due to a degenerate quantum ground state; it is a suitable quantity for studying its equilibria.
- **Quantum phase transition** arises due to quantum fluctuations, which produce instabilities of the system around a critical point. This phenomenon is known in the physics of condensed matter (*spin glasses*).
- **Macroscopic entropy** of extremal black hole can be calculated using the entropy function formalism [Sen 2005].
- **Isometries of a near-horizon geometry of an extremal BH in 4D**:

- spherically symmetric $\longrightarrow SO(2,1) \otimes SO(3)$;
- rotating $\longrightarrow SO(2,1) \otimes U(1)$;
- topological $\longrightarrow SO(2,1) \otimes SO(2,1)$, $SO(2,1) \otimes \mathbb{R}^2$;
- can be generalized to other extremal geometries, such as warped ones [Astefanesei, Miskovic, Olea 2012]

Motivation

- **Our interest – gravity with the cosmological constant**

\- Horizon geometry of an extremal BH is $\mathrm{AdS}_2 \otimes \Sigma_k$

\- 2D transversal section Σ_k can be a $\begin{cases} \text{2-sphere} & k = 1 \\ \text{2D plane} & k = 0 \\ \text{2-hyperboloid} & k = -1 \end{cases}$

- Entropy function formalism is based on a **variational principle** applied to a generic class of functions of the charges, scalar fields and the parameters of the near-horizon geometry.

- Extremization of the entropy function determines **all the near-horizon parameters** without knowledge of a particular solution.

Motivation

- **Recent results about the horizon instabilities**:

- A massless scalar field produces an instability at the horizon of an extreme RN BH [Aretakis 2013]

- The axisymmetric extremal horizons are unstable under linear scalar perturbations [Aretakis 2015], [Lucietti, Murata, Reall 2013]

- Also Kerr horizons are unstable in presence of a scalar [Zimmerman 2016]

- **Non-extremal BH**: Stückelberg scalar has been known to describe both first and second order thermal phase transitions. A question is whether a similar change would also occur at $T = 0$.

- We study phase transitions of a **4D extremal charged black hole** in General Relativity, when it is coupled to a Stückelberg scalar field.

Entropy function formalism

- **Near-horizon geometry of the extremal black hole in 4D has topology** $\mathrm{AdS}_2 \otimes \Sigma_k$

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = v_1 \left(-r^2\, dt^2 + \frac{dr^2}{r^2} \right) + v_2\, d\Omega^2_{(k)}$$

$$\begin{aligned} r &= \text{radial distance from the horizon} \\ v_1 &= \text{radius of 2D anti-de Sitter space } \mathrm{AdS}_2 \\ v_2 &= \text{radius of the transversal section } \Sigma_k \text{ with the metric } \gamma_{nm}(y) \end{aligned}$$

$$d\Omega^2_{(k)} = \begin{cases} d\theta^2 + \sin^2\theta\, d\varphi^2\,, & k = 1 \\ dx^2 + dy^2\,, & k = 0 \\ d\chi^2 + \sinh^2\chi\, d\varphi^2\,, & k = -1 \end{cases}$$

Entropy function formalism

- **Action for gravity coupled to the EM and scalar fields**

  $I = \int d^4x \sqrt{-g}\, L(g, A, \phi)$

- **Extremal BH near the horizon**

  $$\boxed{H: \quad g_{\mu\nu} \rightarrow (v_1\,,\, v_2)\,, \quad A_\mu \rightarrow (e, p)\;, \quad \phi \rightarrow u}$$

  The scalar field, due to the attractor mechanism, depends only on it value on the horizon, u.

  The electromagnetic field on the horizon is $F_{rt} = e$, $F_{34} = \sqrt{\gamma}\,\frac{p}{4\pi}$

- **Action evaluated on the horizon**

  $f(v, e, p, u) = \int_H d^2y \sqrt{-g}\, L(v, e, p, u)$

Entropy function formalism

- The function $f(v, e, p, u)$ satisfies the action principle – it has an extremum on the equations of motion, for given boundary conditions.
- **Boundary conditions**: Asymptotic electric charge q and magnetic charge p are kept fixed.
- **Entropy function:** Legendre transformation of the function f with respect to the electric field
$$E(v, e, p, u) = 2\pi \left[eq - f(v, e, p, u)\right]$$
- **Parameters near the horizon**: calculated as an extremum of the entropy function [Sen 2005]
$$\frac{\partial E}{\partial v_i} = 0\,, \quad \frac{\partial E}{\partial u} = 0\,, \quad \frac{\partial E}{\partial e} = 0\,, \quad \frac{\partial E}{\partial p} = 0$$
- **Black hole entropy:** extremum of the entropy function $S = E_{\mathrm{extr}}$
- Therefore, finding the entropy function $E(v, e, p, u)$ and its maximum, one can calculate the entropy, electric field and AdS_2 and Σ_k radii of the extremal black hole, independently on a particular solution considered.

Charged black hole coupled to a scalar field

- **Action for GR_Λ + Maxwell field + complex scalar field**

$$I=\int d^4x\sqrt{-g}\,\left[\frac{1}{16\pi G}\,(R-2\Lambda)-\frac{1}{4}\,F^2+L_S(\phi\,\mathrm{e}^{i\sigma})\right]$$

- **Stückelberg complex scalar**

$$L_S=-\frac{1}{2}\,\left[(\partial\phi)^2+m^2\phi^2+P(\phi)(\partial\sigma-A)^2\right]$$

$$P(\phi)=\phi^2-\frac{a}{4}\,\phi^4\geq 0 \text{ nonminimal coupling } a\neq 0$$

$a =$ coupling constant

- When $P(\phi)=\phi^2$, the above action describes minimally coupled scalar field $\hat{\phi}=\phi\,\mathrm{e}^{i\sigma}$ of the form $L_S=\left|(\partial-iA)\,\hat{\phi}\right|^2-m^2\,\left|\hat{\phi}\right|^2$.
- When $P(\phi)=P(|\hat{\phi}|)$ is not minimal, the action still possesses a $U(1)$ symmetry.

Charged black hole coupled to a scalar field

- **Equations of motion**

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R + g_{\mu\nu}\, \Lambda = 8\pi G\, T_{\mu\nu}$$
$$\nabla_\mu F^{\mu\nu} = P(\phi)\, (\nabla^\mu \sigma - A^\mu)$$
$$(\Box - m^2)\phi = \frac{1}{2}\, P(\phi)\, (\nabla\sigma - A)^2$$
$$\nabla_\mu \left[P(\phi)\, (\nabla^\mu \sigma - A^\mu)\right] = 0 \;(\textit{not independent})$$

- **Near-horizon parameters**

\- Field equation for $\sigma(x)$ is not independent due to the $U(1)$ symmetry and it can be gauge-fixed to $\sigma = 0$.

$\Rightarrow$ Extremal BH configurations are replaced by five parameters (v_1, v_2, e, p, u)

Charged black hole coupled to a scalar field

- **Lagrangian evaluated on the horizon**

$$L = \frac{1}{8\pi G}\left(\frac{k}{v_2} - \frac{1}{v_1} - \Lambda\right) + \frac{e^2}{2v_1^2} - \frac{p^2}{32\pi^2 v_2^2} - \frac{1}{2}\, m^2 u^2 + \frac{1}{2}\, P(u)\left(\frac{e^2}{v_1} - \frac{p^2 z_k(y)}{16\pi^2 v_2}\right)$$

- **Transversal section volume element**

$$\mathrm{Vol}(\Sigma_k) = \int d^2y\, \sqrt{\gamma}$$

- **Auxiliary function**

$$f = \int d^2y\, \sqrt{\gamma}\, v_1 v_2\, L$$

Charged black hole coupled to a scalar field

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- Explicit dependence on y^m in the function $z_k(y)$ makes f divergent, unless the magnetic charge vanishes, $p = 0$.
- Magnetic field $F_{mn} \neq 0$ breaks a spherical symmetry of the black hole solution.

Charged black hole coupled to a scalar field

- **Charge density** $Q = q/\mathrm{Vol}(\Sigma_k)$
- **Entropy function**

$$E = 2\pi \mathrm{Vol}(\Sigma_k) \left[eQ - \frac{kv_1 - v_2 - \Lambda v_1 v_2}{8\pi G} - \frac{e^2 v_2}{2 v_1} + \frac{v_1 v_2}{2} \left(m^2 u^2 - \frac{e^2}{v_1} P \right) \right]$$

- **Equations of motion** (extremum of E)

$$0 = \frac{\partial E}{\partial v_1} \quad \Rightarrow \quad k - \Lambda v_2 = v_2 \left(\frac{e^2}{v_1^2} + m^2 u^2 \right)$$

$$0 = \frac{\partial E}{\partial v_2} \quad \Rightarrow \quad 1 + \Lambda v_1 = \frac{e^2}{v_1} - v_1 m^2 u^2 + e^2 P$$

$$0 = \frac{\partial E}{\partial e} \quad \Rightarrow \quad Q = v_2 e \left(\frac{1}{v_1} + P \right)$$

$$0 = \frac{\partial E}{\partial u} \quad \Rightarrow \quad 0 = 2 v_1 m^2 u - e^2 P'(u)$$

- **Convention** $4\pi G = 1$; **Choice** $e, Q > 0$ (no loss of generality)

Critical behavior

NORMAL PHASE *RN_Λ black hole*

$u = 0$ it is always a particular solution of the scalar equation
$k = \pm 1$ otherwise we have undetermined geometry

- **General solution for fixed Q**

$$v_1^{(k)}(Q) = \frac{2Q^2}{1-4\Lambda Q^2 + k\sqrt{1-4\Lambda Q^2}},$$
$$v_2^{(k)}(Q) = \frac{2Q^2 k}{1+k\sqrt{1-4\Lambda Q^2}}$$

$$e^{(k)}(Q) = Q\,\frac{1+k\sqrt{1-4\Lambda Q^2}}{1-4\Lambda Q^2+k\sqrt{1-4\Lambda Q^2}}$$

- **Existence of the solution**

$\Lambda < 0$	There are 2 solutions with $k = \pm 1$
$0 < \Lambda < \frac{1}{4Q^2}$	There is 1 solution with $k = +1$
$\Lambda = 0$	Can be reproduced from the limit $\Lambda \to 0$ of the branch '+'
$\Lambda = \frac{1}{4Q^2}$	There is no finite solution for v_1

Critical behavior

- **Extremum of the entropy function**

$$S_k(Q) = 2\pi Q^2 \mathrm{Vol}(\Sigma_k)\, \frac{1-k+k\sqrt{1-4\Lambda Q^2}}{1-4\Lambda Q^2 + k\sqrt{1-4\Lambda Q^2}}$$

- When $\Lambda = 0$, one gets the known result $S = q^2/4$.

HAIRY PHASE

- Solution with hair exists only if $a \neq 0$ (nonlinear interaction), for scalar masses $m \neq 0, 1, \frac{1}{2}$ and the cosmological constant $\Lambda \neq 0$
- **Three solutions for the scalar field** $u = 0\,, \quad u = \pm\sqrt{\frac{2}{a}\left(1 - \frac{v_1\, m^2}{e^2}\right)}$
- When $u \neq 0$, the equations are invariant under the replacement $u \to -u$, so we can chose $u > 0$.

Critical behavior

Critical point

- Two solutions $u = 0$ and $u \geq 0$ co-exist, there can happen a phase transition from one configuration to another
- **Critical point** exists if the hairy phase limit $u \to 0$ exists

$$v_{1c} = \frac{m^2-1}{\Lambda}, \qquad Q_c = \frac{k}{2m^2-1}\sqrt{\frac{m^2(m^2-1)}{\Lambda}}$$

$$v_{2c} = \frac{k\left(m^2-1\right)}{\Lambda(2m^2-1)}, \quad e_c = \sqrt{\frac{m^2(m^2-1)}{\Lambda}}$$

- The following inequalities must be fulfilled:

$$m^2 > 0,\ k\left(2m^2-1\right) > 0,\ \frac{m^2-1}{\Lambda} > 0$$

- **Positive branch** '+' reproduces the known critical results of the parameters, $v_i^+(Q_c) = v_{ic}\,,\ e^+(Q_c) = e_c$
- **Critical entropy** is continuous only for $k = +1$

$$\boxed{S_c = \frac{4\pi^2\left(m^2-1\right)}{\Lambda(2m^2-1)} = S_+(Q_c) > 0}$$

Critical behavior

Near-critical behaviour of the entropy

- We focus to spherical horizons
- We introduce a small parameter $\epsilon = Q - Q_c$
- **Critical exponent** β: Describes how $u = A\epsilon^{\beta} + \cdots \to 0$ when $Q \to Q_c$
- Equations of motion give the critical exponent $\boxed{\beta = 1/2}$ (*mean field theory*)
- **Solution of the fields equations**

$$u = A\,\epsilon^{1/2} + \tilde{A}\,\epsilon^{3/2} + \cdots \qquad v_1 = v_{1c} + B\,\epsilon + \tilde{B}\,\epsilon^2 + \cdots$$

$$e = e_c + D\,\epsilon + \tilde{D}\,\epsilon^2 + \cdots \qquad v_2 = v_{2c} + C\,\epsilon + \tilde{C}\,\epsilon^2 + \cdots$$

- **First coefficients**

$$A = \left(\sqrt{\frac{m^2-1}{\Lambda m^2}} \frac{4\Lambda^2\left(2m^2-1\right)^2}{\Lambda a + 4\left(m^2-1\right)^3} \right)^{1/2} \qquad C = \sqrt{\frac{m^2\left(m^2-1\right)}{\Lambda}} \, \frac{2\Lambda a - 4\left(m^2-1\right)^2}{\Lambda a + 4\left(m^2-1\right)^3}$$

$$B = \sqrt{\frac{m^2\left(m^2-1\right)}{\Lambda}} \, \frac{2\Lambda a\left(2m^2-1\right)^2}{\Lambda a + 4\left(m^2-1\right)^3} \qquad D = \frac{\Lambda a\left(2m^2-1\right)^3}{\Lambda a + 4\left(m^2-1\right)^3}$$

Critical behavior

- **Two different solutions that extremize the entropy**
  $u = 0$ for any Q and $u = A\sqrt{Q - Q_c} + \cdots \geq 0$ for $Q > Q_c$
- **Finding the entropy near the critical point:**

- When $Q > Q_c$, non-linear equations can be solved approximatively in ϵ.
- When $Q < Q_c$, then $u = 0$ and $S_+(Q)$ is known exactly. This result can be compared with a previous one via the Teylor expansion, $Q = Q_c + \epsilon$.

- **Entropy**

$$S|_{u\neq 0} = S_c + 8\pi^2 e_c \epsilon + 4\pi^2 (2m^2 - 1)^3 \,\omega\, \epsilon^2 + O(\epsilon^3)$$
$$S|_{u=0} = S_c + 8\pi^2 e_c \epsilon + 4\pi^2 \left(2m^2 - 1\right)^3 \epsilon^2 + O(\epsilon^3)$$

where $\omega = \frac{\Lambda a}{\Lambda a + 4(m^2 - 1)^3}$

- **Second Law of Thermodynamics:** $\Delta S = S|_{u\neq 0} - S|_{u=0} > 0$
- Gives that a favorable phase satisfies $\omega > 1$ or $m^2 < 1$
- **Quantum phase transition:** allowed values of m and Λ are

$$\frac{1}{2} < m^2 < 1\,, \qquad \Lambda < 0$$

- **Discontinuity typical for phase transitions** $S''(+Q_c) \neq S''(-Q_c)$

Conclusions

- **Extremal AdS_4 black holes** with spherical horizons can develop hair above some Q_c, due to variations of electric charge
- The mass of the Stückelberg scalar has to be in the interval $\frac{1}{2} < m^2 < 1$ and the Stückelberg interaction $a \neq 0$ non-linear
- The phase transition does not occur when $\Lambda = 0$ and $\Lambda > 0$
- All calculations are done **analytically using the entropy function formalism**
- The extreme hairy AdS_4 solution should be studied in **the whole spacetime**
- One should also study the AdS black holes with **hyperbolic horizons**
- Describing this process (naturally) in the **SUGRA context?**
- Quantum phase transition in a dual theory via the **AdS/CFT correspondence**?

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T H A N K Y O U !