Generalization of Brans-Dicke gravity through matter-scalar interaction

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Plan

- Generalization of Brans-Dicke gravity
  - Same assumption as BD : $\Box\phi \sim \mathcal{T}$, $\phi \sim G^{-1}$

  $H_0^2 \sim G_N \rho_0 \Leftrightarrow \frac{G_N^{-1}}{R_{H,0}^2} \sim \rho_0$ ($R_{H,0} \sim H_0^{-1}$ present cosmic horizon)
  - Same assumption as BD : 2nd order eqm
  - Difference : $\mathcal{T}^{\mu}{}_{\nu;\mu} \neq 0$ (the r.h.s. will be determined from consistency $\rightarrow$ complete BD) , with caution
- Attempt for finding the corresponding action of the new theory
  (action of new vacuum theory is found; action of new full matter theory is only found in special cases)
- Corresponding cosmology, applications
  (avoidance of singularity, inflation, late-times acceleration)
- Study of cosmological perturbations

- Brans-Dicke eqm:

$$
\begin{aligned}
G^{\mu}{}_{\nu} &= \frac{8\pi}{\phi}\left(T^{\mu}{}_{\nu}+\mathcal{T}^{\mu}{}_{\nu}\right)\\
T^{\mu}{}_{\nu} &= \frac{2-3\lambda}{16\pi\lambda\phi}\left(\phi^{;\mu}\phi_{;\nu}-\frac{1}{2}\delta^{\mu}{}_{\nu}\phi^{;\rho}\phi_{;\rho}\right)+\frac{1}{8\pi}\left(\phi^{;\mu}{}_{;\nu}-\delta^{\mu}{}_{\nu}\Box\phi\right)\\
\Box\phi &= 4\pi\lambda\mathcal{T}\\
\mathcal{T}^{\mu}{}_{\nu;\mu} &= 0
\end{aligned}
$$

- Second order eqm
- $\phi \sim G^{-1}$
- Simplest eqm for the scalar field (maybe add a potential)
- Probe matter moves on geodesics
- Consistency $\rightarrow$ $T^{\mu}{}_{\nu}$
- We no longer assume the vanishing of $\mathcal{T}^{\mu}{}_{\nu;\mu}$, but let it free to be determined by consistency, and this changes also $T^{\mu}{}_{\nu}$

- Two ways to derive the BD eqm :
  - From consistency of 2nd order eqm – to be followed here with interaction
  - From a uniquely defined action

$$S_{BD} = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\left(\phi R - \frac{\omega_{BD}}{\phi} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right) + \int d^4x\,\sqrt{-g}\,L_m$$

Matter minimally coupled (conserved)
$\phi \sim [G_N]^{-1} \sim [M]^2$
$\Rightarrow\ \frac{\omega_{BD}}{\phi}$ only option for the kinetic term, $\omega_{BD} \sim [M]^0$
The method of dimensional analysis does not help if matter is not conserved (if interactions of L_m with ϕ are allowed, plenty of actions can be written with limit the BD one in the absence of interactions−the number of such actions increases in the presence of G_N or a new mass scale ν)

Proof

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$$G^{\mu}{}_{\nu} = \frac{8\pi}{\phi}(T^{\mu}{}_{\nu} + \mathcal{T}^{\mu}{}_{\nu}) \qquad (1)$$

$$T^{\mu}{}_{\nu} = A(\phi)\phi^{;\mu}\phi_{;\nu} + B(\phi)\delta^{\mu}{}_{\nu}\phi^{;\rho}\phi_{;\rho} + C(\phi)\phi^{;\mu}{}_{;\nu} + E(\phi)\delta^{\mu}{}_{\nu}\Box\phi \qquad (2)$$

- Bianchi of (1) :

$$G^{\mu}{}_{\nu}\phi_{;\mu} - 8\pi T^{\mu}{}_{\nu;\mu} = 8\pi\mathcal{T}^{\mu}{}_{\nu;\mu} \qquad (3)$$

$$\Leftrightarrow \quad T^{\mu}{}_{\nu;\mu} - \frac{1}{\phi}T^{\mu}{}_{\nu}\phi_{;\mu} = \frac{1}{\phi}\mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu} - \mathcal{T}^{\mu}{}_{\nu;\mu} \qquad (4)$$

- Derivative of (2) in (4) $\Rightarrow$ $\Box(\phi_{;\nu})$, $(\Box\phi)_{;\nu}$
- $R^{\mu}{}_{\nu}\phi_{;\mu} = \Box(\phi_{;\nu}) - (\Box\phi)_{;\nu}$, (1) , (2) $\Rightarrow$

$$\Box(\phi_{;\nu}) = (\Box\phi)_{;\nu} + \frac{4\pi}{\phi}(A-2B)\phi^{;\mu}\phi_{;\mu}\phi_{;\nu} - \frac{4\pi}{\phi}(C+2E)\phi_{;\nu}\Box\phi$$
$$+\frac{8\pi}{\phi}C\phi^{;\mu}{}_{;\nu}\phi_{;\mu} + \frac{4\pi}{\phi}(2\mathcal{T}^{\mu}{}_{\nu} - \mathcal{T}\delta^{\mu}{}_{\nu})\phi_{;\mu} \qquad (5)$$

- $$\begin{aligned}&\left[A'+B'+\frac{4\pi}{\phi}C(A-2B)-\frac{1}{\phi}(A+B)\right]\phi^{;\mu}\phi_{;\mu}\phi_{;\nu}\\&+\left[A+E'-\frac{4\pi}{\phi}C(C+2E)-\frac{1}{\phi}E\right]\phi_{;\nu}\Box\phi\\&+\left[A+2B+C'+\frac{8\pi}{\phi}C^2-\frac{1}{\phi}C\right]\phi^{;\mu}{}_{;\nu}\phi_{;\mu}\\&+(C+E)(\Box\phi)_{;\nu}\\&+\left(\mathcal{T}^{\mu}{}_{\nu;\mu}-\frac{1-8\pi C}{\phi}\mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu}-\frac{4\pi}{\phi}C\mathcal{T}\phi_{;\nu}\right)=0\end{aligned}\tag{6}$$

- $$\Box\phi=4\pi\lambda\mathcal{T}\quad\Rightarrow\quad\mathcal{T}=\frac{1}{4\pi\lambda}\Box\phi\tag{7}$$

- (7) in (6) $\Rightarrow$

$$\begin{aligned}&\left[A'+B'+\frac{4\pi}{\phi}C(A-2B)-\frac{1}{\phi}(A+B)\right]\phi^{;\mu}\phi_{;\mu}\phi_{;\nu}\\&+\left[A+E'-\frac{4\pi}{\phi}C(C+2E)-\frac{1}{\phi}E-\frac{1}{\lambda\phi}C\right]\phi_{;\nu}\Box\phi\\&+\left[A+2B+C'+\frac{8\pi}{\phi}C^2-\frac{1}{\phi}C\right]\phi^{;\mu}{}_{;\nu}\phi_{;\mu}\\&+(C+E)(\Box\phi)_{;\nu}\\&+\left(\mathcal{T}^{\mu}{}_{\nu;\mu}-\frac{1-8\pi C}{\phi}\mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu}\right)=0\end{aligned}\tag{8}$$

Basically, we use a method to exploit fully the information that the Bianchi identities set on the undetermined functions which define the eqm.

We cannot vanish the various coefficients, although functionally independent, since we do not know how matter interacts

- Zero matter limit $\mathcal{T}^{\mu}{}_{\nu} = 0 \;\Rightarrow\; \Box\phi = 0$:
  the system should still be meaningful with the same A, B, C, E, a vacuum theory should be defined, and no new equations of motion should arise

$$A' + B' + \frac{4\pi}{\phi} C(A - 2B) - \frac{1}{\phi}(A + B) = 0 \tag{9}$$

$$A + 2B + C' + \frac{8\pi}{\phi} C^2 - \frac{1}{\phi} C = 0 \tag{10}$$

$$\mathcal{T}^{\mu}{}_{\nu;\mu} - \frac{1 - 8\pi C}{\phi} \mathcal{T}^{\mu}{}_{\nu} \phi_{;\mu}$$

$$+ \left[A + E' - \frac{4\pi}{\phi} C(C + 2E) - \frac{1}{\phi} E - \frac{1}{\lambda\phi} C \right] \phi_{;\nu} \Box\phi + (C + E)(\Box\phi)_{;\nu} = 0 \tag{11}$$

- $\mathcal{T}^{\mu}{}_{\nu;\mu} = f(\phi)\mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu} + h(\phi)\mathcal{T}\phi_{;\nu} + m(\phi)\mathcal{T}_{;\nu}$
  general energy-momentum conservation equation with 3 interaction terms $\mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu}$, $\mathcal{T}\phi_{;\nu}$, $\mathcal{T}_{;\nu}$
- Eqs (9), (10) have one freedom, e.g. C
  Eq (11) contains an extra arbitrary function E
  $\Rightarrow$ arbitrariness of 2 functions picked up by hand (infinite consistent theories)
- Only way without arbitrariness : single violating term

$$
\begin{array}{llll}
\mathcal{T}^{\mu}{}_{\nu;\mu} \sim & \mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu} & & \rightarrow \ \textit{unique theory} \\
\mathcal{T}^{\mu}{}_{\nu;\mu} \sim & \mathcal{T}\phi_{;\nu} & & \rightarrow \ \textit{unique theory} \\
\mathcal{T}^{\mu}{}_{\nu;\mu} \sim & & \mathcal{T}_{;\nu} & \rightarrow \ \textit{unique theory}
\end{array}
$$

First theory : $\mathcal{T}^{\mu}{}_{\nu;\mu} \sim \mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu}$

$$A' + B' + \frac{4\pi}{\phi}C(A-2B) - \frac{1}{\phi}(A+B) = 0 \tag{12}$$

$$A + 2B + C' + \frac{8\pi}{\phi}C^2 - \frac{1}{\phi}C = 0 \tag{13}$$

$$A + E' - \frac{4\pi}{\phi}C(C+2E) - \frac{1}{\phi}E - \frac{1}{\lambda\phi}C = 0 \tag{14}$$

$$C + E = 0 \tag{15}$$

$$\mathcal{T}^{\mu}{}_{\nu;\mu} = \frac{1-8\pi C}{\phi}\mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu} \tag{16}$$

The system can be solved exactly

- Equivalently

$$(A+B)' + \frac{4\pi}{\phi}C(A-2B) - \frac{1}{\phi}(A+B) = 0 \tag{17}$$

$$C' - A - \frac{4\pi}{\phi}C^2 - \left(1 - \frac{1}{\lambda}\right)\frac{1}{\phi}C = 0 \tag{18}$$

$$A + B + \frac{6\pi}{\phi}C^2 - \frac{1}{2\lambda\phi}C = 0 \tag{19}$$

$$E = -C \tag{20}$$

$$\mathcal{T}^{\mu}{}_{\nu;\mu} = \frac{1-8\pi C}{\phi}\mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu} \tag{21}$$

- Define $X = A + B$, $Y = A - B$

-

$$X' + \frac{2\pi}{\phi} C(3Y - X) - \frac{1}{\phi} X = 0 \tag{22}$$

$$Y = 2C' - \frac{8\pi}{\phi} C^2 - 2\left(1 - \frac{1}{\lambda}\right) \frac{1}{\phi} C - X \tag{23}$$

$$X = -\frac{6\pi}{\phi} C^2 + \frac{1}{2\lambda\phi} C \tag{24}$$

$$E = -C \tag{25}$$

$$\mathcal{T}^{\mu}{}_{\nu;\mu} = \frac{1 - 8\pi C}{\phi} \mathcal{T}^{\mu}{}_{\nu} \phi_{;\mu} \tag{26}$$

- Differentiating and combining $\Rightarrow$

$$C' + \frac{16\pi}{\phi} C^2 - \frac{2}{\phi} C = 0$$

General solution : $C(\phi) = \frac{\phi^2}{\nu + 8\pi\phi^2}$, $\nu \sim [M]^4$ *integr. const.*
$\Rightarrow$ $A(\phi), B(\phi), E(\phi)$ unique

$$G^{\mu}{}_{\nu} = \frac{8\pi}{\phi}\left(T^{\mu}{}_{\nu} + \mathcal{T}^{\mu}{}_{\nu}\right)$$

$$\begin{aligned} T^{\mu}{}_{\nu} = & \frac{\phi}{2\lambda(\nu+8\pi\phi^2)^2}\Big\{2\big[(1+\lambda)\nu+4\pi(2-3\lambda)\phi^2\big]\phi^{;\mu}\phi_{;\nu} \\ & -\big[(1+2\lambda)\nu+4\pi(2-3\lambda)\phi^2\big]\delta^{\mu}{}_{\nu}\phi^{;\rho}\phi_{;\rho}\Big\} + \frac{\phi^2}{\nu+8\pi\phi^2}\left(\phi^{;\mu}{}_{;\nu} - \delta^{\mu}{}_{\nu}\Box\phi\right) \end{aligned}$$

$$\Box\phi = 4\pi\lambda\mathcal{T}$$

$$\mathcal{T}^{\mu}{}_{\nu;\mu} = \frac{\nu}{\phi(\nu+8\pi\phi^2)}\mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu} \quad \Leftrightarrow \quad \left(\phi^{-1}\sqrt{|\nu+8\pi\phi^2|}\,\mathcal{T}^{\mu}{}_{\nu}\right)_{;\mu} = 0$$

- The system is consistent (Bianchies are satisfied)
- For $\nu = 0$ reduced to the standard Brans-Dicke
- Given that T^μ_ν contains up to second derivatives, and given $\Box\phi = 4\pi\lambda\mathcal{T}$, this theory is unique for this single interaction
- The strength of the interaction in the wave equation is controlled by λ, while in the non-conservation equation by ν
- In the decoupling limit $\lambda \to 0$, Brans-Dicke should reduce to Einstein, which is not the case since $T^{\mu\,(BD)}_\nu = \frac{2-3\lambda}{16\pi\lambda\phi}\{\ldots\}$ (moreover in this limit, a solution of BD does not always reduce to a solution of General Relativity with the same $\mathcal{T}^\mu{}_\nu$). Here, this can be done, redefining $\nu = \lambda^{-2}\nu'$, then for $\lambda \to 0$:
  $G^\mu{}_\nu = 8\pi\phi^{-1}\mathcal{T}^\mu{}_\nu$, $T^\mu{}_\nu \equiv 0$, $\Box\phi = 0$, $(\phi^{-1}\mathcal{T}^\mu{}_\nu)_{;\mu} = 0$
  and for the solution $\phi =$constant we get Einstein

Second theory : $\mathcal{T}^{\mu}{}_{\nu;\mu} \sim \mathcal{T}\phi_{;\nu}$

$$
\begin{aligned}
G^{\mu}{}_{\nu} &= \frac{8\pi}{\phi}(T^{\mu}{}_{\nu} + \mathcal{T}^{\mu}{}_{\nu}) \\
T^{\mu}{}_{\nu} &= \frac{2-3\lambda-4\mu}{16\pi\lambda\phi}\left(\phi^{;\mu}\phi_{;\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}\phi^{;\rho}\phi_{;\rho}\right) + \frac{1}{8\pi}\left(\phi^{;\mu}{}_{;\nu} - \delta^{\mu}{}_{\nu}\Box\phi\right) \\
\Box\phi &= 4\pi\lambda\mathcal{T} \\
\mathcal{T}^{\mu}{}_{\nu;\mu} &= \frac{\mu}{\phi}\mathcal{T}\phi_{;\nu}
\end{aligned}
$$

μ integration constant

For $\mu = 0$ reduces to BD

For $\mu = \frac{1}{2}$ the theory had been found in the past

Third theory : $\mathcal{T}^{\mu}{}_{\nu;\mu} \sim \mathcal{T}_{;\nu}$

$$
\begin{aligned}
G^{\mu}{}_{\nu} &= \frac{8\pi}{\phi}\left(T^{\mu}{}_{\nu} + \mathcal{T}^{\mu}{}_{\nu}\right) \\
T^{\mu}{}_{\nu} &= \frac{2-3\lambda-8\sigma}{16\pi\lambda\phi}\left(\phi^{;\mu}\phi_{;\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}\phi^{;\rho}\phi_{;\rho}\right) + \frac{1}{8\pi}\phi^{;\mu}{}_{;\nu} - \frac{\lambda+2\sigma+2\eta\phi^2}{8\pi\lambda}\delta^{\mu}{}_{\nu}\Box\phi \\
\Box\phi &= 4\pi\lambda\mathcal{T} \\
\mathcal{T}^{\mu}{}_{\nu;\mu} &= (\sigma+\eta\phi^2)\mathcal{T}_{;\nu}
\end{aligned}
$$

- σ, η integration constants. For $\sigma = \eta = 0$ reduces to BD
- no $\phi_{;\mu}$ in conservation equation, so even for slowly varying ϕ, the geodesic equation does not arise

In standard derivation of Brans-Dicke, R^μ_ν in Bianchi is just replaced by $R^\mu{}_\nu \phi_{;\mu} = \Box(\phi_{;\nu}) - (\Box\phi)_{;\nu}$ and R by the trace of the gravitational eqm – instead of replacing G^μ_ν from the gravitational eqm – therefore both $\Box(\phi_{;\nu}), (\Box\phi)_{;\nu}$ appear in the consistency relation

$$\left[A' + B' - \frac{1}{2\phi}(A + 4B)\right]\phi^{;\mu}\phi_{;\mu}\phi_{;\nu}$$
$$+\left[A + E' - \frac{1}{2\phi}(C + 4E) - \frac{1}{8\pi\lambda\phi}\right]\phi_{;\nu}\Box\phi + (A + 2B + C')\phi^{;\mu}{}_{;\nu}\phi_{;\mu}$$
$$+\left(E + \frac{1}{8\pi}\right)(\Box\phi)_{;\nu} + \left(C - \frac{1}{8\pi}\right)\Box(\phi_{;\nu}) + \mathcal{T}^\mu{}_{\nu;\mu} = 0$$

The result is an algebraic system for the unknown coefficients (not differential) with more equations, which give Brans-Dicke.

However, with non-conservation, the correct is to replace $\Box(\phi_{;\nu})$ in terms of $(\Box\phi)_{;\nu}$, $\mathcal{T}^\mu_\nu$, ... $\Rightarrow$ less equations appear (now differential) which give rise to the integration constants ν,...

- In the presence of a potential $V(\phi)$, repeat the process :

$$\Box\phi = V'(\phi) + 4\pi\lambda\mathcal{T}$$

$$G^{\mu}_{\nu} = ...(same)$$

$$\mathcal{T}^{\mu}{}_{\nu;\mu} = \frac{\nu}{\phi(\nu+8\pi\phi^2)}\mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu} - \frac{\phi}{\nu+8\pi\phi^2}\frac{V'}{\lambda}\phi_{;\nu}$$

- A different eqm , e.g. $\Box\phi + F(\phi)\phi^{;\mu}\phi_{;\mu} = 4\pi\lambda\mathcal{T}$ gives different theories

Generalized vacuum Brans-Dicke theories

- In Brans-Dicke, the vacuum theory is obtained by setting $\mathcal{T}^{\mu}_{\nu} = 0$ in the matter eqm ($\Rightarrow \ \Box\phi = 0$, $G^{\mu}_{\nu} = \frac{2-3\lambda}{16\pi\lambda\phi}...$), where λ does not control any coupling.
  If set $\mathcal{T}^{\mu}_{\nu} = 0$ from the beginning of the "wrong" consistency approach, there is continuity with the above limiting process of the matter theory (with the difference that λ appears as integration constant).
- In Complete Brans-Dicke, setting $\mathcal{T}^{\mu}_{\nu} = 0$ gives probably the most interesting vacuum theories (more general that the vacuum BD), but does not exhaust the vacuum theories with $\Box\phi = 0$ (which are derived with the "correct" consistency process).

The vacuum consistency condition is eq. (6). Set $\Box\phi = 0$.
Consistency equations :

$$A' + B' + \frac{4\pi}{\phi} C(A - 2B) - \frac{1}{\phi}(A + B) = 0$$
$$C' + \frac{8\pi}{\phi} C^2 - \frac{1}{\phi} C + A + 2B = 0$$

$\Rightarrow$ one arbitrary function of ϕ
Of course, the unique T^{μ}_{ν} found before in the matter theory satisfies these eqs, and this is what I will call vacuum theory in the next

Vacuum Action

- $$S_g = \frac{\eta}{2(8\pi)^{3/2}} \int d^4x \sqrt{-g} \left[\sqrt{|\nu + 8\pi\phi^2|}\, R - \frac{8\pi}{\lambda} \frac{\nu + 4\pi(2-3\lambda)\phi^2}{|\nu + 8\pi\phi^2|^{3/2}} g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} \right]$$

$$\xrightarrow{\nu=0} \quad S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R - \frac{2-3\lambda}{2\lambda\phi} g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} \right)$$

- Symmetry transformation : $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $\chi = \chi(\phi)$

$$\Omega^2 = \sqrt{\frac{|\nu + 8\pi\phi^2|}{|\nu + 8\pi\chi^2|}}$$

- $\epsilon = \mathrm{sgn}(\nu + 8\pi\chi^2) > 0$:

$$\chi = \frac{s}{8\pi} \left(\theta \left| 4\pi\phi + \sqrt{2\pi}\sqrt{\nu + 8\pi\phi^2} \right|^{\pm 1} - \frac{2\pi\nu}{\theta} \left| 4\pi\phi + \sqrt{2\pi}\sqrt{\nu + 8\pi\phi^2} \right|^{\mp 1} \right)$$

- $\epsilon = \mathrm{sgn}(\nu + 8\pi\chi^2) < 0$: $\chi = \pm\sqrt{\frac{|\nu|}{8\pi}} \sin\left[\arcsin\left(\sqrt{\frac{8\pi}{|\nu|}}\, \phi \right) - c_1 \right]$

Total Action ?

- Simplest candidate
$$S = S_g + \int d^4x \sqrt{-g}\, J(\phi) L_m$$
To give the correct gravitational eqm under $\delta_g S \Rightarrow$
$J(\phi) = \frac{1}{\sqrt{8\pi}} \frac{\sqrt{|\nu + 8\pi\phi^2|}}{|\phi|}$
- However, $\delta_\phi S \Rightarrow \Box\phi = 4\pi\lambda\mathcal{T} + \frac{\lambda\nu}{\phi^2} L_m$
Thus, this total Lagrangian is valid only if on-shell the numerical value of L_m vanishes, e.g. for relativistic perfect fluids, action functionals have been constructed where the matter Lagrangian is proportional to the pressure (for pressureless dust this on-shell value vanishes)
- Even in the Jordan frame the matter is not minimally-coupled, due to the existence of the interaction
- It is possible that a general complete Lagrangian of a different form exists

Canonical form of the action (meaningful also in vacuum)

- Einstein term + non-canonical kinetic term :

$$S = \frac{\eta}{16\pi}\int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{8\pi}{\lambda(\nu+8\pi\phi^2)}\tilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \frac{2(8\pi)^{\frac{3}{2}}}{\phi\sqrt{|\nu+8\pi\phi^2|}}L_m(\tilde{\omega}^2\tilde{g}_{\kappa\lambda},\Psi)\right]$$

$$\tilde{g}_{\mu\nu} = \left(\frac{|\nu+8\pi\phi^2|}{8\pi}\right)^{\frac{1}{2}} g_{\mu\nu}$$

- Einstein term + canonical kinetic term :

- $\epsilon = \mathrm{sgn}(\nu+8\pi\chi^2) > 0$:

$$S = \frac{\eta}{16\pi}\int d^4x\,\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{1}{2}\epsilon_\lambda\tilde{g}^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} + \frac{2\eta(8\pi)^3}{\left|e^{\sqrt{2|\lambda|}\,\sigma} - 4\pi^2\nu^2 e^{-\sqrt{2|\lambda|}\,\sigma}\right|}L_m(\tilde{\omega}^2\tilde{g}_{\kappa\lambda},\Psi)\right]$$

$$\sigma-\sigma_0 = \sqrt{\frac{2}{|\lambda|}}\,\ln\left|4\pi\phi+\sqrt{2\pi}\sqrt{\nu+8\pi\phi^2}\right| \quad , \quad \tilde{\omega}^2 = \frac{8\pi}{\left|e^{\sqrt{\frac{|\lambda|}{2}}\,\sigma} + 2\pi\nu e^{-\sqrt{\frac{|\lambda|}{2}}\,\sigma}\right|}$$

- $\epsilon = \mathrm{sgn}(\nu + 8\pi\chi^2) < 0$:

$$S = \frac{\eta}{16\pi} \int d^4x \, \sqrt{-\tilde{g}} \, \Big[\tilde{R} + \frac{1}{2} \epsilon_\lambda \tilde{g}^{\mu\nu} \sigma_{,\mu} \sigma_{,\nu} + \frac{4(8\pi)^2}{|\nu| \, \sin\left(\sqrt{2|\lambda|}\,\sigma\right)} L_m(\tilde{\omega}^2 \tilde{g}_{\kappa\lambda}, \Psi) \Big]$$

$$\sigma - \sigma_0 = \sqrt{\frac{2}{|\lambda|}} \, \arcsin\left(\sqrt{\frac{8\pi}{|\nu|}} \, \phi \right) \quad , \quad \tilde{\omega}^2 = \frac{\sqrt{8\pi}}{\sqrt{|\nu|} \, \cos\left(\sqrt{\frac{|\lambda|}{2}} \, \sigma \right)}$$

Cosmology

- Complete Brans-Dicke

$$H^2+\frac{\kappa}{a^2}=\frac{8\pi}{3\phi}\rho-\frac{8\pi\phi}{\nu+8\pi\phi^2}H\dot{\phi}+\frac{4\pi}{3\lambda}\,\frac{\nu+4\pi(2-3\lambda)\phi^2}{(\nu+8\pi\phi^2)^2}\dot{\phi}^2$$

$$2\dot{H}+3H^2+\frac{\kappa}{a^2}=-\frac{8\pi}{\phi}\Big[p+\frac{\phi}{2\lambda}\,\frac{(1+2\lambda)\nu+4\pi(2-3\lambda)\phi^2}{(\nu+8\pi\phi^2)^2}\dot{\phi}^2 +\frac{\phi^2}{\nu+8\pi\phi^2}(2H\dot{\phi}+\ddot{\phi})\Big]$$

$$\ddot{\phi}+3H\dot{\phi}+4\pi\lambda(3p-\rho)=0$$

$$\dot{\rho}+3H(\rho+p)=\frac{\nu}{\phi(\nu+8\pi\phi^2)}\rho\,\dot{\phi}$$

- Brans-Dicke

$$H^2+\frac{\kappa}{a^2}=\frac{8\pi}{3\phi}\rho-H\frac{\dot{\phi}}{\phi}+\frac{2-3\lambda}{12\lambda}\frac{\dot{\phi}^2}{\phi^2}$$

$$2\dot{H}+3H^2+\frac{\kappa}{a^2}=-\frac{1}{\phi}\left(8\pi p+\frac{2-3\lambda}{4\lambda\phi}\dot{\phi}^2+2H\dot{\phi}+\ddot{\phi}\right)$$

$$\ddot{\phi}+3H\dot{\phi}+4\pi\lambda(3p-\rho)=0 \quad , \quad \dot{\rho}+3H(\rho+p)=0$$

- Non-conservation equation is integrated :

$$\rho = \frac{\rho_*}{a^{3(1+w)}} \frac{|\phi|}{\sqrt{|\nu + 8\pi\phi^2|}}$$

Early-times evolution

- Scalar field equation is integrated :

$$\dot{\phi} a^3 = c$$

c integration constant, $\phi(a)$ monotonic function

- Hubble equation :

$$\left(\frac{da}{d\phi}+\frac{4\pi\phi a}{\nu+8\pi\phi^2}\right)^2-\frac{4\pi}{3\lambda}\,\frac{a^2}{\nu+8\pi\phi^2}\left(1+\frac{2\epsilon\lambda\rho_*}{c^2}a^2\sqrt{|\nu+8\pi\phi^2|}\right)+\frac{\kappa a^6}{c^2}=0$$

and is integrated for any κ (here $\kappa = 0$)

$$\psi = a^2\sqrt{|\nu+8\pi\phi^2|}$$

$$\left(\frac{d\psi}{d\phi}\right)^2-\frac{16\pi}{3\lambda}\,\frac{\psi^2}{\nu+8\pi\phi^2}\left(1+\frac{2\epsilon\lambda\rho_*}{c^2}\psi\right)+\frac{4\kappa}{c^2}\,\frac{\psi^4}{|\nu+8\pi\phi^2|}=0$$

- *Case I*: If $\nu + 8\pi\phi^2 > 0$, $\lambda\phi > 0$

$$a^2(\phi) = \frac{2c^2}{|\lambda|\rho_*} \frac{1}{\sqrt{\nu + 8\pi\phi^2}} \frac{\sigma \left|4\pi\phi + \sqrt{2\pi}\sqrt{\nu + 8\pi\phi^2}\right|^{\pm\sqrt{\frac{2}{3|\lambda|}}}}{\left[1 - \sigma\left|4\pi\phi + \sqrt{2\pi}\sqrt{\nu + 8\pi\phi^2}\right|^{\pm\sqrt{\frac{2}{3|\lambda|}}}\right]^2}$$

$$t = \frac{1}{c}\int a(\phi)^3 d\phi$$

$\sigma > 0$ integration constant
$\nu > 0\,, \lambda > 0$:
Universe can emerge at zero cosmic time at a finite volume and avoids the cosmological singularity both in density and curvature (absent in Brans-Dicke). Additionally, it can exist a transient accelerating era (inflation) with exit into deceleration.

- *Case II*: If $\nu + 8\pi\phi^2 < 0$, $\lambda\phi > 0$

$$a^2(\phi) = \frac{c^2}{2|\lambda|\rho_*} \frac{1}{\sqrt{|\nu| - 8\pi\phi^2}} \left\{1 + \tan^2\left[\sigma \pm \frac{1}{\sqrt{6|\lambda|}} \arcsin\left(\sqrt{\frac{8\pi}{|\nu|}}\,\phi\right)\right]\right\}$$

σ integration constant

$\nu < 0\,, \lambda > 0:$

Again, there exist non-singular solutions in all volume, energy density, curvature, starting with acceleration and entering into deceleration (even for very small $\lambda > 0$). Also, this branch ($\nu < 0$) provides correct phenomenology at late times (so we have a unified picture for all times with a unique mechanism of energy transfer between matter and scalar field)

- *Case III*: If $\nu + 8\pi\phi^2 > 0$, $\lambda\phi < 0$

$$a^2 = \frac{c^2}{2|\lambda|\rho_*} \frac{1}{\sqrt{\nu + 8\pi\phi^2}} \left[1 + \tan^2\left(\sigma \pm \frac{1}{\sqrt{6|\lambda|}} \ln\left|4\pi\phi + \sqrt{2\pi}\sqrt{\nu + 8\pi\phi^2}\right|\right)\right]$$

σ integration constant

$\nu > 0\,, \lambda < 0:$

Again, non-singular, with finite a, ρ, R and a transient acceleration within deceleration

- *Case IV*: If $\nu + 8\pi\phi^2 < 0$, $\lambda\phi < 0$.

$$a^2 = \frac{c^2}{2|\lambda|\rho_*} \frac{1}{\sqrt{|\nu| - 8\pi\phi^2}} \sinh^{-2}\Big[\sigma \mp \frac{1}{\sqrt{6|\lambda|}} \arcsin\Big(\sqrt{\frac{8\pi}{|\nu|}}\,\phi\Big)\Big]$$

$\nu < 0\,, \lambda < 0\,:$

Again, non-singular, with finite a, ρ, R and decelerating.

Due to the interaction term, entropy production can occur at early times.

From $dU + pdV = TdS$, $U = \rho V$ $\Rightarrow$

$$\frac{T}{V}\dot{S} = \frac{\nu}{\phi(\nu+8\pi\phi^2)}\rho\dot{\phi}$$

Thus, it can be $\dot{S} > 0$ for the previous solutions. Initially, S is shared between all relativistic species, but as universe cools down, massive particles freeze out and S is only shared to photons. These photos propagate in universe and observed today with high entropy/baryon and $T \sim 1/a$. Of course, not too much S and matter should be produced to comply with observations.

Late-times evolution (numerical study)

-

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi}{3\phi}\left(\rho + \rho_{DE}\right)$$

$$2\dot{H} + 3H^2 + \frac{\kappa}{a^2} = -\frac{8\pi}{\phi}\left(p + p_{DE}\right)$$

$$\rho_{DE} \equiv -\frac{3\phi^2}{\nu+8\pi\phi^2}H\dot{\phi} + \frac{\phi}{2\lambda}\,\frac{\nu+4\pi(2-3\lambda)\phi^2}{(\nu+8\pi\phi^2)^2}\dot{\phi}^2$$

$$p_{DE} \equiv \frac{\phi}{2\lambda}\,\frac{(1+2\lambda)\nu+4\pi(2-3\lambda)\phi^2}{(\nu+8\pi\phi^2)^2}\dot{\phi}^2 + \frac{\phi^2}{\nu+8\pi\phi^2}(2H\dot{\phi} + \ddot{\phi})$$

▸

$$\left(\frac{1}{\phi}\,\rho\right)^{\cdot} + \frac{3H}{\phi}(\rho + p) = -\frac{8\pi}{\nu + 8\pi\phi^2}\rho\dot{\phi} = -Q$$

$$\left(\frac{1}{\phi}\,\rho_{DE}\right)^{\cdot} + \frac{3H}{\phi}(\rho_{DE} + p_{DE}) = \frac{8\pi}{\nu + 8\pi\phi^2}\rho\dot{\phi} = Q$$

similar+differ from standard relations :

$$\dot{\rho} + 3H(\rho + p) = -Q$$

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = Q$$

- $(\rho + \rho_{DE})/\phi$ strictly conserved instead of $\rho + \rho_{DE}$
- the form of Q here is determined by the theory itself

- Ignore the separable conservation of baryonic matter (small) – no fittings with data
- Ignore the radiation (small)
- Flat case, $\kappa = 0$
- Choose the units with $\phi_0 = 1$
- From $\Omega_{DE,0} \approx 0.7 \;\; \Rightarrow \;\; \dot{\phi}_0(\lambda, \nu)$
- From $\Omega_{m,0} \approx 0.3 \;\; \Rightarrow \;\; \rho_*(\nu)$
- Solve numerically the system $(\frac{\dot{a}}{a})^2 = \ldots$, $\ddot{\phi} = \ldots$, with $\rho = \ldots$
- derive phenomenological quantities $\rho_{DE}, \Omega_m, \Omega_{DE}, w_{DE}, q(z)$

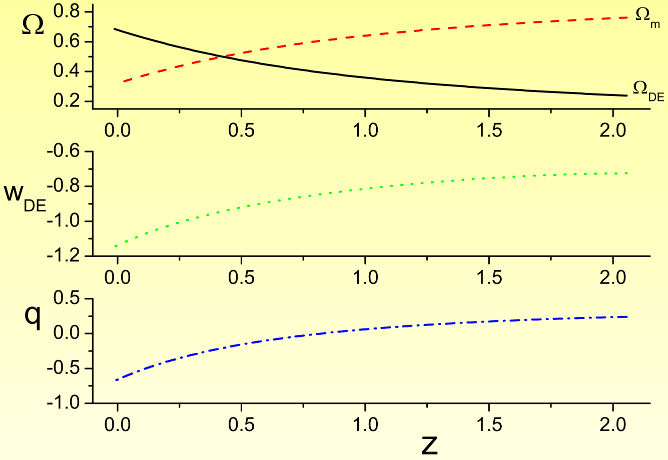


Figure: $\lambda = 10$, $\nu = -100$ *(more generally : $\lambda > 0$, $\nu < 0$, $\nu + 8\pi\phi^2 < 0$)*
Consistent with Case II of Radiation era
Or also $\lambda < 0$, $\nu < 0$, $\nu + 8\pi\phi^2 > 0$, $|\nu| \sim 8\pi\phi^2$

- Standard kinetic terms of Brans-Dicke cannot lead to acceleration (at least for realistic values of ω) and some potential is added; here only the modified kinetic terms and the modified conservation equation give an interesting cosmology
- Energy transfer can be either from the scalar field to dark matter or opposite
- Additionally, $\phi(t)$ increases up to very large redshifts, i.e. $G(t)$ decreases (as expected since today it has a small value)
- No physical divergence (e.g. $a, H, ...$) at finite time because of the pole of the quantity $\nu + 8\pi\phi^2$
- Additionally, $\frac{\dot{\phi}}{\phi H}\big|_0 \lesssim 10^{-2}$, necessary to be consistent with the bounds of variation of Newton's constant

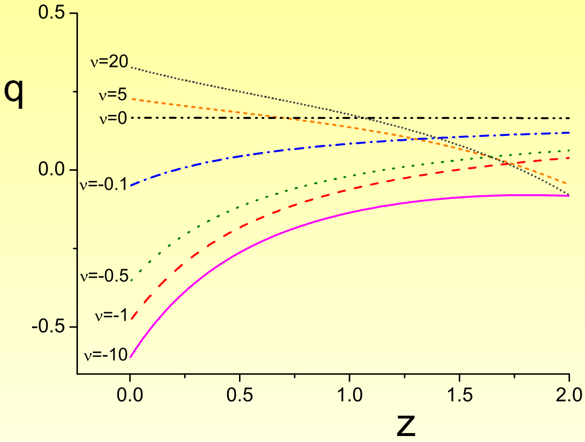


Figure: $\lambda = -10$

Evolution of q for fixed $\lambda < 0$ and various ν

Role of ν in obtaining late-times acceleration : for intermediate $\nu < 0$ we have a recent passage into acceleration

Cosmological Perturbations

- Background metric ($\kappa = 0$) ($\dot{} = \frac{d}{d\tau}$) ($\mathcal{H} \equiv \frac{\dot{a}}{a} = aH$)

$$d\bar{s}^2 = -a^2 d\tau^2 + a^2(\tau)\delta_{ij}dx^i dx^j$$

$\phi = \varphi + \delta\phi$

- Evolution of linear perturbations (forward) from deep within matter domination (neglect radiation) : $z_{\rm i} = 1000$, $N_{\rm i} = \ln a_{\rm i} = -6.91$ ($' = \frac{d}{dN}$)
-

$$\frac{4\pi}{3\lambda}\frac{\nu+4\pi(2-3\lambda)\varphi^2}{(\nu+8\pi\varphi^2)^2}\varphi'^2 - \frac{8\pi\varphi}{\nu+8\pi\varphi^2}\varphi' + \frac{8\pi\rho_* e^{-N}}{3\mathcal{H}^2\sqrt{|\nu+8\pi\varphi^2|}} - 1 = 0$$

$$\frac{2}{\mathcal{H}}\mathcal{H}' + \frac{4\pi}{\lambda}\frac{(1+2\lambda)\nu+4\pi(2-3\lambda)\varphi^2}{(\nu+8\pi\varphi^2)^2}\varphi'^2 + \frac{8\pi\varphi}{\nu+8\pi\varphi^2}\left(\frac{4\pi\lambda\rho_*\varphi e^{-N}}{\mathcal{H}^2\sqrt{|\nu+8\pi\varphi^2|}} - \varphi'\right) + 1 = 0$$

$$\varphi'' + \left(2 + \frac{\mathcal{H}'}{\mathcal{H}}\right)\varphi' - \frac{4\pi\lambda\rho_*\varphi e^{-N}}{\mathcal{H}^2\sqrt{|\nu+8\pi\varphi^2|}} = 0$$

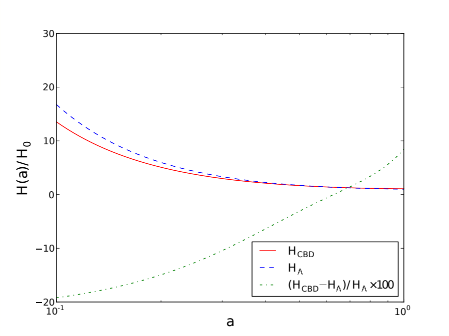
- Two ways to be integrated numerically:
  - $e^{-N}\mathcal{H}^{-2}$ replaced from the constraint $\rightarrow\ \varphi'' = \ldots$
    Initial conditions $\varphi_{\rm i}, \varphi'_{\rm i}$
  - $\varphi'^{\,2}_{\rm i} = \ldots\ ,\ \mathcal{H}' = \ldots\ \rightarrow$ first order system for $\varphi, \mathcal{H}$
    Initial conditions $\varphi_{\rm i}, \mathcal{H}_{\rm i}$
- $\Omega_{\rm DE\,,\,i} \ll 1\ \rightarrow\ |\varphi'_{\rm i}| \ll 1\ \rightarrow\ \rho_* = (3/8\pi) a_{\rm i}^3 H_{\rm i}^2 \sqrt{|\nu + 8\pi\varphi_{\rm i}^2|}$
  Initially, $\varphi \approx \varphi_{\rm i}\ \rightarrow\ H^2 \approx a_{\rm i}^3 H_{\rm i}^2 a^{-3}$ (Einstein in matter era)
  Parametrize $H_{\rm i}$ in terms of the dimensionless $\hat{\Omega}_{\rm m}$:
  $a_{\rm i}^3 H_{\rm i}^2 = \hat{H}_0^2 \hat{\Omega}_{\rm m}$
- $\lambda = 1$
- $\nu = -100$ (in units G_N^2)

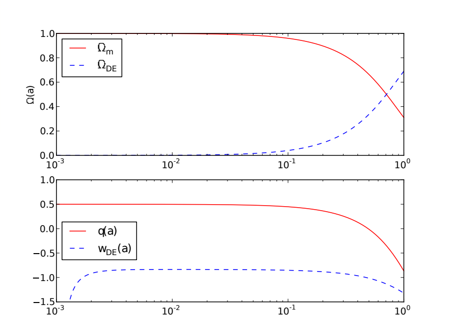
- $\Omega_\mathrm{m} = \frac{\hat{H}_0^2 \hat{\Omega}_\mathrm{m} \sqrt{|\nu + 8\pi\varphi_\mathrm{i}^2|}}{e^{3N} H^2 \sqrt{|\nu + 8\pi\varphi^2|}} \quad \rightarrow \quad \Omega_\mathrm{m}(a{=}1) = \hat{\Omega}_\mathrm{m} \sqrt{\frac{|\nu + 8\pi\varphi_\mathrm{i}^2|}{|\nu + 8\pi\varphi_0^2|}} \frac{\hat{H}_0^2}{H^2(a{=}1)}$

  constraint for φ_0

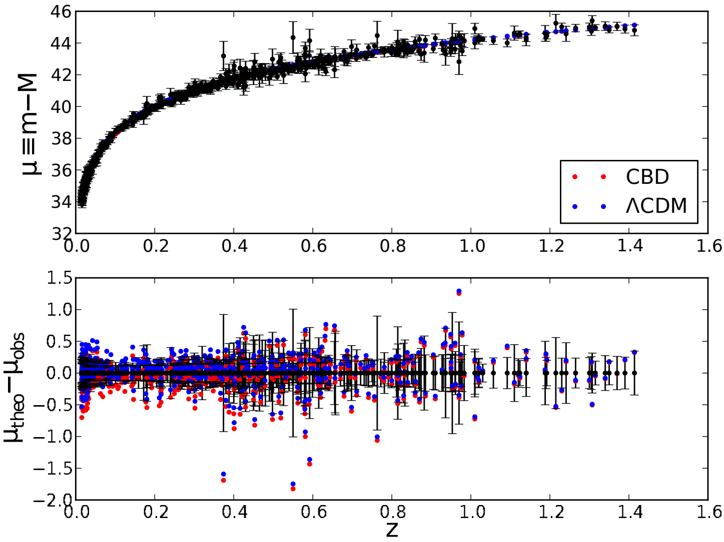
  $H(a{=}1) \approx \hat{H}_0 \;,\; \Omega_\mathrm{m}(a{=}1) \approx 0.3$

  $\hat{\Omega}_\mathrm{m}, \varphi_\mathrm{i} \;\; : \;\; \varphi(t=0) = \varphi_0$ satisfies the constraint


H(a)/H0
a
HCBD
HΛ
(HCBD−HΛ)/HΛ ×100


Ω(a)
Ωm
ΩDE
q(a)
wDE(a)

- Plot of the background quantities for $\hat{\Omega}_{\rm m} = 0.17$, $\varphi_{\rm i} = 0.029$
- Compare $H(a)$ to ΛCDM, with larger separation at earlier times
- Plot of $\Omega(a)$'s, stable matter dominated phase gradually overtaken dy DE
- Plot of $q(a)$, transition from deceleration to acceleration recently
- Plot of $w_{DE}(a)$, phantom behaviour today



μ ≡ m − M
CBD
ΛCDM
μtheo − μobs
z

- Plot of the distance moduli $\mu \equiv m - M$ of our model and of ΛCDM, compared to Union2.1 compilation from the Supernova Cosmology Project
- Our model fits the data with remarkable precision, comparable to ΛCDM, without the presence of a potential
- Viable background history $\rightarrow$ evolution of linear (scalar) perturbations

- Perturbed metric $ds^2 = g_{00} d\tau^2 + 2g_{0i} d\tau dx^i + g_{ij} dx^i dx^j$

$$g_{00} = -a^2(1 + 2AY)$$
$$g_{0i} = -a^2 B Y_i$$
$$g_{ij} = a^2(\delta_{ij} + 2H_L Y \delta_{ij} + 2H_T Y_{ij})$$

$A(\tau), B(\tau), H_L(\tau), H_T(\tau)$
$Y(\vec{k}, \vec{x}) \propto e^{i\vec{k}\cdot\vec{x}}$, $Y_i = -i\frac{k_i}{k} Y$, $Y_{ij} = \left(\frac{1}{3}\delta_{ij} - \frac{k_i k_j}{k^2}\right) Y$
$\phi = \varphi(\tau) + \chi(\tau) Y$

- Perturbed energy-momentum tensor

$$\begin{aligned} T^0{}_0 &= -\rho(1 + \delta Y) \\ T^0{}_i &= (\rho + p)(v - B) Y_i \\ T^i{}_j &= (p + \varpi Y)\delta^i_j + \frac{3}{2}(\rho + p)\sigma Y^i{}_j \end{aligned}$$

$\delta(\tau), v(\tau), \varpi(\tau), \sigma(\tau)$
$\delta\rho = \rho\delta = \rho\delta Y$, $\frac{u^i}{u^0} = v Y^i$, $\delta p = \varpi Y$, $\frac{3}{2}(\rho + p)\sigma(\tau)$

- Linearized non-conservation equations

$$\dot{\underset{\sim}{\delta}} + (1+w)\big(kv+3\dot{H}_L\big) + 3\mathcal{H}\left(\frac{\delta p}{\delta\rho}-w\right)\underset{\sim}{\delta} = \frac{\nu}{\varphi(\nu+8\pi\varphi^2)}\dot{\chi} - \frac{\nu(\nu+24\pi\varphi^2)}{\varphi^2(\nu+8\pi\varphi^2)^2}\dot{\varphi}\chi$$

$$\begin{aligned}\dot{v} - \dot{B} + (1-3w)\mathcal{H}(v-B) + \frac{\dot{w}}{1+w}(v-B) - \frac{\delta p/\delta\rho}{1+w}k\underset{\sim}{\delta} - kA + k\sigma \\ = -\frac{\nu}{\varphi(\nu+8\pi\varphi^2)}\frac{w}{1+w}k\chi\end{aligned}$$

$\frac{\delta p}{\delta\rho}\,\underset{\sim}{\delta} = \frac{\varpi}{\rho}$, $w = \dot{w} = 0$

- Linearized gravitational equations

0 - 0 component

$$\varphi\left[3\mathcal{H}^2 A - k\mathcal{H}B - 3\mathcal{H}\dot{H}_L - k^2\left(H_L + \frac{H_T}{3}\right)\right] - \frac{3}{2}\mathcal{H}^2\chi = 4\pi(-\rho a^2 \underset{\sim}{\delta} + \tau_1)$$

$$\tau_1 = \frac{24\pi\varphi^2 - \nu}{2\lambda(\nu + 8\pi\varphi^2)^3}\left[\nu + 4\pi(2-3\lambda)\varphi^2\right]\dot{\varphi}^2\chi$$
$$+\frac{\varphi\dot{\varphi}}{\lambda(\nu+8\pi\varphi^2)^2}\left\{\left[\nu+4\pi(2-3\lambda)\varphi^2\right](\dot{\varphi}A - \dot{\chi}) - 4\pi(2-3\lambda)\varphi\dot{\varphi}\chi\right\}$$
$$+\frac{6\nu\varphi\dot{\varphi}}{(\nu+8\pi\varphi^2)^2}\mathcal{H}\chi + \frac{\varphi^2}{\nu+8\pi\varphi^2}\left[k^2\chi + 3\mathcal{H}\dot{\chi} - \dot{\varphi}\left(6\mathcal{H}A - kB - 3\dot{H}_L\right)\right]$$

0 - i component

$$\varphi\left(\mathcal{H}A - \dot{H}_L - \frac{1}{3}\dot{H}_T\right) = \frac{4\pi\varphi^2}{\nu+8\pi\varphi^2}\left(\dot{\chi} - \mathcal{H}\chi - \dot{\varphi}A\right) + \frac{4\pi}{k}(1+w)\rho a^2\left(v - B\right)$$
$$+\frac{4\pi\varphi}{\lambda(\nu+8\pi\varphi^2)^2}\left[\nu(1+\lambda) + 4\pi(2-3\lambda)\varphi^2\right]\dot{\varphi}\chi$$

$\underline{i - j\ (i \neq j) \text{ component}}$

$$\varphi\left[-k^2 A - k\big(\dot{B} + \mathcal{H}B\big) + \ddot{H}_T - k^2\left(H_L + \frac{H_T}{3}\right) + \mathcal{H}\big(2\dot{H}_T - kB\big)\right]$$

$$= \frac{8\pi\varphi^2}{\nu + 8\pi\varphi^2}\left[k^2\chi + \dot{\varphi}\big(kB - \dot{H}_T\big)\right] + 12\pi(1+w)\rho a^2\sigma$$

$\underline{i-i\text{ component}}$

$$2\varphi\left[\left(\mathcal{H}^2+2\dot{\mathcal{H}}-\frac{k^2}{3}\right)A-\frac{k}{3}\left(\dot{B}+2\mathcal{H}B\right)+\mathcal{H}\dot{A}-\ddot{H}_L-2\mathcal{H}\dot{H}_L-\frac{k^2}{3}\left(H_L+\frac{H_T}{3}\right)\right]$$
$$-\left(\mathcal{H}^2+2\dot{\mathcal{H}}\right)\chi=8\pi\left(a^2\varpi+\tau_2\right)$$

$$\tau_2=\frac{\nu-24\pi\varphi^2}{2\lambda(\nu+8\pi\varphi^2)^3}\left[(1+2\lambda)\nu+4\pi(2-3\lambda)\varphi^2\right]\dot{\varphi}^2\chi$$
$$+\frac{\varphi^2}{\nu+8\pi\varphi^2}\left[\frac{2k}{3}\dot{\varphi}B-2(\ddot{\varphi}+\mathcal{H}\dot{\varphi})A-\dot{\varphi}\dot{A}+\ddot{\chi}+\mathcal{H}\dot{\chi}+\frac{2k^2}{3}\chi+2\dot{\varphi}\dot{H}_L\right]$$
$$-\frac{\varphi\dot{\varphi}}{\lambda(\nu+8\pi\varphi^2)^2}\left\{\left[(1+2\lambda)\nu+4\pi(2-3\lambda)\varphi^2\right](\dot{\varphi}A-\dot{\chi})-4\pi(2-3\lambda)\varphi\dot{\varphi}\chi\right\}$$
$$-\frac{2\nu\varphi}{(\nu+8\pi\varphi^2)^2}\left[\mathcal{H}\dot{\varphi}+4\pi\lambda(3w-1)\rho a^2\right]\chi$$

- Linearized scalar field equation
  <u>$\delta\phi$ equation</u>

$$\ddot{\chi}+2\mathcal{H}\dot{\chi}+k^2\chi-2\ddot{\varphi}A-\dot{\varphi}\left(4\mathcal{H}A+\dot{A}-kB-3\dot{H}_L\right)=4\pi\lambda\left(1-3\frac{\delta p}{\delta\rho}\right)\rho a^2\underset{\sim}{\delta}$$

As for the background, also for the perturbed equations, one is constraint

Conformal Newtonian Gauge : $H_T = B = 0$, $A = \Psi$, $H_L = -\Phi$

- conservation

$$\dot{\underset{\sim}{\delta}} + (1+w)\big(kv - 3\dot{\Phi}\big) + 3\mathcal{H}\left(\frac{\delta p}{\delta \rho} - w\right)\underset{\sim}{\delta} = \frac{\nu}{\varphi(\nu+8\pi\varphi^2)}\dot{\chi} - \frac{\nu(\nu+24\pi\varphi^2)}{\varphi^2(\nu+8\pi\varphi^2)^2}\dot{\varphi}\chi$$

$$\dot{v} + (1-3w)\mathcal{H}v + \frac{\dot{w}}{1+w}v - \frac{\delta p/\delta\rho}{1+w}k\underset{\sim}{\delta} - k\Psi + k\sigma = -\frac{\nu}{\varphi(\nu+8\pi\varphi^2)}\frac{w}{1+w}k\chi$$

- $0-0$ component

$$\varphi\big(3\mathcal{H}^2\Psi + 3\mathcal{H}\dot{\Phi} + k^2\Phi\big) - \frac{3}{2}\mathcal{H}^2\chi = 4\pi(-\rho a^2\underset{\sim}{\delta} + \tau_1)$$

$$\begin{aligned}\tau_1 = {} & \frac{24\pi\varphi^2 - \nu}{2\lambda(\nu+8\pi\varphi^2)^3}\left[\nu+4\pi(2-3\lambda)\varphi^2\right]\dot{\varphi}^2\chi \\ & + \frac{\varphi\dot{\varphi}}{\lambda(\nu+8\pi\varphi^2)^2}\left\{\left[\nu+4\pi(2-3\lambda)\varphi^2\right](\dot{\varphi}\Psi - \dot{\chi}) - 4\pi(2-3\lambda)\varphi\dot{\varphi}\chi\right\} \\ & + \frac{6\nu\varphi\dot{\varphi}}{(\nu+8\pi\varphi^2)^2}\mathcal{H}\chi + \frac{\varphi^2}{\nu+8\pi\varphi^2}\left[k^2\chi + 3\mathcal{H}\dot{\chi} - \dot{\varphi}\big(6\mathcal{H}\Psi + 3\dot{\Phi}\big)\right]\end{aligned}$$

0 - i component

$$\varphi\big(\mathcal{H}\Psi + \dot{\Phi}\big) = \frac{4\pi\varphi^2}{\nu+8\pi\varphi^2}\big(\dot{\chi} - \mathcal{H}\chi - \dot{\varphi}\Psi\big)$$
$$+\frac{4\pi\varphi}{\lambda(\nu+8\pi\varphi^2)^2}\big[\nu(1+\lambda) + 4\pi(2-3\lambda)\varphi^2\big]\dot{\varphi}\chi + \frac{4\pi}{k}(1+w)\rho a^2 v$$

$i-j$ $(i \neq j)$ component

$$\varphi\big(\Phi - \Psi\big) = \frac{8\pi\varphi^2}{\nu+8\pi\varphi^2}\chi + \frac{12\pi}{k^2}(1+w)\rho a^2\sigma$$

$i-i$ component

$$2\varphi\left[\left(\mathcal{H}^2+2\dot{\mathcal{H}} - \frac{k^2}{3}\right)\Psi + \frac{k^2}{3}\Phi + \ddot{\Phi} + 2\mathcal{H}\dot{\Phi} + \mathcal{H}\dot{\Psi}\right] - \big(\mathcal{H}^2+2\dot{\mathcal{H}}\big)\chi$$
$$= 8\pi\big(a^2\varpi + \tau_2\big)$$

$$\tau_2 = \frac{\nu-24\pi\varphi^2}{2\lambda(\nu+8\pi\varphi^2)^3}\big[(1+2\lambda)\nu+4\pi(2-3\lambda)\varphi^2\big]\dot{\varphi}^2\chi + \ldots..$$

- $\delta\phi$ equation

$$\ddot{\chi} + 2\mathcal{H}\dot{\chi} + k^2\chi - 2\ddot{\varphi}\Psi - \dot{\varphi}\left(4\mathcal{H}\Psi + \dot{\Psi} + 3\dot{\Phi}\right) = 4\pi\lambda\left(1-3\frac{\delta p}{\delta\rho}\right)\rho a^2 \underset{\sim}{\delta}$$

- no anisotropic contributions $(\sigma = 0)$
  <u>$i - j$ $(i \neq j)$ component</u>

$$\Phi - \Psi = \frac{\chi}{D(\varphi)} \quad , \quad D(\varphi) = \frac{\nu + 8\pi\varphi^2}{8\pi\varphi}$$

GR : $\Phi = \Psi$

- lensing potential : $\Phi_+ = \frac{\Phi+\Psi}{2}$
  comoving density perturbation : $\Delta = \underset{\sim}{\delta} + \frac{3\mathcal{H}}{k}(1+w)v$
- Derive differential equations of $\Phi_+, \chi, \underset{\sim}{\delta}, v$
- <u>0 - i component</u>

$$\Phi'_+ = -\left(1+\frac{\varphi'}{2D}\right)\Phi_+ + \frac{1}{2D^2}\left(D'+\frac{\varphi'}{2}\right)\chi + \frac{4\pi\varphi'}{\lambda(\nu+8\pi\varphi^2)^2}\left[\nu(1+\lambda)+4\pi(2-3\lambda)\varphi^2\right]\chi + \frac{4\pi}{kH\varphi}(1+w)\rho a v$$

0 - i component

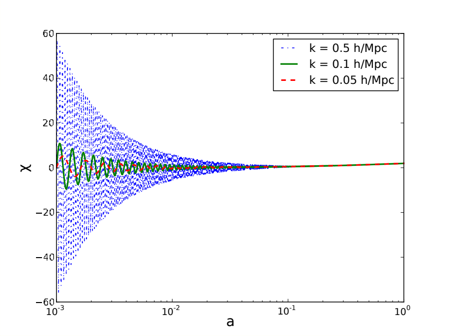
$$\frac{\varphi'}{\lambda D}\chi' = -\frac{8\pi\rho}{H^2}\Delta + 3\chi - \frac{4\pi(\nu-24\pi\varphi^2)}{\lambda(\nu+8\pi\varphi^2)^3}\left[\nu+4\pi(2-3\lambda)\varphi^2\right]\varphi'^2\chi$$
$$+\frac{8\pi}{\nu+8\pi\varphi^2}\left\{\frac{3\nu\varphi'}{4\pi D}+3\varphi^2+\frac{3\varphi^2\varphi'}{2D}+\frac{2\varphi^2 D'\varphi'}{2D^2}-\frac{2-3\lambda}{2\lambda D}\varphi\varphi'\left(\frac{\varphi\varphi'}{2D}+\varphi'+3\varphi\right)\right.$$
$$\left.-\frac{\nu\varphi'}{8\pi\lambda D}\left[\frac{\varphi'}{2D}+3(1+\lambda)\right]\right\}\chi - \frac{3\varphi\varphi'}{D}\Phi'_+$$
$$+\frac{8\pi\varphi'}{\nu+8\pi\varphi^2}\left(\frac{2-3\lambda}{2\lambda D}\varphi^2\varphi'+\frac{\nu}{8\pi\lambda D}\varphi'-3\varphi^2\right)\Phi_+ - \frac{2k^2\varphi}{a^2H^2}\Phi_+$$

conservation equations : also in terms of $\Phi_+, \chi, \underset{\sim}{\delta}, v$

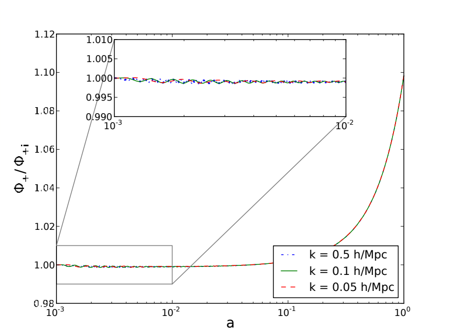
- $\delta p = 0$ in the scales of interest (confirmed in the sub-horizon approximation).

- Initial conditions for perturbations
  — Set $\Phi_{+\mathrm{i}} = -1$, $\chi_\mathrm{i} = 0$ as if we had minimal deviations from GR
  — Set $\Phi'_{+\mathrm{i}} = 0$ since in GR Φ, Ψ are constants initially $\rightarrow$

$$v_\mathrm{i} = \frac{2k}{3a_\mathrm{i} H_\mathrm{i}} \Phi_{+\mathrm{i}} \ , \quad \Delta_\mathrm{i} = -\frac{2k^2}{3a_\mathrm{i}^2 H_\mathrm{i}^2} \Phi_{+\mathrm{i}}$$



k = 0.5 h/Mpc
k = 0.1 h/Mpc
k = 0.05 h/Mpc
χ
a



k = 0.5 h/Mpc
k = 0.1 h/Mpc
k = 0.05 h/Mpc
Φ+/Φ+i
a

- χ, Φ_+ are scale-dependent, particularly at early times (contrary to GR)
- Oscillatory behaviour of χ mainly at early times, more pronounced for smaller scales - higher k's (also observed in other scalar-tensor theories).
  Understood from $\ddot{\chi} = ...$: equation of a damped harmonic oscillatory with a driving term, progressively the oscillations get damped by the Hubble friction term.
  More analysis is needed to see if these oscillations leads to instabilities at early times.
- Today, the equilibrium position of χ is shifted from zero to a positive value due to the driving term in $\ddot{\chi} = ...$, which tries to displace χ from the equilibrium position set by the initial conditions (as $\underset{\sim}{\delta}$ grows, the driving term will become more important)

- Φ_+ oscillates at early times (could contribute to early-times integrated Sachs-Wolfe effect with impact on CMB)
  $|\Phi_+|$ grows at late times, despite the accelerating background (yielding late-times integrated Sachs-Wolfe effect opposite to that of ΛCDM; needs further study). In GR Φ_+ decays with the onset of cosmic acceleration due to expansion
- Φ, Ψ oscillate around -1 at early times
  Ψ becomes positive at late times $\rightarrow$ significant for δ

Sub-Horizon approximation

- Length scales of wavemodes much smaller than Hubble radius, $k \gg aH$
- Quasistatic approximation : discard time derivatives of perturbations compared to spatial variation
- $0-0$ component :

$$\Phi = -\frac{4\pi}{\varphi}\,\frac{a^2}{k^2}\rho\underset{\sim}{\delta} + \frac{4\pi\varphi}{\nu+8\pi\varphi^2}\chi$$

  algebraic in χ instead of differential
- $i-i$ component :

$$\Phi - \Psi = \frac{8\pi\varphi}{\nu+8\pi\varphi^2}\chi + \frac{12\pi}{\varphi}\,\frac{a^2}{k^2}\varpi$$

  $\Rightarrow \quad \varpi = 0$
- $\delta\phi$ equation :

$$\chi = 4\pi\lambda\frac{a^2}{k^2}\rho\underset{\sim}{\delta} - 12\pi\lambda\frac{a^2}{k^2}\varpi$$

$$\frac{k^2}{a^2}\Phi = -\frac{4\pi}{\varphi}\left(\frac{\nu + 8\pi\varphi^2\,(1-\lambda/2)}{\nu + 8\pi\varphi^2}\right)\rho\underset{\sim}{\delta}$$

$$\frac{k^2}{a^2}\Psi = -\frac{4\pi}{\varphi}\left(\frac{\nu + 8\pi\varphi^2\,(1+\lambda/2)}{\nu + 8\pi\varphi^2}\right)\rho\underset{\sim}{\delta}$$

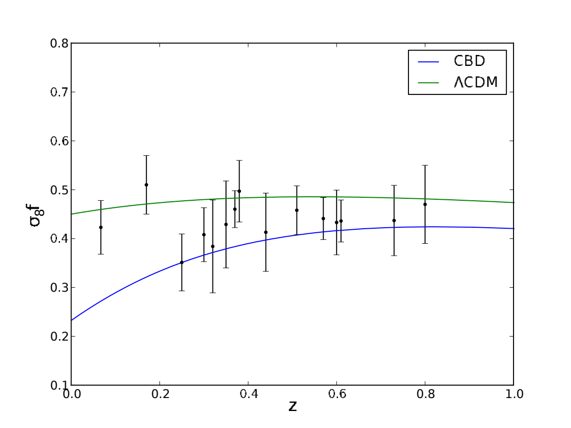
$$\Phi_+ = -\frac{1}{\lambda\varphi}\chi$$

$\rightarrow$ if χ grows at late times (or is scale independent) then Φ_+ also grows (or is scale independent)

Growth Rate in sub-horizon approximation

- $$\underset{\sim}{\delta}'' + \left(\frac{\mathcal{H}'}{\mathcal{H}} + 1\right)\underset{\sim}{\delta}' - \frac{4\pi}{\mathcal{H}^2}\,\frac{\nu + 4\pi(2+\lambda)\varphi^2}{\varphi(\nu + 8\pi\varphi^2)}\rho a^2 \underset{\sim}{\delta} = 0$$
- $$f' + f^2 + \left(\frac{\mathcal{H}'}{\mathcal{H}} + 1\right) f - \frac{4\pi}{\mathcal{H}^2}\,\frac{\nu + 4\pi(2+\lambda)\varphi^2}{\varphi(\nu + 8\pi\varphi^2)}\rho a^2 = 0$$

  $f = \frac{d \ln \underset{\sim}{\delta}}{d \ln a}$ linear growth rate
  $G_{\mathrm{eff}} = \frac{\nu + 4\pi(2+\lambda)\varphi^2}{\varphi(\nu + 8\pi\varphi^2)}$ (effective gravitational coupling of perturbations) passes to negative values recently (does not happen in ΛCDM, GR)
  $\Psi \propto G_{\mathrm{eff}}$
- $\Rightarrow\ \underset{\sim}{\delta} \propto k^2$ (as in sub-horizon of GR)
- $\Rightarrow\ \chi, \Phi_+, \Phi, \Psi$ scale-independent in agreement with late-times behaviour
- $\dot{v} + \mathcal{H} v - k\Psi = 0 \quad \Rightarrow \quad v \propto k$



Plot $f\sigma_8(z)$ numerically for recent z against ΛCDM and data

Amplitude of fluctuations $\sigma_8(z) = \sigma_8^0 \frac{\delta(z,k)}{\delta(0,k)}$

- Our model predicts less growth than ΛCDM due to that recently $G_{\mathrm{eff}} < 0$ $(\Psi > 0)$
  G_{eff} persists independently of parameters/initial conditions for any reasonable cosmology (more study is needed)

Conclusions

- Allowing non-conservation of matter in BD (with the same simple wave equation for the scalar), consistency gives three kinds of violating terms. Assuming a single interaction term each time, we extend BD theory to three uniquely defined interacting theories
- New massive parameters appears as integration constants, which when vanished give back BD.
- The general family of vacuum solutions with $\Box\phi = 0$ can be found, but the most interesting member of this family is the vanishing matter limit of the full matter theory.
- Although for special cases the action has been found (where the matter Lagrangian is non-minimally coupled even in the Jordan frame), it remains open to find the generic action (if exists).
- General solutions have been found in radiation cosmology with complete avoidance of initial singularity, a transient accelerating period and entropy production.

- At late-times, acceleration arises in agreement with the correct behavior of the density parameters and the dark energy equation of state. This happens with a sort of unified description of the universe history (inflation, matter domination and late-times acceleration) under the same mechanism of energy transfer between matter and the scalar field.
- Variation of ϕ is very slight over all history in agreement with the bounds of variation of G.
- Nice fitting with Supernovae
- All these, with just modifying the kinetic terms and the conservation equation, no extra ingredients (potentials, varying ω_{BD}, non-minimally couplings, e.t.c.)
- Initial and scale-dependent oscillations of scalar field perturbation are damped and lead to non-vanishing present value

- Lensing potential exhibits unusual growth at late times, in agreement with sub-horizon approximation
- Less growth is predicted compared to ΛCDM due to $\Psi > 0$ recently