Dynamical black holes in an expanding universe

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OVERVIEW

1. Motivation/context
2. Apparent horizons and their problems
3. A selection of exact solutions for cosmological black holes in various theories of gravity
4. Conclusions/open problems

MOTIVATION/CONTEXT

Real black holes are not static but dynamical, due to:

- astrophysical environment–companion in a binary system (*LIGO*), accretion disks, spherical accretion, ...
- cosmological background
- Hawking radiation and evaporation (for small black holes)

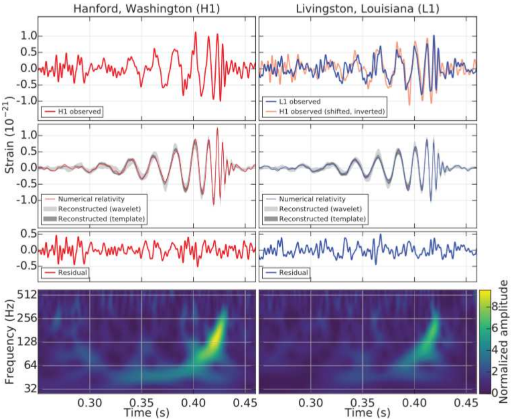
A problem of principle

- Cosmological asymptotics negligible for astrophysical BHs (*not* for primordial BHs), but important in principle.
- Theories of gravity alternative to GR aiming at explaining the cosmic acceleration without dark energy (*e.g.*, $f(R)$ gravity) contain a built-in, time-dependent $\Lambda \rightarrow$ BHs are not asympt. flat. Want to understand BHs in these theories.
- Testing ideas about backreaction of inhomogeneities, living in a giant void, ... (Boleiko & Celerier review).
- BH mechanics and thermodynamics were developed for *stationary* BHs with *event horizons* (null). Realistic BHs are dynamical and have *apparent horizons* (time-/space-like).
- Primordial BHs would have size $\sim H^{-1}$, very dynamical horizons. How fast do they accrete/grow? $\rightarrow$ dark matter

- Accretion of dark/phantom energy onto BHs (Babichev *et al.* '04; Chen & Jing '05; Izquierdo & Pavon '06; Pacheco & Horvath '07; Maeda, Harada & Carr '08; Gao, Chen, VF, Shen '08; Guariento *et al.* '08; Sun '08, '09; Gonzalez & Guzman '09; He *et l.* '09; Babichev *et al.* '11; Nouicer '11; Chadburn & Gregory '13)
- Studying the spatial variation of fundamental constants (*e.g.*, G in scalar-tensor gravity).

A practical problem

- Numerical simulations of BH mergers generate banks of templates of gravitational waveforms for *LIGO* detection; they use apparent/trapping horizons.



Event horizons useless for practical purposes. “BHs” identified with outermost marginally trapped surfaces and AHs (*e.g.*, Thornburg ’07, Baumgarte & Shapiro ’03, Chu, Pfeiffer, Cohen ’11).

in astrophysics, we use AHs, not event horizons

AHs AND THEIR PROBLEMS

Horizon $=$ *a frontier between things observable and things unobservable* (Rindler 1956)

The horizon, product of strong gravity, characterizes a black hole.
BH thermodynamics: if "background" is not Minkowski, internal energy in 1st law must be defined carefully (quasi-local energy, related to the notion of horizon).

Mini-review of null geodesic congruences, trapped surfaces: Congruence of null geodesics (tangent $l^a = dx^a/d\lambda$, affine parameter λ); metric h_{ab} in the 2-space orthogonal to l^a is determined by the following: pick another null vector field n^a such that $l^c n_c = -1$, then

$$h_{ab} \equiv g_{ab} + l_a n_b + l_b n_a$$

h_{ab} purely spatial, $h^a{}_b$ is a projection operator on the 2-space orthogonal to l^a. The choice of n^a is not unique but the geometric quantities of interest do not depend on it once l^a is fixed. Let $\eta^a =$ geodesic deviation, define

$$B_{ab} \equiv \nabla_b \eta_a \,,$$

orthogonal to the null geodesics. The transverse part of the deviation vector is

$$\tilde{\eta}^a \equiv h^a{}_b \, \eta^b = \eta^a + (n^c \eta_c) l^a$$

the orthogonal component of $l^c \nabla_c \eta^a$, denoted by a tilde, is

$$\widetilde{(l^c \nabla_c \eta^a)} = h^a{}_b h^c{}_d B^b{}_c \, \tilde{\eta}^d \equiv \tilde{B}^a{}_d \, \tilde{\eta}^d$$

Decompose transverse tensor $\widetilde{B}_{ab}$ as

$$\widetilde{B}_{ab} = \widetilde{B}_{(ab)} + \widetilde{B}_{[ab]} = \left(\frac{\theta}{2} \, h_{ab} + \sigma_{ab} \right) + \omega_{ab} \, ,$$

where expansion $\theta = \nabla_c \, l^c$ propagates according to the Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{2} - \sigma^2 + \omega^2 - R_{ab} l^a l^b$$

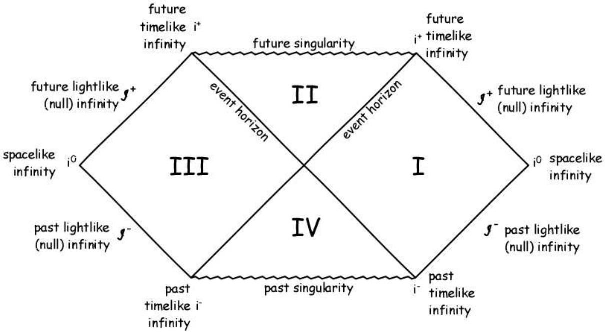
A compact and orientable surface has two independent directions orthogonal to it, corresponding to ingoing and outgoing null geodesics with tangents l^a and n^a, respectively.

- A *normal surface* corresponds to $\theta_l > 0$ and $\theta_n < 0$.
- A *trapped surface* corresponds to $\theta_l < 0$ and $\theta_n < 0$. The outgoing, in addition to the ingoing, future-directed null rays converge here instead of diverging and outward-propagating light is dragged back by strong gravity.
- A *marginally outer trapped surface (MOTS)* corresponds to $\theta_l = 0$ (where l^a is the outgoing null normal to the surface) and $\theta_n < 0$.
- A *marginally outer trapped tube (MOTT)* is a 3-dimensional surface which can be foliated entirely by marginally outer trapped (2-dimensional) surfaces.

Event horizons

An *event horizon* is a connected component of the boundary $\partial\left(J^{-}(\mathscr{I}^{+})\right)$ of the causal past $J^{-}(\mathscr{I}^{+})$ of future null infinity $\mathscr{I}^{+}$.

Causal boundary separating a region from which nothing can come out to reach a distant observer from a region in which signals can be sent out and eventually arrive to this observer.
Generated by the null geodesics which fail to reach infinity.
Provided that it is smooth, it is a null hypersurface.
To define and locate an event horizon, one must know all the future history of spacetime: a globalconcept, has teleological nature.



This event horizon "knows" about events belonging to a spacetime region very far away and in its future but not causally connected to it ("clarvoyance")
(Ashtekar & Krishnan, Ben Dov '07, BengtssonSenovilla '11, Bengtsson '11)

Apparent horizons

A *future apparent horizon* is *the closure of a 3-surface which is foliated by marginal surfaces*; defined by the conditions on the time slicings (Hayward '93)

$$\theta_l = 0\,,$$
$$\theta_n < 0\,,$$

where θ_l and θ_n are the expansions of the future-directed outgoing and ingoing null geodesic congruences, respectively (outgoing null rays momentarily stop expanding and turn around at the horizon). Inequality distinguishes between BHs and white holes.

AHs defined quasi-locally but *they depend on the choice of the foliation* (non-symmetric slicings of the Schwarzschild spacetime exist for which there is no AH (Wald & Iyer '91; Schnetter & Krishnan '06). In non-stationary situations, AHs $\neq$ EHs

Overlooked problem: the very existence of a (dynamical) BH depends on the observer!

In spherical symmetry, the AHs coincide in all *spherical foliations* (VF, G.F.R. Ellis, J. Firouzjaee, A. Helou, I. Musco 2017)

In GR, a black hole apparent horizon lies inside the event horizon provided that the null curvature condition $R_{ab}\, l^a l^b \geqslant 0 \; \forall$ null vector l^a is satisfied. But Hawking radiation itself violates the weak and the null energy conditions, as do quantum matter and non-minimally coupled scalars.

Trapping horizons

A *future outer trapping horizon (FOTH)* is *the closure of a surface* (usually a 3-surface) *foliated by marginal surfaces such that on its 2-dimensional "time slicings"* (Hayward 1993)

$$\theta_l = 0 \,,$$

$$\theta_n < 0 \,,$$

$$\mathcal{L}_n \theta_l = n^a \nabla_a \theta_l < 0$$

Last condition distinguishes between inner and outer Hs and between AHs and trapping Hs (sign distinguishes between future and past horizons).

BH trapping horizons have been associated with thermodynamics; claimed that it is the trapping horizon area and not the area of the event horizon which should be associated with entropy in black hole thermodynamics (Haijcek 1987; Hiscock 1989; Collins 1992; Nielsen) – controversial (Sorkin 1997; Corichi & Sudarsky 2002; Nielsen & Firouzjaee 2012).

The Parikh-Wilczek (2000) “tunneling” approach is in principle applicable also to apparent and trapping horizons.

Spherical symmetry

Misner-Sharp-Hernandez mass defined in GR and for spherical symmetry, coincides with the Hawking-Hayward quasi-local energy (Hawking '68; Hayward '94). Use areal radius R, write

$$ds^2 = h_{ab} dx^a dx^b + R^2 d\Omega_{(2)}^2 \qquad (a, b = 1, 2)$$

then

$$1 - \frac{2M}{R} \equiv \nabla^c R \nabla_c R$$

Formalism of Nielsen and Visser '06, general spherical metric is

$$ds^2 = -e^{-2\phi(t,R)} \left[1 - \frac{2M(t,R)}{R} \right] dt^2 + \frac{dR^2}{1 - \frac{2M(t,R)}{R}} + R^2 d\Omega_{(2)}^2$$

where $M(t, R)$ *a posteriori* is the Misner-Sharp-Hernandez mass.

Recast in Painlevé-Gullstrand coordinates as

$$ds^2 = -\frac{e^{-2\phi}}{(\partial\tau/\partial t)^2}\left(1-\frac{2M}{R}\right)d\tau^2 + \frac{2e^{-\phi}}{\partial\tau/\partial t}\sqrt{\frac{2M}{R}}\,d\tau dR + dR^2 + R^2 d\Omega_{(2)}^2$$

with $\phi(\tau, R)$ and $M(\tau, R)$ implicit functions. Use

$$c\,(\tau, R) \equiv \frac{e^{-\phi(t,R)}}{(\partial\tau/\partial t)},$$

$$v\,(\tau, R) \equiv \sqrt{\frac{2M(t,R)}{R}}\,\frac{e^{-\phi(t,R)}}{\partial\tau/\partial t} = c\sqrt{\frac{2M}{R}},$$

then line element becomes

$$ds^2 = -\left[c^2\,(\tau, R) - v^2\,(\tau, R)\right]d\tau^2 + 2v\,(\tau, R)\,d\tau dR + dR^2 + R^2 d\Omega_{(2)}^2$$

Expansions of radial null geodesic congruences are

$$\theta_{l,n} = \pm \frac{2}{R} \left(1 \mp \sqrt{\frac{2M}{R}} \right)$$

A sphere of radius R is trapped if $R < 2M$, marginal if $R = 2M$, untrapped if $R > 2M$. AHs located by

$$\frac{2M(\tau, R_{AH})}{R_{AH}(\tau)} = 1 \quad \Longleftrightarrow \quad \nabla^c R \nabla_c R \mid_{AH} = 0 \Longleftrightarrow \quad g^{RR} \mid_{AH} = 0$$

Inverse metric is

$$(g^{\mu\nu}) = \frac{1}{c^2} \begin{pmatrix} 1 & -v \\ -v & -(c^2 - v^2) \end{pmatrix}$$

Condition $g^{RR} = 0$ is a very convenient recipe to locate the apparent horizons in spherical symmetry.

Schwarzschild-de Sitter/Kottler spacetime

(Kottler 1918)

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2m}{R} - H^2 R^2\right) dt^2 + \left(1 - \frac{2m}{R} - H^2 R^2\right)^{-1} dR^2 \\
&\quad + R^2 d\Omega_{(2)}^2
\end{aligned}
$$

locally static for $R_1 < R < R_2$

AHs located by $g^{RR} = 1 - \frac{2m}{R} - H^2 R^2 = 0$. Formal roots of this cubic are

$$
\begin{aligned}
R_1 &= \frac{2}{\sqrt{3}H} \sin\psi \,, \\
R_2 &= \frac{1}{H} \cos\psi - \frac{1}{\sqrt{3}H} \sin\psi \,, \\
R_3 &= -\frac{1}{H} \cos\psi - \frac{1}{\sqrt{3}H} \sin\psi \,,
\end{aligned}
$$

with $\sin(3\psi) = 3\sqrt{3}\, mH$

$m, H > 0 \rightarrow R_3 < 0$ and there are at most two AHs. When R_1 and R_2 are real, R_1 is a BH AH, R_2 is a cosmological AH (both are null horizons).
Both apparent horizons exist only if $0 < \sin(3\psi) < 1$.
If $\sin(3\psi) = 1$ they coincide (extremal Nariai BH).
For $\sin(3\psi) > 1$ there is a naked singularity (interpretation: the BH horizon becomes larger than the cosmological one).

McVittie solution

(McVittie 1933) solution generalizes SdS/Kottler and represents a central object embedded in FLRW. Many papers over > 80 years, but not fully understood (also versions with negative cosmological constant and electrically charged). Focus on spatially flat FLRW “background”.
Simplifying assumption: McVittie’s no-accretion condition $G^1_0 = 0$. Original motivation: cosmological expansion on local system, also Swiss-cheese model (Einstein & Straus)

$$ds^2 = -\frac{\left(1 - \frac{m(t)}{2\bar{r}}\right)^2}{\left(1 + \frac{m(t)}{2\bar{r}}\right)^2} dt^2 + a^2(t)\left(1 + \frac{m(t)}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2 d\Omega^2_{(2)}\right) ,$$

no-accretion condition $\rightarrow$

$$\frac{\dot{m}}{m} + \frac{\dot{a}}{a} = 0$$

with solution

$$m(t) = \frac{m_0}{a(t)}\,, \qquad m_0 = \text{const.}$$

Reduces to Schwarzschild if $a \equiv 1$ and to FLRW if $m = 0$, singularity at $\bar{r} = m/2$ and $\bar{r} = 0$. Consider $\bar{r} > m/2$: energy density of the source fluid is finite but

$$P\left(t, \bar{r}\right) = -\frac{1}{8\pi}\left[3H^2 + \frac{2\dot{H}\left(1 + \frac{m}{2\bar{r}}\right)}{1 - \frac{m}{2\bar{r}}}\right] \rightarrow \infty$$

as $\bar{r} \rightarrow m/2$ with ${R^a}_a = 8\pi\left(3P - \rho\right)$ except for de Sitter "background" $\dot{H} = 0$.

AHs (Nolan, Li & Wang '06, VF, Zambrano & Nandra): rewrite metric using $R \equiv a(t)\bar{r}\left(1+\frac{m}{2\bar{r}}\right)^2$:

$$ds^2 = -\left(1-\frac{2m_0}{R}-H^2R^2\right)dt^2 + \frac{dR^2}{1-\frac{2m_0}{R}} - \frac{2HR\,dtdR}{\sqrt{1-\frac{2m_0}{R}}} + R^2 d\Omega^2_{(2)}$$

eliminate cross-term in $dtdR$ by defining new time $T(t,R)$ as $dT = \frac{1}{F}(dt+\beta dR)$ with $F(t,R)$ integrating factor and

$$\beta(t,R) = \frac{HR}{\sqrt{1-\frac{2m_0}{R}}\left(1-\frac{2m_0}{R}-H^2R^2\right)},$$

then

$$ds^2 = -\left(1-\frac{2m_0}{R}-H^2R^2\right)F^2dT^2 + \frac{dR^2}{1-\frac{2m_0}{R}-H^2R^2} + R^2 d\Omega^2_{(2)}$$

where $\bar{r} = m/2 \leftrightarrow R = 2m\,a(t) = 2m_0$ (non-expanding).

The McVittie metric admits arbitrary FLRW "backgrounds" generated by fluids with any constant equation of state. Restrict to dust at spatial infinity ($w = 0$), pressure is

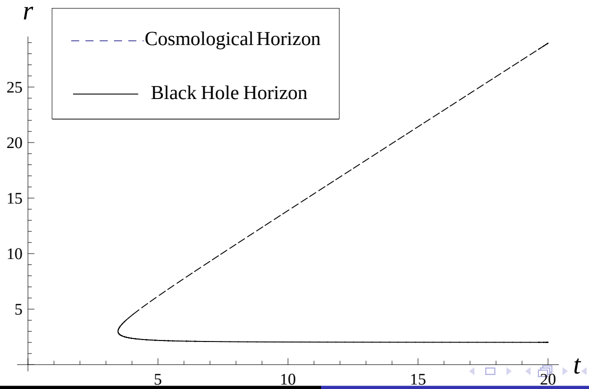
$$P(t, R) = \rho(t) \left(\frac{1}{\sqrt{1 - \frac{2m}{R}}} - 1 \right)$$

AHs at

$$g^{RR} = 1 - \frac{2m}{R} - H^2(t) R^2 = 0$$

Same cubic as in the SdS/Kottler case but now with time-dependent $H(t)$. Roots $R_{1,2}(t)$ given by the same expression but with time-dependent coefficient $H(t)$ – location of AHs depends on t. Both horizons exist if $mH(t) < 1/(3\sqrt{3})$, inequality satisfied only if $t > t_*$. The critical time for dust "background" is $t_* = 2\sqrt{3}\, m$.

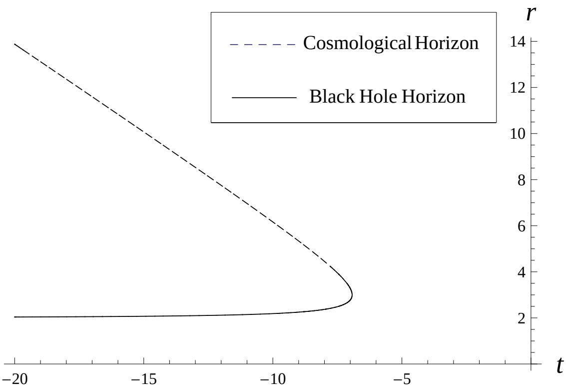
1. For $t < t_*$ it is $m > \frac{1}{3\sqrt{3}\,H(t)}$ and both $R_1(t)$ and $R_2(t)$ are complex. There are no AHs.
2. The critical time $t = t_*$ corresponds to $m = \frac{1}{3\sqrt{3}\,H(t)}$. $R_{1,2}(t)$ coincide at a real value, single AH at $R_* = \frac{1}{\sqrt{3}\,H(t_*)}$.
3. For $t > t_*$ it is $m < \frac{1}{3\sqrt{3}\,H(t)}$, there are two AHs at real $R_{1,2}(t) > 0$.



$H(t)$ diverges near the Big Bang, when the mass coefficient m stays supercritical at $m > \frac{1}{3\sqrt{3}\,H(t)}$. A BH horizon cannot be accommodated in this small universe and at $t < t_*$ there is a naked singularity at $R = 2m_0$. At t_* an (instantaneous) BH AH and a cosmological AH appear together at $R_1(t_*) = R_2(t_*) = \frac{1}{\sqrt{3}\,H(t_*)}$, in analogy with the Nariai BH. As $t > t_*$, this single horizon splits into an evolving black hole apparent horizon surrounded by an evolving cosmological horizon. The black hole apparent horizon shrinks, asymptoting to the $2m_0$ singularity as $t \to +\infty$.

Phantom FLRW "background" with $w < -1$

$$a(t) = \frac{A}{\left(t_{\rm rip} - t\right)^{\frac{2}{3|w+1|}}}\,, \qquad H(t) = \frac{2}{3|w+1|}\,\frac{1}{t_{\rm rip} - t}$$



Idealized interior solution for McVittie describes a relativistic star of uniform density in a FLRW “background” (Nolan), TOV equation in (VF & Jacques 2008).

Recent works on McVittie spacetime study conformal structure (Kleban *et al.*, Lake & Abdelqader 2011, da Silva *et al.* 2012, ...), which means integrating numerically the null geodesics or deriving general analytical results upon assuming something on the expansion. Lake & Abdelqader 2011: null geodesics asymptote to the singularity without entering it. Depending on the form of the scale factor, a bifurcation surface may appear which splits the spacetime boundary into a black hole horizon in the future and a white hole horizon in the past. A reflection of the McVittie no-accretion condition? da Silva, Fontanini, Guariento 2012 find that the presence of this white hole horizon depends crucially on the expansion history of the universe.

McV is also a solution of cuscuton theory, a special Horava-Lifschitz theory, and of shape dynamics (Abdalla *et al.* 2014; Gomes *et al.* 2011).

Generalized McVittie solutions

Remove no-accretion restriction from McVittie solutions (VF & A. Jacques 2008); "Synge approach" but reasonable matter sources exist.

$$ds^2 = -\frac{B^2(t,\bar{r})}{A^2(t,\bar{r})}dt^2 + a^2(t)A^4(t,\bar{r})\left(d\bar{r}^2 + \bar{r}^2 d\Omega_{(2)}^2\right),$$

$$m(t) \geqslant 0\,, \quad A(t,\bar{r}) = 1 + \frac{m(t)}{2\bar{r}}\,, \qquad B(t,\bar{r}) = 1 - \frac{m(t)}{2\bar{r}}$$

Mixed Einstein tensor

$$
\begin{aligned}
G_0^0 &= -\frac{3A^2}{B^2}\left(\frac{\dot{a}}{a}+\frac{\dot{m}}{\bar{r}A}\right)^2 , \\
G_0^1 &= \frac{2m}{\bar{r}^2 a^2 A^5 B}\left(\frac{\dot{m}}{m}+\frac{\dot{a}}{a}\right) , \\
G_1^1 &= G_2^2 = G_3^3 = -\frac{A^2}{B^2}\left\{2\frac{d}{dt}\left(\frac{\dot{a}}{a}+\frac{\dot{m}}{\bar{r}A}\right)+\left(\frac{\dot{a}}{a}+\frac{\dot{m}}{\bar{r}A}\right)\right. \\
& \left. \cdot\left[3\left(\frac{\dot{a}}{a}+\frac{\dot{m}}{\bar{r}A}\right)+\frac{2\dot{m}}{\bar{r}AB}\right]\right\}
\end{aligned}
$$

For the special subclass with $m = m_0 =$const., the quantity

$$
C \equiv \frac{\dot{a}}{a}+\frac{\dot{m}}{\bar{r}A} = \frac{\dot{M}}{M} - \frac{\dot{m}}{m}\frac{B}{A}
$$

reduces to $\dot{M}/M$ where $M(t) \equiv m_0 a(t)$ ("comoving mass" subclass).

At $\bar{r} = m/2$, C reduces to

$$C_\Sigma = \frac{\dot{a}}{a} + \frac{\dot{m}}{m} = \frac{\dot{M}}{M}$$

McVittie solutions correspond to $C_\Sigma = 0$, comoving mass solutions to $C = C_\Sigma = H$ everywhere.
Ricci scalar

$$R^a{}_a = \frac{3A^2}{B^2}\left(2\dot{C} + 4C^2 + \frac{2\dot{m}C}{\bar{r}AB}\right)$$

diverges at $\bar{r} = m/2$ unless m is a constant.

Single perfect fluid: only McVittie solutions.
Imperfect fluids can be matter sources for GMcV.
Imperfect fluid and no radial mass flow:

$$T_{ab} = (P + \rho)\, u_a u_b + P g_{ab} + q_a u_b + q_b u_a$$

purely spatial vector q^c describes a radial energy flow,

$$u^\mu = \left(\frac{A}{B}, 0, 0, 0\right) , \qquad q^\alpha = (0, q, 0, 0) \ , \qquad q^c u_c = 0$$

and $u^c u_c = -1$.

$$\rho\left(t,\bar{r}\right) = \frac{3A^2}{8\pi B^2}\left(\frac{\dot{a}}{a}+\frac{\dot{m}}{\bar{r}A}\right)^2 \geqslant 0\,,$$

$$P\left(t,\bar{r}\right) = \frac{-A^2}{8\pi B^2}\left\{2\frac{d}{dt}\left(\frac{\dot{a}}{a}+\frac{\dot{m}}{\bar{r}A}\right) + \left(\frac{\dot{a}}{a}+\frac{\dot{m}}{\bar{r}A}\right)\left[3\left(\frac{\dot{a}}{a}+\frac{\dot{m}}{\bar{r}A}\right)+\frac{2\dot{m}}{\bar{r}AB}\right]\right\}$$

Generalized Raychaudhuri equation

$$\dot{C} = -\frac{3C^2}{2} - \frac{\dot{m}}{\bar{r}AB}\,C - 4\pi\,\frac{B^2}{A^2}\,P = -4\pi\,\frac{B^2}{A^2}\left(P+\rho\right) - \frac{\dot{m}C}{\bar{r}AB}$$

Imperfect fluid and radial mass flow

Radial mass flow and energy current:

$$u^\mu = \left(\frac{A}{B}\sqrt{1+a^2A^4u^2}, u, 0, 0\right) , \qquad q^\mu = (0, q, 0, 0)$$

$$q = -\left(P+\rho\right)\frac{u}{2}$$

accretion rate is

$$\dot{M} = -\frac{1}{2}aB^2\sqrt{1+a^2A^4u^2}\left(P+\rho\right)\mathcal{A}u \, ,$$

Energy density is

$$8\pi\rho = \frac{A^2}{B^2}\left[3C^2 + \left(\dot{C}+\frac{\dot{m}C}{\bar{r}AB}\right)\frac{2a^2A^4u^2}{1+a^2A^4u^2}\right]$$

GMcV geometry is also a solution of Horndeski theory (Afshordi *et al.* 2014).

The "comoving mass" solution

Choice $M(t) = m_0\, a(t)$ selects a special subclass which is a *late-time attractor* of generalized McVittie solutions (Gao, Chen, VF & Shen 2008). AHs given *analytically* by

$$R_{c,b} = \frac{1}{2H}\left(1 \pm \sqrt{1 - 8m_0 \dot{a}}\right),$$

"Comoving mass" solutions are generic under certain assumptions, in the sense that all other generalized McVittie solutions approach them at late times (VF, Gao, Chen & Shen 2009). Coincides with non-rotating Thakurta (1981) solution, see (Culetu 2013; Mello, Maciel & Zanchin 2017)

Same phenomenology of AHs ("C-curve") appears in other solutions: some LTB models (solutions of GR), generalized McVittie solutions (solutions of GR and of Horndeski gravity).

Husain-Martinez-Nuñez 1994 new phenomenology of AHs. This spacetime describes an inhomogeneous universe with a spatially flat FLRW "background" sourced by a free, minimally coupled, scalar field

$$
\begin{aligned}
ds^2 &= (A_0\eta + B_0)\left[-\left(1-\frac{2C}{r}\right)^{\alpha} d\eta^2 + \frac{dr^2}{\left(1-\frac{2C}{r}\right)^{\alpha}}\right. \\
&\quad \left. + r^2\left(1-\frac{2C}{r}\right)^{1-\alpha} d\Omega^2_{(2)}\right], \\
\phi(\eta, r) &= \pm\frac{1}{4\sqrt{\pi}} \ln\left[D\left(1-\frac{2C}{r}\right)^{\alpha/\sqrt{3}} (A_0\eta + B_0)^{\sqrt{3}}\right]
\end{aligned}
$$

where $A_0, B_0, C, D \geqslant 0$ constants, $\alpha = \pm\sqrt{3}/2$, $\eta > 0$

Additive constant B_0 is irrelevant and can be dropped if $A_0 \neq 0$. When $A_0 = 0$, the HMN metric degenerates into the static Fisher spacetime (Fisher 1948)

$$ds^2 = -V^\nu(r)\, d\eta^2 + \frac{dr^2}{V^\nu(r)} + r^2 V^{1-\nu}(r) d\Omega^2_{(2)}$$

where $V(r) = 1 - 2\mu/r$, μ and ν are parameters, and the Fisher scalar field is

$$\psi(r) = \psi_0 \ln V(r)$$

(known as Janis-Newman-Winicour-Wyman solution, rediscovered many times, naked singularity at $r = 2C$, asympt. flat). The general HMN metric is conformal to the Fisher metric with conformal factor $\Omega = \sqrt{A_0\eta + B_0}$ equal to the scale factor of the "background" FLRW space and with only two possible values of the parameter ν. Set $B_0 = 0$.

Metric is asympt. FLRW for $r \to +\infty$ and is FLRW if $C = 0$ (in which case the constant A_0 can be eliminated by rescaling η).
Ricci scalar is

$$R^a{}_a = 8\pi \nabla^c \phi \nabla_c \phi = \frac{2\alpha^2 C^2 \left(1 - \frac{2C}{r}\right)^{\alpha - 2}}{3 r^4 A_0 \eta} - \frac{3A_0^2}{2\left(A_0 \eta\right)^3 \left(1 - \frac{2C}{r}\right)^{\alpha}}$$

Spacetime singularity at $r = 2C$ (for both values of α). ϕ also diverges there, Big Bang singularity at $\eta = 0$.

$2C < r < +\infty$ and $r = 2C$ corresponds to zero areal radius

$$R(\eta, r) = \sqrt{A_0 \eta}\, r \left(1 - \frac{2C}{r}\right)^{\frac{1-\alpha}{2}}$$

Using comoving time t

$$\begin{aligned} t &= \int d\eta\, a(\eta) = \frac{2\sqrt{A_0}}{3}\, \eta^{3/2}, \quad \eta = \left(\frac{3}{2\sqrt{A_0}}\, t\right)^{2/3} \\ a(t) &= \sqrt{A_0 \eta} = a_0\, t^{1/3} \end{aligned}$$

HMN solution in comoving time reads

$$ds^2 = -\left(1-\frac{2C}{r}\right)^{\alpha} dt^2 + a^2(t)\left[\frac{dr^2}{\left(1-\frac{2C}{r}\right)^{\alpha}} + \frac{r^2 d\Omega_{(2)}^2}{\left(1-\frac{2C}{r}\right)^{\alpha-1}}\right]$$

$$\phi(t,r) = \pm\frac{1}{4\sqrt{\pi}} \ln\left[D\left(1-\frac{2C}{r}\right)^{\alpha/\sqrt{3}} a^{2\sqrt{3}}(t)\right]$$

Areal radius increases with r for $r > 2C$. In terms of areal radius R, setting

$$A(r) \equiv 1-\frac{2C}{r}\,, \qquad B(r) \equiv 1-\frac{(\alpha+1)C}{r}$$

we have $R(t,r) = a(t) r A^{\frac{1-\alpha}{2}}(r)$ and a time-radius cross-term is eliminated by introducing a new time T with differential

$$dT = \frac{1}{F}\left(dt + \beta dR\right) ,$$

choosing

$$\beta(t,R) = \frac{HRA^{\frac{3(1-\alpha)}{2}}}{B^2(r) - H^2R^2A^{2(1-\alpha)}}$$

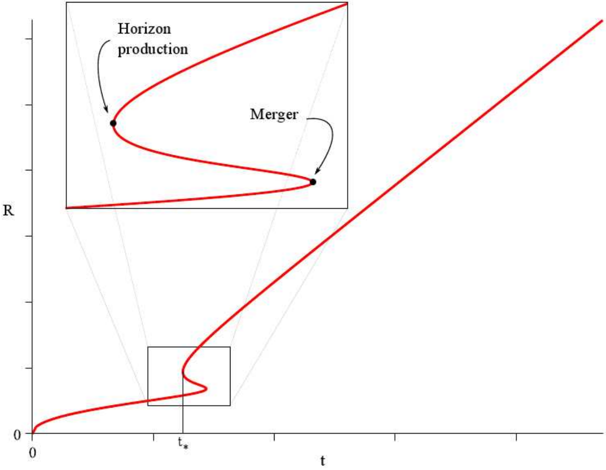
one has

$$\begin{aligned} ds^2 &= -A^{\alpha}(r)\left[1 - \frac{H^2R^2A^{2(1-\alpha)}(r)}{B^2(r)}\right]F^2dt^2 + R^2d\Omega^2_{(2)} \\ &+ \frac{H^2R^2A^{2-\alpha}(r)}{B^2(r)}\left[1 + \frac{A^{1-\alpha}(r)}{B^2(r) - H^2R^2A^{2(1-\alpha)}(r)}\right]dR^2 \end{aligned}$$

AHs located by $g^{RR} = 0$, or

$$\frac{1}{\eta} = \frac{2}{r^2}\Big[r - (\alpha+1)C\Big]\left(1 - \frac{2C}{r}\right)^{\alpha-1}$$

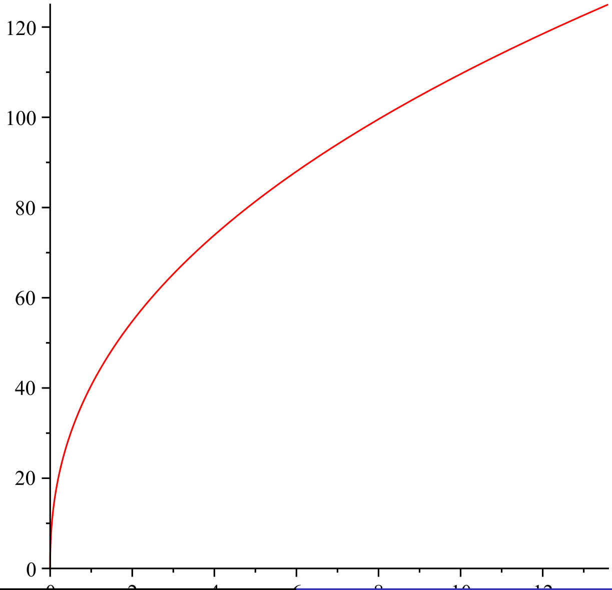
For $r \to +\infty$ ($R \to +\infty$), eq. reduces to $R \simeq H^{-1}$, cosmological AH in FLRW.



Horizon
production
Merger
R
0
0
t_*
t

If $\alpha = \sqrt{3}/2$, between the Big Bang and a critical time t_* there is only one expanding AH, then two other AHs are created at t_*. One is a cosmological AH which expands forever and the other is a BH horizon which contracts until it meets the first (expanding) BH AH. When they meet, these two annihilate and a naked singularity appears at $R = 0$ in a FLRW universe. "S-curve" phenomenology appears also in Lemaître-Tolman-Bondi spacetimes (dust fluid) (Booth *et al.*) (multiple "S"s are possible, *e.g.*, 5 may appear). The scalar field is regular on AHs.

For $\alpha = -\sqrt{3}/2$ there is only one cosmological AH and the universe contains a naked singularity at $R = 0$.



AHs are *spacelike*: normal vector always lies inside the light cone in an (η, r) diagram. In agreement with a general result of (Booth, Brits & Gonzalez) that a trapping horizon created by a massless scalar field must be spacelike.

Singularity at $R = 0$ is timelike for both values of α. Created with the universe in the Big Bang, not in a collapse process.

Clifton's (2006) solution of $f(R^c{}_c) = (R^c{}_c)^n$ gravity and some Clifton-Mota-Barrow (2005) solutions of Brans-Dicke gravity with fluid exhibit the same S-curve phenomenology of AHs (VF 2009; VF, V. Vitagliano. T. Sotiriou, S. Liberati 2012)

Other GR solutions

GR: Swiss-Cheese models (Einstein & Straus), Lemaìtre-Tolman-Bondi BHs (dust), members of the large Barnes family (1973), Sultana-Dyer solution. Several other analytical solutions of GR describing central inhomogeneities in FLRW "backgrounds" (Fonarev 1995, Vaidya 1977; Patel & Trivedi 1982; Roberts 1989, Burko 1997; Balbinot 1988; Cox 2003; Lindesay 2007, 2013). Many do not have reasonable matter sources ($\rho < 0$ somewhere). Usually obtained by conformally transforming or Kerr-Schild-transforming stationary BH solutions (Krasinski, Krasinski & Hellaby 2004, McClure & Dyer 2006; McClure *et al.* 2007, 2008).

CLIFTON-MOTA-BARROW SOLUTIONS OF BRANS-DICKE GRAVITY WITH PERFECT FLUID

Perfect fluid with $P^{(m)} = (\gamma - 1)\,\rho^{(m)}$, free Brans-Dicke field. Spherical, inhomogeneous, asympt. FLRW

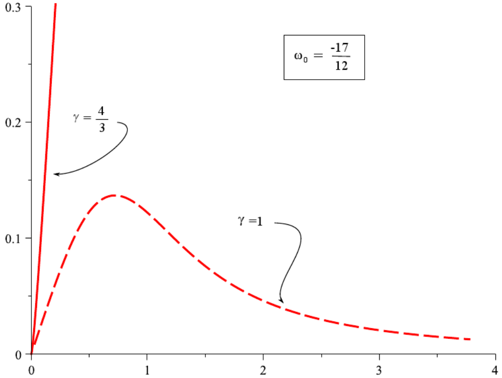
$$ds^2 = -e^{\nu(\bar{r})}dt^2 + a^2(t)e^{\mu(\bar{r})}(d\bar{r}^2 + \bar{r}^2 d\Omega^2_{(2)})\,,$$

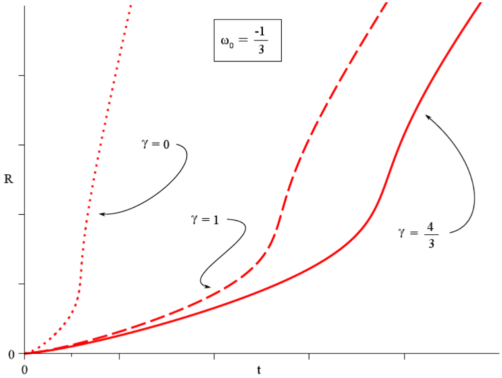
where

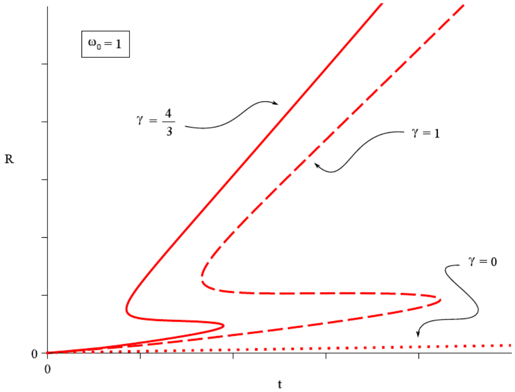
$$\begin{aligned}
e^{\nu(\bar{r})} &= \left(\frac{1 - \frac{m}{2\alpha\bar{r}}}{1 + \frac{m}{2\alpha\bar{r}}}\right)^{2\alpha} \equiv A^{2\alpha}\,,\\
e^{\mu(\bar{r})} &= \left(1 + \frac{m}{2\alpha\bar{r}}\right)^4 A^{\frac{2}{\alpha}(\alpha-1)(\alpha+2)}\,,\\
a(t) &= a_0\left(\frac{t}{t_0}\right)^{\frac{2\omega_0(2-\gamma)+2}{3\omega_0\gamma(2-\gamma)+4}} \equiv a_* t^{\beta}
\end{aligned}$$

$$
\begin{aligned}
\phi(t, \bar{r}) &= \phi_0 \left(\frac{t}{t_0} \right)^{\frac{2(4-3\gamma)}{3\omega_0 \gamma(2-\gamma)+4}} A^{-\frac{2}{\alpha}(\alpha^2 - 1)} \,, \\
\alpha &= \sqrt{\frac{2(\omega_0 + 2)}{2\omega_0 + 3}} \,, \\
\rho^{(m)}(t, \bar{r}) &= \rho_0^{(m)} \left(\frac{a_0}{a(t)} \right)^{3\gamma} A^{-2\alpha}
\end{aligned}
$$

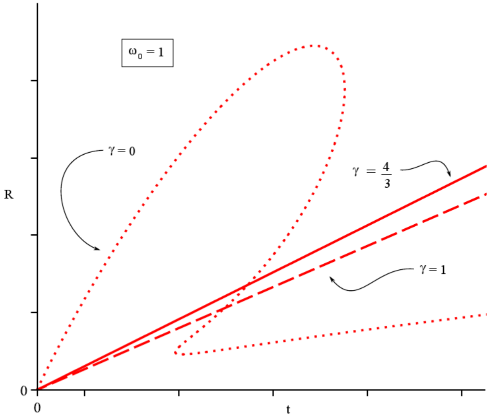
Areal radius of AHS shows a rich variety of behaviours as the 3 parameters vary (VF, V. Vitagliano, T. Sotiriou, S. Liberati 2012)




$\omega_0 = \frac{-1}{3}$
R
t
0
0
$\gamma = 0$
$\gamma = 1$
$\gamma = \frac{4}{3}$



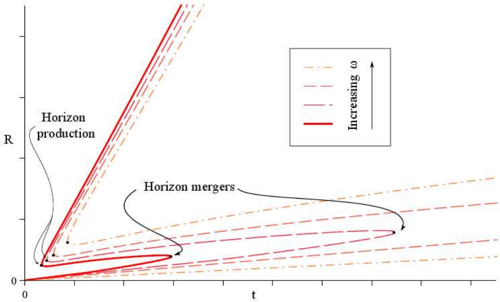
$\omega_0 = 1$
$\gamma = \frac{4}{3}$
$\gamma = 1$
$\gamma = 0$
R
0
0
t



$\omega_0 = 1$
$\gamma = 0$
$\gamma = \frac{4}{3}$
$\gamma = 1$
R
0
0
t

IS THERE A RELATION BETWEEN S-CURVE AND C-CURVE?

As $\omega \to \infty$, the Clifton-Mota-Barrow solution asymptotes to the comoving mass/non-rotating Thakurta solution of GR. The S-curve of the AHs reduces to a C-curve because the lower bend of the S-curve is pushed to infinity (VF & A. Prain 2015).



Is the C-curve always a limit of an S-curve?

CONCLUSIONS/OPEN PROBLEMS

- Ok to find new solutions, harder to interpret them
- Are AHs the "right" surfaces to characterize BHs? *LIGO* seems to say "yes".
- Foliation-dependence problem?
- Evolving horizons exhibit rich phenomenology and dynamics, but there seem to be 2 main classes for $R_{AH}(t)$: "C-curve" (McVittie) and "S-curve" (HMN).
- What is the relation between these 2 classes?
- Are AHs the "right' surfaces for BH thermodynamics? Adiabatic approximation needed or else non-equilibrium thermodynamics.
- Cosmic expansion vs local dynamics: sometimes AHs expand (even comoving), sometimes they resist the expansion (even contract). Is there any general rule?

THANK YOU

$Euχαρίστω$