A class of rotating hairy black holes in arbitrary dimensions

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arXiv:1707.03483

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Ninth Aegean Summer School. Sifnos, Greece.
September, 2017

Motivation

- In the last years a number of static black holes dressed with scalar field have been found in different dimensions.
- Black holes circumvent the no-hair conjecture $\rightarrow$ Cosmological constant, potentials and nonminimal couplings.
- In this way, for a self- interacting real scalar field nonminimally coupled to the Ricci invariant ($\xi R \Phi^2$), exact static black hole solutions has been found

$2+1\ D$	[Martínez and Zanelli, 1996, Henneaux et al., 2002, Xu and Zhao, 2013]
$3+1\ D$	[Bocharova et al., 1970, Bekenstein, 1974, Troncoso, R. et al., 2006] [Dotti et al., 2008, Charmousis et al., 2009, Anabalón and Cisterna, 2012] [Bardoux et al., 2012, Caldarelli et al., 2013, Astorino, 2013, Ayón-Beato et al., 2015]
$D>4$	[Nadalini et al., 2008, Martínez, 2009, Bravo-Gaete and Hassaïne, 2013b] [Bravo-Gaete and Hassaïne, 2013a, Giribet et al., 2014, Fan and Lu, 2015]

Motivation

- Exact rotating hairy black holes with a conformally coupled scalar field are almost absent in the literature
  [Natsuume and Okamura, 2000, Anabalón and Maeda, 2010, Bardoux et al., 2014]
  [Astorino, 2015] $\Longrightarrow$ Nonminimal case in arbitrary dimensions has not been reported.

- The lack of stationary hairy solutions arises from the complexity of the field equations

- The objective is to fill this gap and describe a new class of rotating hairy black holes in arbitrary dimensions.

Outline

1. Static hairy black holes
2. Rotating hairy black holes
3. Mass and angular momentum
4. Thermodynamics
   - Local stability
   - Global stability

Section

Static hairy black holes

- The action

$$I[g_{\mu\nu}, \Phi] = \int d^n x \sqrt{-g} \left[\frac{R}{2\kappa} - \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{\xi}{2} R \Phi^2 - V(\Phi) \right].$$

- Field equations

$$\begin{aligned} G_{\mu\nu} &= \kappa T_{\mu\nu}, \\ \Box\Phi &= \xi R \Phi + \frac{dV(\Phi)}{d\Phi}. \end{aligned}$$

- The energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi + \xi [g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu + G_{\mu\nu}] \Phi^2 - g_{\mu\nu} V(\Phi).$$

The black hole solution

It is found that choosing the non-minimal parameter as

$$\xi = \frac{n-1}{4n}, \tag{1}$$

the solution is given by

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\sigma^2, \quad \Phi = \Phi(r), \\ f(r) &= \frac{r^2}{l^2} + \alpha r^2 \log\left(1 - \left(\frac{a}{r}\right)^{n-1}\right), \\ \Phi(r) &= \sqrt{\frac{4n}{\kappa(n-1)}}\left(\frac{a}{r}\right)^{\frac{n-1}{2}}, \end{aligned}$$

where $a \rightarrow$ integration constant, $l \rightarrow$ n-dimensional AdS radius $\alpha \rightarrow$ coupling constant and $d\sigma^2 \rightarrow$ line element of an $n-2$ dimensional Ricci flat base manifold

- The self-intreraction potential takes the form

$$V(\Phi) = -\frac{(n-2)(n-1)}{2\kappa l^2} + \frac{\alpha(n-1)^2\Phi^2\left(-\kappa\Phi^2 + 4n^2 + n\left(\kappa\Phi^2 - 8\right)\right)}{8n\left(n\left(\kappa\Phi^2 - 4\right) - \kappa\Phi^2\right)} - \frac{\alpha(n-2)(n-1)\log\left(1 - \frac{\kappa(n-1)\Phi^2}{4n}\right)}{2\kappa},$$

- The Ricci scalar for the metric reads

$$R = (n-1)\left[-\frac{n}{l^2} + \alpha\left(\frac{a}{r}\right)^{n-1}\frac{(2n-1)(\frac{a}{r})^{n-1} - n}{(1-(\frac{a}{r})^{n-1})^2} - \alpha n\log\left(1 - \left(\frac{a}{r}\right)^{n-1}\right)\right],$$

which determines the existence of curvature singularities at $r = 0$ and $r = a$.

Some properties

- Ranging the radial coordinate as $r > a$, the exterior curvature singularity is set to the origin.

- The lapse $f(r)$ and scalar field $\Phi(r)$ are real and regular functions everywhere.

- Moreover, $f(r)$ is a monotonically increasing function and has a single non-zero positive root provided $\alpha > 0$. This root is given by

$$r_+ = h\,a, \quad \text{with} \quad h = \left(1 - \exp\left(-\frac{1}{\alpha l^2}\right)\right)^{-\frac{1}{n-1}} > 1.$$

- Thus, r_+ hides the curvature singularity at $r = a$.

The effective mass of the scalar field

- Expanding the potential around $\Phi = 0$, we get

$$V(\Phi) \xrightarrow[\Phi \to 0]{} -\frac{(n-1)(n-2)}{2\kappa l^2} - \frac{(n-1)^3 \alpha\kappa}{64n}\Phi^4 + \mathcal{O}(\Phi^6).$$

- Note that the potential does not contain a mass term.

- The non-minimal term ξR appearing in the scalar field equation

$$\Box\Phi = \xi R\Phi + \frac{dV(\Phi)}{d\Phi}$$

provides such a massive term.

The BF saturating mass

- The scalar field tends to zero for large r, and the metric locally approaches the AdS spacetime
$$f(r) \xrightarrow[r\to\infty]{} \frac{r^2}{l^2} - \alpha\frac{a^{n-1}}{r^{n-3}} + \mathcal{O}(r^{-2n+4}).$$

- Consequently, the Ricci scalar approaches to $R \to -n(n-1)l^{-2}$, so that at infinity the mass term is
$$\boxed{m_{eff}^2 = \xi R = -\frac{(n-1)^2}{4l^2},}$$
which exactly matches the Breitenlohner-Freedman mass bound in n spacetime dimensions. This is in agreement with the effective mass found in [Fan and Lu, 2015].

Section

Rotating hairy black holes

- The idea is to built the rotating black hole from the static one.

- The starting point is to consider the set of $(n-2)-$dimensional Ricci flat spaces having the form $\mathcal{M}_{n-2} = \mathbb{R} \times \mathcal{M}_{n-3}$ and then split the line element as

$$d\sigma^2 = d\phi^2 + d\Sigma^2,$$

  where $d\Sigma^2$ denotes the line element of $\mathcal{M}_{n-3}$, which is independent of the r, ϕ coordinates.

- The second and crucial step is to consider the following improper gauge transformation:

$$t \to \frac{1}{\sqrt{1-\omega^2}} (t - l\omega\phi) , \qquad \phi \to \frac{1}{\sqrt{1-\omega^2}} \left(\phi - \frac{\omega}{l} t\right) ,$$

  where $\omega^2 < 1$ (real coordinate transformation).

Some relevant properties

- Applying the boost, the stationary axisymmetric line element is obtained

$$ds^2 = -N^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + H(r)(d\phi + N^{\phi}(r)dt)^2 + r^2 d\Sigma^2,$$

$$N^2(r) = \frac{r^2(1-\omega^2)}{r^2 - l^2\omega^2 f(r)}, \quad N^{\phi}(r) = -\frac{r^2 - l^2 f(r)}{r^2 - l^2\omega^2 f(r)}\frac{\omega}{l}, \quad H(r) = \frac{r^2 - l^2\omega^2 f(r)}{1-\omega^2}.$$

- In the asymptotic region, $r \to \infty$, we have

$$\begin{aligned} N^2(r) &= 1 + \mathcal{O}(r^{-n+1}), \\ N^{\phi}(r) &= -\frac{\alpha l \omega a^{n-1}}{(1-\omega^2)r^{n-1}} + \mathcal{O}(r^{-2n+2}), \\ H(r) &= r^2 + \frac{\alpha l^2\omega^2 a^{n-1}}{(1-\omega^2)r^{n-3}} + \mathcal{O}(r^{-2n+4}). \end{aligned}$$

- In consequence, the rotating black hole is an asymptotically locally AdS spacetime in n dimensions.

Section

Conserved quantities

- The Regge-Teitelboim approach [Regge and Teitelboim, 1974] to determine the mass and angular momentum of this class of rotating hairy black holes is shown. In general, the canonical generator of an asymptotic symmetry is given by

$$H[\xi^\mu] = \int d^{n-1}x \left(\xi^\perp \mathcal{H}_\perp + \xi^i \mathcal{H}_i\right) + Q[\xi^\mu].$$

  - $\xi^\mu = (\xi^\perp, \xi^i)$ $\rightarrow$ Asymptotic Killing vectors
  - $\mathcal{H}_\perp, \mathcal{H}_i$ $\rightarrow$ Hamiltonian constraints
  - $Q[\xi^\mu]$ $\rightarrow$ Surface term that ensures a well-defined generator

- Therefore, on-shell evaluation of the canonical generator implies that $H[\xi^\mu] = Q[\xi^\mu]$ is the conserved charge associated to ξ^μ.

- We consider a minisuperspace containing the set of stationary metric and a scalar field previously considered. In this way, the expressions for the constraints are

$$\mathcal{H}_{\perp} = -\sqrt{\frac{H\gamma}{f}}r^{n-3}\left[(1-\kappa\xi\Phi^2)\left(\frac{^{(n-1)}R}{2\kappa}\right)-\frac{1}{2}fr^{n-3}\Phi'^2 - V\right] + \sqrt{\frac{f}{H\gamma}}\left(\frac{4\kappa}{1-\kappa\xi\Phi^2}\right)\pi_{r\phi}\pi^{r\phi} - [\xi\sqrt{fH\gamma}r^{n-3}(\Phi^2)']',$$

$$\mathcal{H}_{\phi} = -2\pi_{\phi}{}^{r}{}_{|r},$$

where $'$ stands for d/dr, $^{(n-1)}R$ is the Ricci scalar of the spatial section of the metric and γ is the determinant of the base space $\mathcal{M}_{n-3}$.

- The only nonvanishing canonical momentum of the gravitational field is given by

$$\pi_{\phi}^{r} = -\frac{H}{N}\sqrt{H\gamma}r^{n-3}\left(\frac{1-\kappa\xi\Phi^2}{4\kappa}\right)(N^{\phi})'.$$

Variation of the charge

- Then, the expression for the variation of the charge is

$$
\begin{aligned}
&\delta Q[\xi^{\mu}] = \\
&= 2\pi\Sigma \lim_{r\to\infty}\Bigg\{\sqrt{\frac{H\gamma}{f}}\left[\left(-\frac{f}{H}\left(\delta H' - \frac{H'}{2H}\delta H\right) - \left(\frac{H'}{2H} + \frac{n-3}{r}\right)\delta f\right)\left(\frac{1-\kappa\xi\Phi^2}{2\kappa}\right)\xi^{\perp}\right. \\
&+ \left.\frac{f}{H}\left[\left(\frac{1-\kappa\xi\Phi^2}{2\kappa}\right)\xi^{\perp}\right]'\delta H + f\left[\xi^{\perp}(\xi(\delta\Phi^2)' - \delta\Phi\Phi') - \xi^{\perp}_{,r}\xi\delta\Phi^2\right]\right] + 2\xi^{\phi}\delta\pi_{\phi}^{r}\Bigg\},
\end{aligned}
$$

- Additionally, the components of the asymptotic symmetries are given in terms of the spacetime components ∂_t and ∂_ϕ as follows,

$$
\begin{aligned}
\xi^{\perp} &= N\sqrt{f}\partial_t, \\
\xi^{\phi} &= \partial_\phi + N^{\phi}\partial_t.
\end{aligned}
$$

M and J

- The black holes considered here have two symmetries.
  - $\partial_t \rightarrow M = Q(\partial_t)$
  - $\partial_\phi \rightarrow J = -Q(\partial_\phi)$
- Using the previous expressions, the conserved charges are

$$M = \frac{\pi}{\kappa}\Sigma\alpha a^{n-1}\left(\frac{\omega^2+n-2}{1-\omega^2}\right), \quad J = \frac{\pi}{\kappa}\Sigma\alpha a^{n-1} l\omega\left(\frac{n-1}{1-\omega^2}\right),$$

such that the ground state

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2 d\sigma^2, \quad \Phi = 0,$$

possesses $M = 0$ and $J = 0$.

Some comments about the bounds

- $\omega = 0 \implies J = 0, \quad \omega^2 < 1 \implies M > 0.$
- Mass is bounded from below $M > |J/l|$.
- This bound guarantees the existence of an event horizon as can be seen from

$$r_+ = h \left[\frac{\kappa}{2\pi\alpha l\Sigma} \frac{\sqrt{M^2 l^2 (n-1)^2 - 4J^2(n-2)} - Ml(n-3)}{n-2} \right]^{\frac{1}{n-1}} .$$

- Additionally, the angular velocity reads

$$\omega = \frac{Ml(n-1) - \sqrt{M^2 l^2 (n-1)^2 - 4J^2(n-2)}}{2J},$$

in agreement with the condition $\omega^2 < 1$ by virtue of the bound $M > |J/l|$.

Section

Thermodynamical quantities

- The temperature can be determined by means of the surface gravity k at the horizon,

$$T = \frac{k}{2\pi} = \frac{\alpha h(n-1)a\sqrt{1-\omega^2}}{4\pi\left(h^{n-1}-1\right)}.$$

- Due to the nonminimal coupling, the standard Bekenstein-Hawking entropy acquires an extra factor [Visser, 1993, Ashtekar et al., 2003] and reads

$$S = \left(1-\kappa\xi\Phi(r_+)^2\right)\frac{2\pi A_+}{\kappa} = \frac{4\pi^2}{\kappa}\frac{a^{n-2}\Sigma}{\sqrt{1-\omega^2}}\left(\frac{h^{n-1}-1}{h}\right),$$

where A_+ is the area of the horizon.

- $\omega^2 < 1$ and $h > 1 \implies T > 0$ and $S > 0$.
- First law

$$dM = TdS + \Omega dJ$$

- Exhibits a Smarr formula

$$M = \frac{n-2}{n-1} TS + \Omega J.$$

Local stability

- Grand canonical ensemble $\rightarrow T, \Omega$ are fixed.
- Thermodynamical stability $\rightarrow$ Gibbs free energy $G(T,\Omega)$. The local stability criteria demand to analyze the concavity of this function, which implies the following stability conditions:

1. $$\frac{\partial^2 G}{\partial T^2} \leq 0,$$
2. $$\frac{\partial^2 G}{\partial \Omega^2} \leq 0,$$
3. $$\frac{\partial^2 G}{\partial T^2}\frac{\partial^2 G}{\partial \Omega^2} - \left(\frac{\partial^2 G}{\partial T \partial \Omega}\right)^2 \geq 0.$$

Gibbs free energy

- By using the Smarr relation, the Gibbs free energy $G = G(T,\Omega) = M - TS - \Omega J$, reduces to

$$G = -\frac{TS}{n-1} = -\frac{\pi}{\kappa}\Sigma\alpha a^{n-1},$$

which in terms of T and Ω is

$$G(T,\Omega) = -\frac{\pi\Sigma\alpha}{\kappa}\left[\frac{4\pi\left(h^{n-1}-1\right)}{\alpha h(n-1)}\frac{T}{\sqrt{1-\Omega^2 l^2}}\right]^{n-1}.$$

Stability criteria are satisfied as it can be seen explicitly from

1. $$\frac{\partial^2 G}{\partial T^2} = -\frac{4^{n-1}\alpha(n-2)(n-1)\pi^n\Sigma\left(h^{n-1}-1\right)^{n-1}T^{n-3}}{\kappa\left(\alpha h(n-1)\sqrt{1-\Omega^2 l^2}\right)^{n-1}} \leq 0,$$

2. $$\frac{\partial^2 G}{\partial \Omega^2} = -\frac{4^{n-1}\alpha l^2(n-1)\pi^n\Sigma\left(n\Omega^2 l^2+1\right)\left(h^{n-1}-1\right)^{n-1}T^{n-1}}{\kappa\left(1-\Omega^2 l^2\right)^2\left(\alpha h(n-1)\sqrt{1-\Omega^2 l^2}\right)^{n-1}} \leq 0,$$

3. $$\frac{\partial^2 G}{\partial T^2}\frac{\partial^2 G}{\partial \Omega^2} - \left(\frac{\partial^2 G}{\partial T \partial \Omega}\right)^2 =$$
$$= \frac{4^{2n-2}\alpha^4 h^2 l^2(n-1)^4\pi^{2n}\Sigma^2\left(h^{n-1}-1\right)^{2n-2}T^{2n-4}}{\kappa^2\left(\alpha h(n-1)\sqrt{1-\Omega^2 l^2}\right)^{2n}} \times \left(1+\frac{n-3}{1-\Omega^2 l^2}\right) \geq 0.$$

- Therefore, the rotating hairy black holes are thermodynamically locally stable in any dimension $n \geq 3$.

- The specific heat at fixed angular velocity,

$$C_\Omega = \left(\frac{\partial M}{\partial T}\right)_\Omega = 4\pi^2 \Sigma r_+^{n-2} \frac{(1-h^{1-n})(\omega^2+n-2)}{\kappa(1-\omega^2)^{3/2}} > 0.$$

- In consequence, the rotating hairy black hole always attains equilibrium with a heat bath.

Global stability

- The action principle for a fixed temperature also admits a rotating vacuum solution [Lemos, 1995]

$$ds_0^2 = -\frac{l^2 f - r^2\omega_0^2}{l^2(1-\omega_0^2)}dt^2 + \frac{2\omega_0\left(l^2 f - r^2\right)}{l(1-\omega_0^2)}dtd\phi + \frac{r^2 - l^2\omega_0^2 f}{1-\omega^2}d\phi^2 + \frac{dr^2}{f} + r^2 d\Sigma^2,$$

  - $f(\rho) = \dfrac{\rho^2}{l^2} - \dfrac{bl^{n-3}}{\rho^{n-3}}$,
  - ω_0 $\rightarrow$ boost parameter,
  - $b > 0$ $\rightarrow$ integration constant and
  - $d\Sigma^2$ $\rightarrow$ line element of a $n-3$ dimensional Ricci-flat hypersurface.

- This raises the question of whether one black hole can decay into the other.

- Global stability can be analyzed by comparing the free energy of the hairy black hole and the vacuum solution.

- The temperature, angular velocity and entropy for the vacuum solution are

$$T_0 = \frac{(n-1)b^{1/(n-1)}\sqrt{1-\omega_0^2}}{4\pi l}, \quad \Omega_0 = \frac{\omega_0}{l}, \quad S_0 = \frac{4\pi^2}{\kappa}\frac{\Sigma b^{(n-2)/(n-1)} l^{n-2}}{\sqrt{1-\omega_0^2}}.$$

- The mass and angular momentum can be computed using the expressions found previously

$$M_0 = \frac{\pi}{\kappa}\Sigma b l^{n-3}\left(\frac{\omega_0^2+n-2}{1-\omega_0^2}\right), \qquad J_0 = \frac{\pi}{\kappa}\Sigma b l^{n-2}\omega_0\left(\frac{n-1}{1-\omega_0^2}\right).$$

- Therefore, the Gibbs free energy of the vacuum solution is

$$G_0 = -\frac{\pi \Sigma b l^{n-3}}{\kappa}.$$

- Comparing the black holes with equal temperature and angular velocity yields

$$a = \frac{(h^n - h) b^{1/(n-1)}}{h^2 \alpha l}, \quad \omega = \omega_0,$$

respectively.

- Computing the difference between the free energies of the hairy and vacuum black holes gives

$$\boxed{\Delta G = G - G_0 > 0}$$

- The vacuum solution is the configuration thermodynamically preferred over the hairy one.

Concluding remarks

- A class of exact rotating black hole solutions of gravity nonminimally coupled to a self-interacting scalar field in arbitrary dimensions has been presented.
- These geometries are asymptotically locally AdS spacetimes and possess an event horizon with a Ricci-flat geometry that covers the curvature singularity at the origin.
- The scalar field is real and regular everywhere.
- The effective mass term saturates the Breitenlohner-Freedman bound for arbitrary dimension, which ensures the perturbative stability of the global AdS spacetime under scalar perturbations.

Concluding remarks

- It is found that the mass is bounded from below by the angular momentum as $M > |J/l|$, being consistent with the existence of an event horizon.
- The thermodynamical analysis was carried out in the grand canonical ensemble.
- The first law is satisfied and a Smarr formula is exhibited.
- The rotating hairy black hole always attains equilibrium with a heat bath.
- The hairy black hole is likely to decay into the vacuum black hole for any temperature.

$$E\upsilon\chi\alpha\rho\iota\sigma\tau\acute{\omega}\,!$$

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