No hair theorems and compact objects in Horndeski theories

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Fact: GR is a unique theory

- Theoretical consistency: In 4 dimensions, consider $\mathcal{L} = \mathcal{L}(\mathcal{M}, g, \nabla g, \nabla\nabla g)$. Then Lovelock's theorem in $D = 4$ states that GR with cosmological constant is the unique metric theory, emerging from

$$S_{(4)} = \int_{\mathcal{M}} d^4x \sqrt{-g^{(4)}}\,[R - 2\Lambda],$$

giving,

  - Equations of motion of 2^{nd}-order
  - given by a symmetric two-tensor, $G_{\mu\nu} + \Lambda g_{\mu\nu}$
  - and admitting Bianchi identities.

Under these hypotheses GR is the unique massless-metric 4 dimensional theory of gravity!

Modified gravity theories

- GR agrees to growing accuracy with local gravity experiments
- Cosmological observations point towards the presence of a tiny cosmological constant fueling the late time acceleration of the Universe.
- Astrophysical observations point towards an unknown and undiscovered dark matter component

Could it be that GR is not only modified at the UV but also at the IR?
$\rightarrow$ Modified gravity theories

- Extra dimensions
- 4-dimensional modification of GR: Scalar-tensor theories, vector-tensor, Massive gravity, bigravity
- Breaking of Lorentz invariance: Horava theory, Einstein-Aether theories
- Theories modifying geometry: torsion, choice of geometric connexion

Scalar-tensor theories

- are the simplest modification of gravity with one additional degree of freedom
- Admit a uniqueness theorem due to Horndeski 1973.
- contain or are limits of other modified gravity theories. $f(R)$, $f(\mathcal{G})$, massive gravity etc.
- Can have late time de Sitter behavior.
- Have non trivial black hole solutions in Horndeski theory
- Have insightful screening mechanisms (Vainshtein) providing a "classical" limit to GR
- Include theories that can screen classically a big cosmological constant

Jordan-Brans-Dicke theory [review by Sotiriou 2014]

Simplest scalar tensor theory

$$S_{\rm BD} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(\varphi R - \frac{\omega_0}{\varphi}\nabla^\mu\varphi\nabla_\mu\varphi - m^2(\varphi-\varphi_0)^2\right) + S_m(g_{\mu\nu},\psi)$$

- ω_0 Brans Dicke coupling parameter fixing scalar strength
- $\phi = \phi_0$ constant gives GR solutions (with a cosmological constant) but spherically symmetric solutions are not unique!
- For spherical symmetry we find,

$$\gamma \equiv \frac{h_{ij}|_{i=j}}{h_{00}} = \frac{2\omega_0 + 3 - exp\left[-\sqrt{\frac{2\varphi_0}{2\omega_0+3}}mr\right]}{2\omega_0 + 3 + exp\left[-\sqrt{\frac{2\varphi_0}{2\omega_0+3}}mr\right]}$$

- where $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$
- $\omega_0 > 40000$... Need higher order kinetic terms in order to screen the scalar mode locally : Vainshtein mechanism [Review: Babichev, Deffayet]

Galileons/Horndeski [Horndeski 1973]

What is the most general scalar-tensor theory

with second order field equations [Horndeski 1973]?

Horndeski has shown that the most general action with this property is

$$S_H = \int d^4x \sqrt{-g}\,(L_2 + L_3 + L_4 + L_5)$$

$$
\begin{aligned}
L_2 &= K(\phi, X), \\
L_3 &= -G_3(\phi, X)\Box\phi, \\
L_4 &= G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\
L_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]
\end{aligned}
$$

the G_i are free functions of ϕ and $X \equiv -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$ and $G_{iX} \equiv \partial G_i/\partial X$.

- In fact same action as covariant Galileons [Deffayet, Esposito-Farese, Vikman]. Galileons are scalars with Galilean symmetry for flat spacetime. Examples:
  $G_4 = 1 \longrightarrow R$.
  $G_4 = X \longrightarrow G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$

Galileons/Horndeski [Horndeski 1973]

$$S_H = \int d^4x \sqrt{-g}\,(L_2 + L_3 + L_4 + L_5)$$

$$L_2 = K(X),$$
$$L_3 = -G_3(X)\Box\phi,$$
$$L_4 = G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right],$$
$$L_5 = G_5(X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]$$

the G_i are free functions of ϕ and $X \equiv -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$ and $G_{iX} \equiv \partial G_i/\partial X$.

- Horndeski theory includes Shift symmetric theories where G_i's depend only on X and $\phi \to \phi + c$.
  Associated with the symmetry there is a Noether current, J^μ which is conserved $\nabla_\mu J^\mu = 0$.
  Presence of this symmetry permits a very general no hair argument

Black holes have no hair [recent review Herdeiro and Radu 2015]

During gravitational collapse...
Black holes eat or expel surrounding matter
their stationary phase is characterized by a limited number of charges
and no details
black holes are bald...

No hair arguments/theorems dictate under some reasonable hypotheses that adding degrees of freedom lead to singular solutions...

For example in vanilla scalar-tensor theories black hole solutions are GR black holes with constant scalar.

Warning : beyond GR Birkhoff's theorem will not function.
Spherical symmetry thus does not guarantee staticity.
Scalar tensor black holes radiate monopole gravity waves.
There is no reason for metric and scalar not to radiate for spherical symmetry

No hair [Hui, Nicolis] [Sotiriou, Zhou] [Babichev, CC, Lehébel]

Static no hair theorem

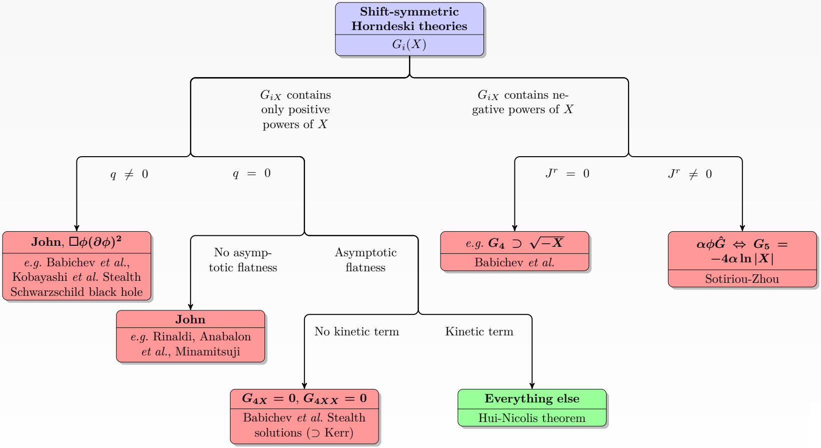
Consider shift symmetric Horndeski theory with G_2, G_3, G_4, G_5 arbitrary functions of X. We have a Noether current J^μ which is conserved, $\nabla_\mu J^\mu = 0$.
We now suppose that:

1. spacetime and scalar are spherically symmetric and static,
$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 dK^2, \ \phi = \phi(r)$$
2. spacetime is asymptotically flat, $\phi' \to 0$ as $r \to \infty$ and the norm of the current J^2 is finite on the horizon,
3. there is a canonical kinetic term X in the action,
4. and the G_i functions are such that their X-derivatives contain only positive or zero powers of X.

Under these hypotheses, ϕ is constant and thus the only black hole solution is locally isometric to Schwarzschild.

Most interesting part of no go theorem: Breaking any of these hypotheses leads to black hole solutions!

Hair versus no hair [figure: Lehébel]



Introducing time dependence, $q \neq 0$

Spherical symmetry certainly does not impose staticity. In fact no hair theorems may be pointing out to an inconsistency in this direction.

- Furthermore, for self accelerating [Babichev, Esposito-Farese] or self tuning solutions [Charmousis] one has a time dependence for the scalar in FRW coordinates
- In spherical symmetry this leads to a time and radially depending scalar already for flat spacetime.
- So let us allow time dependence for the scalar as a first step while keeping for a static and spherically symmetric spacetime.

The question of time dependence, $qt + \psi(r)$

Consistency theorem [Babichev, CC, Hassaine]

Consider an arbitrary shift symmetric Horndeski theory and a scalar-metric ansatz with $q \neq 0$. The unique solution to the scalar field equation $\mathcal{E}_\phi = 0$ and the "matter flow" metric equation $\mathcal{E}_{tr} = 0$ is given by $J^r = 0$.

- We are killing two birds with one stone.
- The current now reads, $J^\mu J_\mu = -h(J^t)^2 + (J^r)^2/f$ and is regular. Time dependence renders theorem irrelevant.
- Given the higher order nature of Horndeski theory this theorem basically tells us that if $\phi = qt + \psi(r)$ then there exist $\phi' \neq 0$ solutions to the field equations.
- One can prove for some theories that if $\phi = \phi(t, r)$ then the only compatible ϕ are $\phi = qt + \psi(r)$ and also $\phi = \phi_1(r^2 - t^2)$ for flat spacetime (Fab 4 self tuning solution)

General solution

Consider, $L = R - \eta(\partial\phi)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\Lambda$ For static and spherically symmetric spacetime.

The general solution of theory L for static and spherically symmetric metric and $\phi = \phi(t, r)$ is given as a solution to the following third order algebraic equation with respect to $\sqrt{k(r)}$:

$$(q\beta)^2\left(\kappa + \frac{r^2}{2\beta}\right)^2 - \left(2\kappa + (1 - 2\beta\Lambda)\frac{r^2}{2\beta}\right)k(r) + C_0 k^{3/2}(r) = 0$$

All metric and scalar functions given with respect to k.
For general shift symmetric G_2, G_4 the result can be extended, [Kobayashi, Tanahashi]

Let us now give some specific examples for the different cases...

Scalar with constant velocity $q \neq 0$

Consider the action,

$$S = \int d^4x\sqrt{-g}\left[\zeta R - 2\Lambda - \eta\left(\partial\phi\right)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right],$$

- Scalar field equation and conservation of current,

$$\nabla_\mu J^\mu = 0, \quad J^\mu = \left(\eta g^{\mu\nu} - \beta G^{\mu\nu}\right)\partial_\nu\phi.$$

- Take $ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$, and $\phi = \phi(t, r)$ then
- $\phi = \psi + qt \longrightarrow J^r = 0$
- $\beta G^{rr} - \eta g^{rr} = 0$ or $\phi' = 0$

For a higher order theory $J^r = 0$ does not necessarily imply $\phi = const$.

$J^r = 0$ means that we kill primary hair since, $\nabla_\mu J^\mu = 0 \rightarrow \sqrt{-g}(\beta G^{rr} - \eta g^{rr})\partial_r\phi = c$

- We now solve for the remaining field eqs...

Solving the remaining EoM

- From (rr)-component get ψ'

$$\psi' = \pm \frac{\sqrt{r}}{h(\beta+\eta r^2)} \left(q^2 \beta(\beta+\eta r^2) h' - \frac{\zeta\eta + \beta\Lambda}{2} (h^2 r^2)' \right)^{1/2} .$$

- and finally (tt)-component gives $h(r)$ via,

$$h(r) = -\frac{\mu}{r} + \frac{1}{r} \int \frac{k(r)}{\beta + \eta r^2} dr,$$

with

$$q^2 \beta(\beta+\eta r^2)^2 - \left(2\zeta\beta + (2\zeta\eta - \lambda)\, r^2\right) k + C_0 k^{3/2} = 0,$$

Any solution to the algebraic eq for $k = k(r)$ gives full solution to the system!

Asymptotically flat limit : $\Lambda = 0$, $\eta = 0$

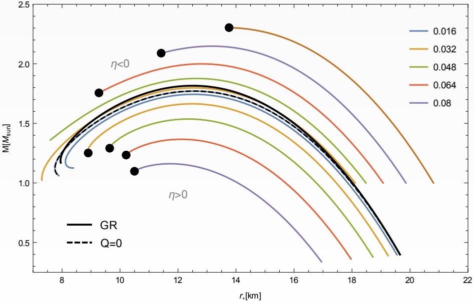
- Consider $S = \int d^4x\sqrt{-g}\,[\zeta R + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi]$
- Algebraic equation to solve: $q^2\beta^3 - 2\zeta\beta k + C_0 k^{3/2} = 0 \to k = constant$!
- $f(r) = h(r) = 1 - \mu/r$
- $\phi_\pm = qt \pm q\mu\left[2\sqrt{\frac{r}{\mu}} + \log\frac{\sqrt{r}-\sqrt{\mu}}{\sqrt{r}+\sqrt{\mu}}\right] + \phi_0$
- Consider $v = t + \int (fh)^{-1/2}dr$ then $ds^2 = -hdv^2 + 2\sqrt{h/f}\,dvdr + r^2d\Omega^2$
  Regular chart for horizon, EF coordinates ([Jacobson], [Ayon-Beato, Martinez & Zanelli])
- $\phi_+ = q\left[v - r + 2\sqrt{\mu r} - 2\mu\log\left(\sqrt{\frac{r}{\mu}} + 1\right)\right] + \text{const}$
- Scalar regular at future black hole horizon!

Schwarzschild geometry with a non-trivial regular scalar field.

Exterior geometry for star

Star solutions [Cisterna, Delsate, Rinaldi], [Maselli, Silva, Minamitsuji, Berti]

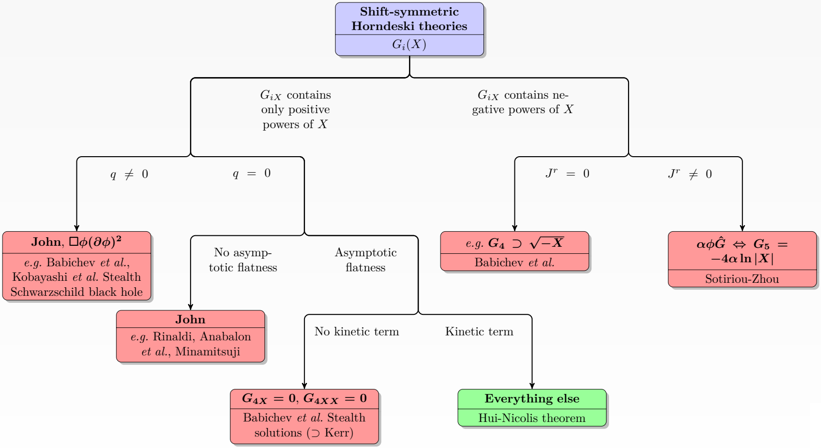
- Consider $S = \int d^4x\sqrt{-g}\,[\zeta R + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi]$
- Take stealth solution for exterior and consider PF matter for interior with ρ and P that does not couple to scalar.
- $J^r = 0$, and therefore $G^{rr} = 0$ which effects star interior.
- For fixed star radius $\beta > 0$ ($\beta < 0$) gives heavier (lighter) stars than GR.
- No GR limit for $q \to 0$



Self tuning de Sitter black hole

- $$S = \int d^4x \sqrt{-g} \left[\zeta R - 2\Lambda - \eta (\partial\phi)^2 + \beta G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right]$$
- $f = h = 1 - \frac{\mu}{r} + \frac{\eta}{3\beta} r^2$ de Sitter Schwarzschild!
- $\psi' = \pm \frac{q}{h} \sqrt{1-h}$ and $\phi(t,r) = q\, t + \psi(r)$
- The effective cosmological constant is not the vacuum cosmological constant. In fact,
- Self tuning relation : $q^2 \eta = \Lambda - \Lambda_{eff} > 0$
- Hence for any $\Lambda > \Lambda_{eff}$ fixes q, integration constant.
- where $\Lambda_{eff} = -\frac{\eta}{\beta}$ is fixed by effective theory.
- Solution hides vacuum cosmological constant leaving a smaller effective cosmological constant [Gubitosi, Linder]

Hair versus no hair [Lehébel]



The special case of the Gauss-Bonnet invariant

[Sotiriou, Zhou] [Duncan et.al] [Mavromatos et.al]

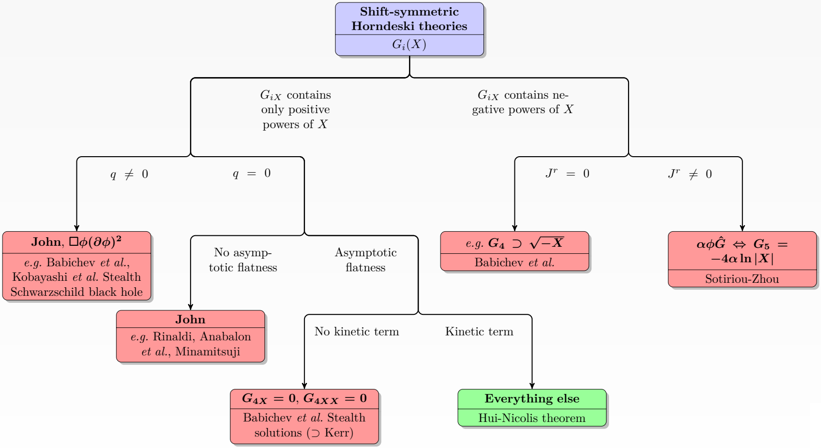
The Gauss-Bonnet term, $\hat{G} = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - 4R^{\mu\nu}R_{\mu\nu} + R^2$, is a topological invariant in 4 dimensions.
Variation with respect to the metric gives the 4 dim Lovelock identity, $H_{\mu\nu} = -2P_{\mu cde}R_{\nu}{}^{cde} + \frac{g_{\mu\nu}}{2}\hat{G} = 0$. If we couple to scalar then $\phi\hat{G}$ ceases to be trivial.
It can be obtained in Horndeski theory via $G_5 \sim \ln X$
The theory

$$\mathcal{L}^{\mathrm{GB}} = \frac{R}{2} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + \alpha\phi\hat{G}$$

is non trivial and shift symmetric. Here, $\hat{G}$ (is independent of ϕ) and acts as a source to the scalar which cannot be set to zero.

- $\Box\phi + \alpha\hat{G} = 0$
- Numerical solution can be found where the scalar and mass integration constants are fixed so that the solution is regular at the horizon.
- The mass of the black hole has a minimal size fixed by the GB coupling α. The singularity is attained at positive r.
- The solution has infinite current norm at the horizon because $J^r \neq 0$
- Solutions with $q \neq 0$ and regular Noether current are in a different branch and are singular.
- Are there other cases where scalar is sourced by non trivial geometry?

Hair versus no hair



Square root Lagrangian [Babichev, CC, Lehébel]

Current for spherical symmetry:

$$J^r = -f\phi' G_{2X} - f\frac{rh'+4h}{rh}XG_{3X} + 2f\phi'\frac{fh-h+rfh'}{r^2h}G_{4X} +$$
$$+4f^2\phi'\frac{h+rh'}{r^2h}XG_{4XX} - fh'\frac{1-3f}{r^2h}XG_{5X} + 2\frac{h'f^2}{r^2h}X^2G_{5XX}$$

where $X \equiv -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$

Keeping the kinetic term $G_2 \sim X$ the GB term $\hat{G} \sim \ln X$ kills ϕ dependence and acts as a source to the scalar field equation.

Same can happen if we take eg. $G_4 \sim \sqrt{-X}$. Consider the theory,

$$G_2 = \eta X, \qquad G_4 = \zeta + \beta\sqrt{-X}$$

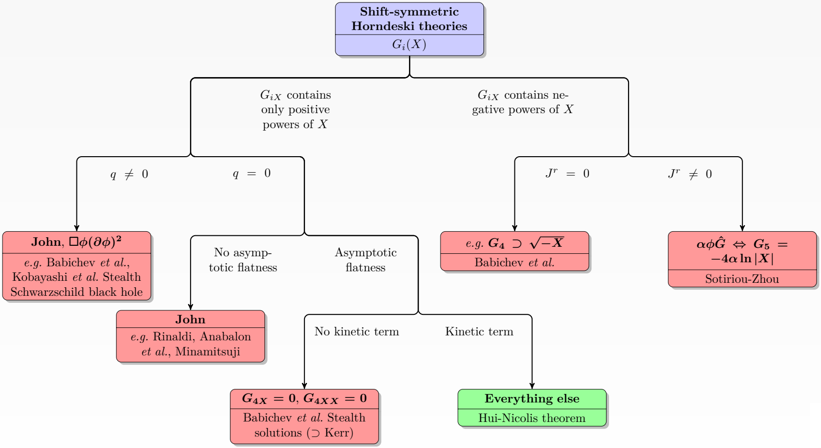
Solve $J^r = 0$ to get $\phi' = \pm\frac{\sqrt{2}\beta}{\eta r^2\sqrt{f}}$.

We have a black hole solution, $ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$

$$f(r) = h(r) = 1 - \frac{\mu}{r} - \frac{\beta^2}{2\zeta\eta r^2}$$

- Solution has a RN type of black hole potential with real or imaginary charge which is fixed in this case
- Solution is a black hole even in the absence of mass for $\eta > 0$
- Solution is never flat. The theory cannot have trivial solutions.

Hair versus no hair



Purely quartic model, G_4 [Babichev, CC, Lehébel]

Suppose now that we keep G_4 analytic but do away with the kinetic term.
Consider again spherical symmetry ($q = 0$)

- Field equations dictate that $X = X_0$
- Regularity, $G_{4X}(X_0) = 0$ and $G_{4XX}(X_0) = 0$.

Any theory of the type, $G_4(X) = \zeta + \sum_{n\geq 2} \beta_n (X - X_0)^n$ will admit a stealth Schw. solution.
Further examination shows that, the Kerr metric is also an exact solution of the theory with

$$\phi(r,\theta) = \sqrt{-2X_0}\left[a\sin\theta - \sqrt{a^2 - 2mr + r^2} - m\ln\left(\sqrt{a^2 - 2mr + r^2} - m + r\right)\right]$$

Slowly rotating solutions [Maselli, Silva, Minamitsuji, Berti]

Using the Hartle Thorne perturbative approximation in which frame-dragging is assumed linear in angular velocity

$$ds^2 = -h(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) - 2\omega(r) r^2 \sin^2\theta \mathrm{d}t\mathrm{d}\varphi,$$

We get an ode to linear order:

$$2(1-\beta X)\left[\omega'' + \frac{\omega'}{2}\left(\frac{f'}{f} + \frac{8}{r} - \frac{h'}{h}\right)\right] - 2\beta X'\omega' = 0$$

which agrees with GR for X constant.
What happens for $X \neq$ const.
We can integrate once,

$$(1-\beta X)\omega' = \frac{C_1\sqrt{k}}{r^4(1+\frac{r^2}{2\beta})}$$

but, one can show by using remaining field equations that correction is always identical to GR [Lehébel].

Kaluza-Klein reduction

Start from Einstein $\rightarrow$ Lovelock theory in vacuum [CC, Gouteraux and Kiritsis]

$$S_{(4+n)} = \int d^{4+n}x\sqrt{-g^{(4+n)}}\left[R - 2\Lambda + \alpha\hat{\mathcal{G}}\right], \hat{\mathcal{G}} = R_{ABCD}R^{ABCD} - 4R_{AB}R^{AB} + R^2$$

Consider a simple metric Anzatz and show it is a consistent truncation (see eg. [Kanitscheider and Skenderis]),

$$\mathrm{d}s^2_{(4+n)} = \mathrm{d}\bar{s}^2_{(p+1)} + \mathrm{e}^{\phi}\mathrm{d}\tilde{K}^2_{(n)} \,.$$

Compactify on some curved n dimensional constant curvature manifold $\tilde{K}$.

$$\bar{S}_{galileon} = \int \mathrm{d}^{p+1}x\,\sqrt{-\bar{g}}\,e^{\frac{n}{2}\phi}\left\{\bar{R} - 2\Lambda + \hat{\alpha}\bar{\mathcal{G}} + \frac{n}{4}(n-1)\partial\phi^2 - \hat{\alpha}n(n-1)\bar{G}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right.$$
$$- \frac{\hat{\alpha}}{4}n(n-1)(n-2)\partial\phi^2\Box\phi + \frac{\hat{\alpha}}{16}n(n-1)^2(n-2)\left(\partial\phi^2\right)^2$$
$$\left. + e^{-\phi}\tilde{R}\left[1 + \hat{\alpha}\bar{R} + \hat{\alpha}4(n-2)(n-3)\partial\phi^2\right] + \hat{\alpha}\tilde{\mathcal{G}}e^{-2\phi}\right\},$$

- Tilded quantities are related to compactified $\tilde{K}$ geometry and are constants. They yield 4 dim potentials.
- Barred quantities are 4 dimensional. Notice that Lovelock densities interact with the scalar field.
- Coefficient n is extended to the real line.
- We have a generalised scalar tensor theory which admits 2nd order field equations.

Take Lovelock black hole with m 2-spheres $H = S^{(2)} \times S^{(2)} ... \times S^{(2)}$.
Compactify on $m-1$ of these keeping one $S^{(2)}$ in 4 dims.
We obtain,

$$\begin{aligned}
\mathrm{d}\bar{s}^2_{(4)} &= -V(R)\mathrm{d}t^2 + \frac{\mathrm{d}R^2}{V(R)} + \frac{R^2}{n+1}\mathrm{d}S^2\,, \\
V(R) &= \kappa + \frac{R^2}{\tilde{\alpha}_r}\left[1 \mp \sqrt{1 - \frac{2\tilde{\alpha}_r^2\kappa^2}{(n-1)R^4} + \frac{4\tilde{\alpha}_r\mathfrak{m}}{R^{3+n}}}\right], \\
\tilde{\alpha}_r &= 2\hat{\alpha}n(n+1), \qquad \kappa = 1 \\
e^{\phi/n} &= \frac{R^2}{n+1}\,,
\end{aligned}$$

- Taking $n \to 0$ gives us Schwarzchild.
- Taking $\tilde{\alpha}_r \to 0$ gives Einstein Dilaton solution [Chan Horne and Mann]
- Solution has solid deficit angle. Solution is similar to the external field of the gravitational monopole.
- Solution for zero mass is therefore singular and has non-trivial topology distinguishing it from GR.
- For large R and small $\tilde{\alpha}_r$ we have $V(R) \sim 1 + \frac{\tilde{\alpha}_r}{(n-1)R^2} - \frac{2\mathfrak{m}}{R^{n+1}} + \ldots$
- Higher order terms give rise to an extra horizon, a bit like in RN geometry.
- For $\mathfrak{m} = 0$ and $0 < n < 1$ we hide the singularity at $R = 0$
- Higher order term cloaks an otherwise naked singularity

Conclusions

- Starting from a no hair theorem we have seen how to construct hairy black holes.
- Similar theorem exists for neutron stars [Lehébel].
- Using Lovelock solutions we can construct black holes in Horndeski theory.
- Many questions about stability of solutions; staticity of spacetime quite unclear
- Higher order terms essential for novel branches of black holes
- One can construct solutions with EM fields and black hole solutions with primary hair by adding additional scalar fields
- Techniques for shift symmetric Horndeski can be extended to Maxwell-Proca theories [Heisenberg].