Caustic Free Completion of k-essence

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Modification of gravity

Why modify gravity?

- cosmological constant problems,
- non-renormalizability problem,
- benchmarks for testing General Relativity (LIGO detection of a gravitational wave)
- theoretical curiosity.

Many ways to modify gravity:

- f(R), scalar-tensor theories,
- Galileons, Horndeski (and beyond) theory, KGB, Fab-four,
- higher-dimensions,
- DGP,
- Horava, Khronometric
- massive gravity

[Christos's talk today]

- Most general scalar-tensor theory leading to equations of motions with no more than 2 derivatives;
- Cancellation of Lambda (Fab-Four), Self-tuning, Self-acceleration;
- Vainshtein mechanism

Horndeski theory (Galileons)

The most generic scalar-tensor theory in 4D, whose equations of motion contain no more than second derivatives

Horndeski'1974

$$S = \int d^4x \, F\left[g, \partial g, \partial^2 g, \varphi, \partial\varphi, \partial^2\varphi\right]$$

$\Downarrow$?

$$E[g, \partial g, \partial^2 g, \varphi, \partial\varphi, \partial^2\varphi] = 0$$

Horndeski theory

[Christos's talk today]

No more than 2 derivatives in EOMs to avoid the Ostrogradski ghost !
When the equations of motion are of higher order, in general it means a new degree of freedom which is a ghost

[See also Marco's talk]

Examples of Horndeski

and beyond

Horndeski	$\mathcal{L}$
canonical field	$\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi)$
k-essence	$K(\partial_\mu\varphi\partial^\mu\varphi,\, \varphi)$
DGP-like term	$\frac{1}{2}\left(\partial\varphi\right)^2\Box\varphi$
and more	$-\frac{1}{4}(\Box\varphi)^2\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\Box\varphi\partial_\mu\varphi\partial_\nu\varphi\partial^\mu\partial^\nu\varphi + \ldots$

beyond Horndeski	$\mathcal{L}$
pressureless fluid	$\frac{\Sigma}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - 1\right)$

Possible pathologies

- Ghosts - negative energies (even if Ostrogradski ghost does not appear.)
- Gradient instabilities - catastrophic exponential instability
- Formation of caustics

Pressureless perfect fluid

Non-relativistic 1+1 case

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = 0$$ Euler equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$ Continuity equation

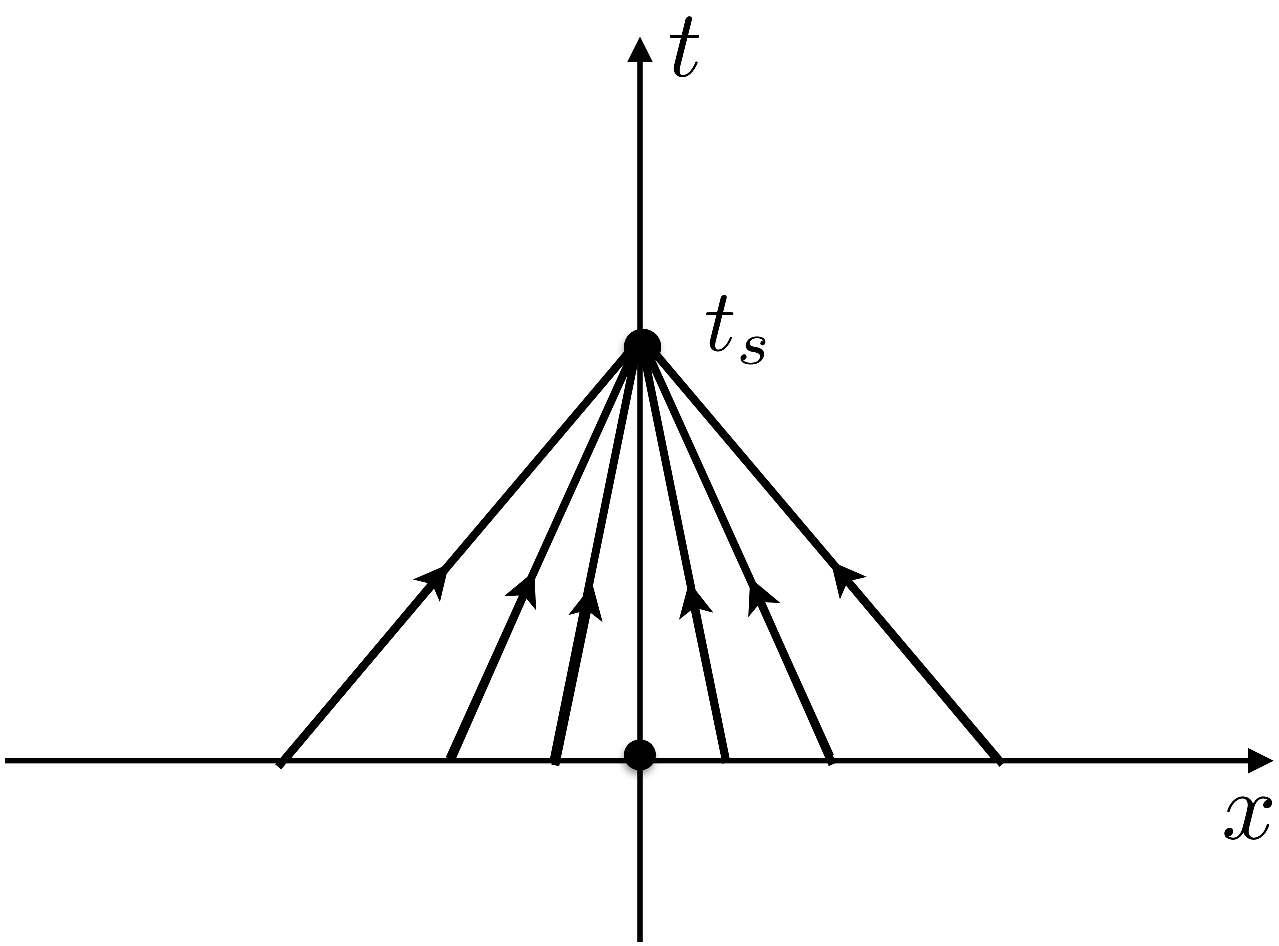
Initial conditions $$v(t=0) = -\frac{x}{T_s}$$

The solution $$v = -\frac{x}{T_s - t} + \mathcal{O}\left(\frac{x^2}{(T_s - t)^2}\right)$$

$$\left.\frac{\partial v}{\partial x}\right|_{x=0} = -\frac{1}{T_s - t} = \infty$$

Caustic formation

Characteristics cross



$$\frac{dx}{dt} = v$$

For generic initial conditions characteristics cross
=> caustics

Pressureless fluid arises in:

- Low energy limit of Horava gravity
- Ghost condensate
- Mimetic theory
- ...

Linear theory

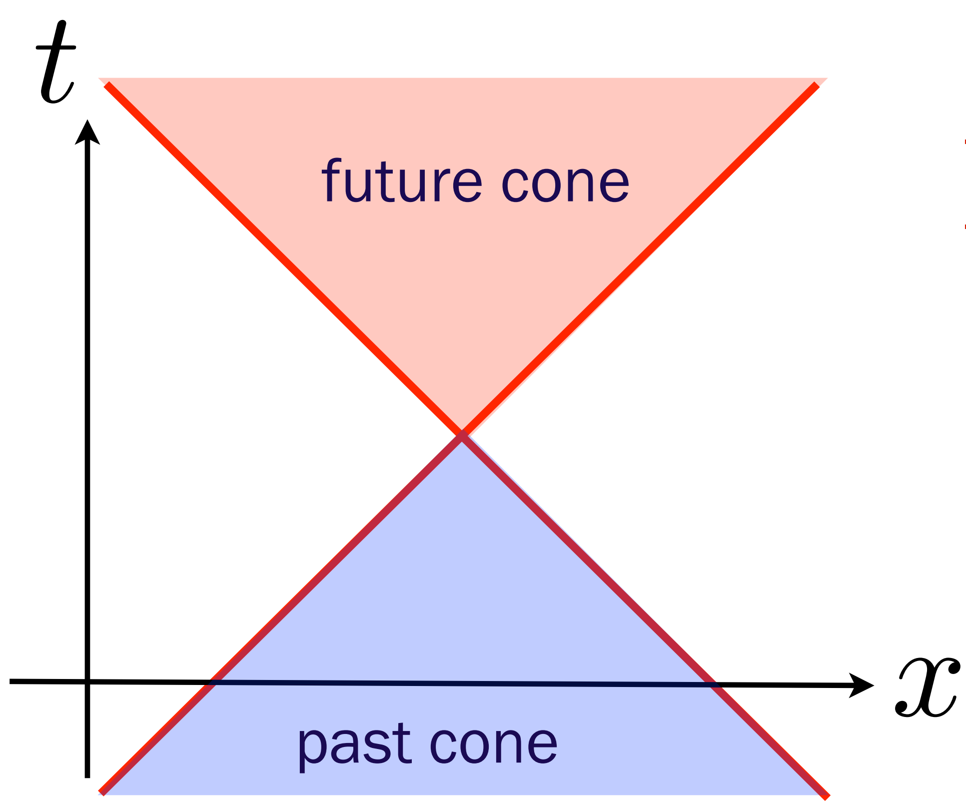
causal structure

Canonical kinetic term + quadratic mass:

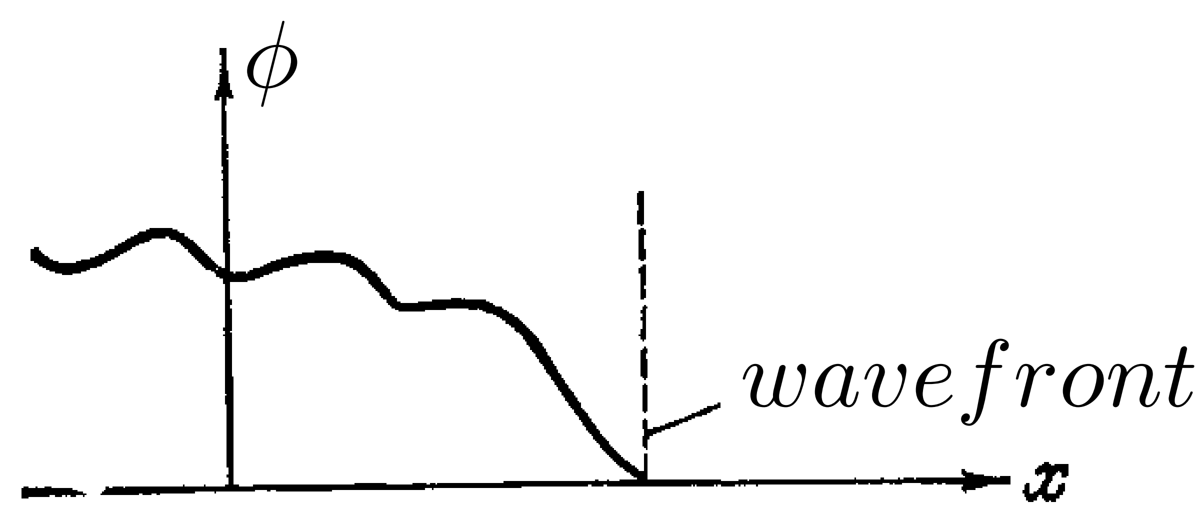
$$S = \int d^4x \left(\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 \right)$$

1+1 case

$$\ddot{\varphi} - \varphi'' = m^2 \varphi$$



- All characteristics are straight lines (45 degrees)
- Characteristics are determined by the kinetic part



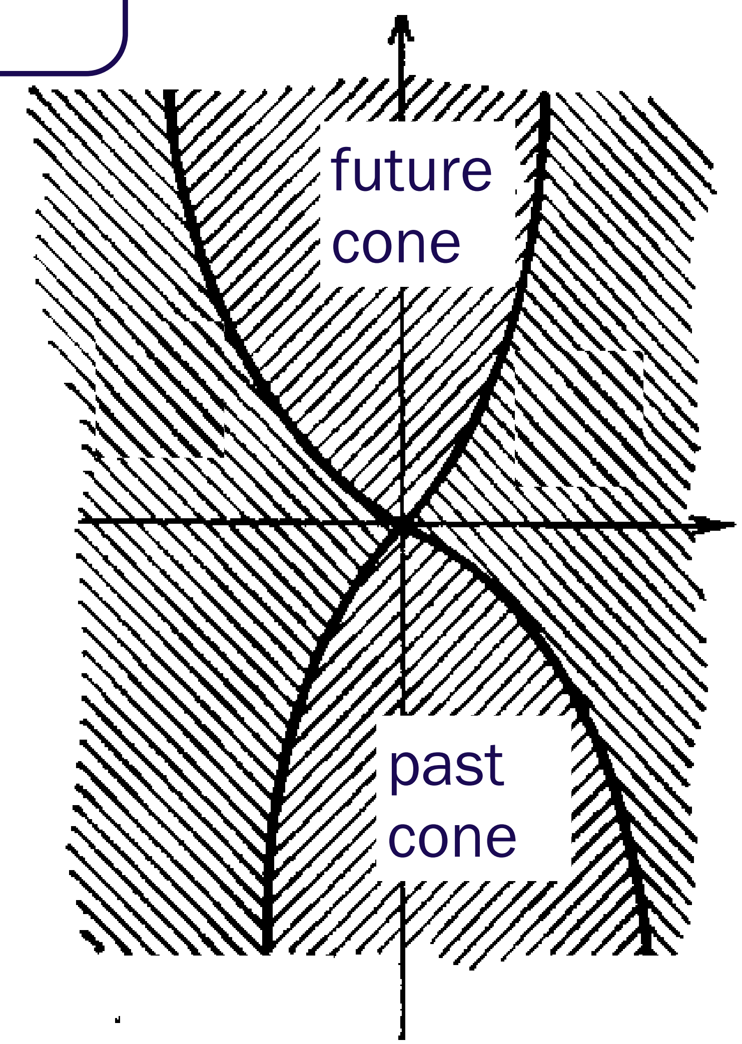
- Perturbations propagate along characteristics
- Signals (wave fronts) propagate along characteristics
- Cones of influence are defined by characteristics

k-essence

equations of motion & causal structure

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}(X) \qquad X = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$$

Variation with respect to the scalar field gives:
- quasi-linear equations
- second order in derivatives

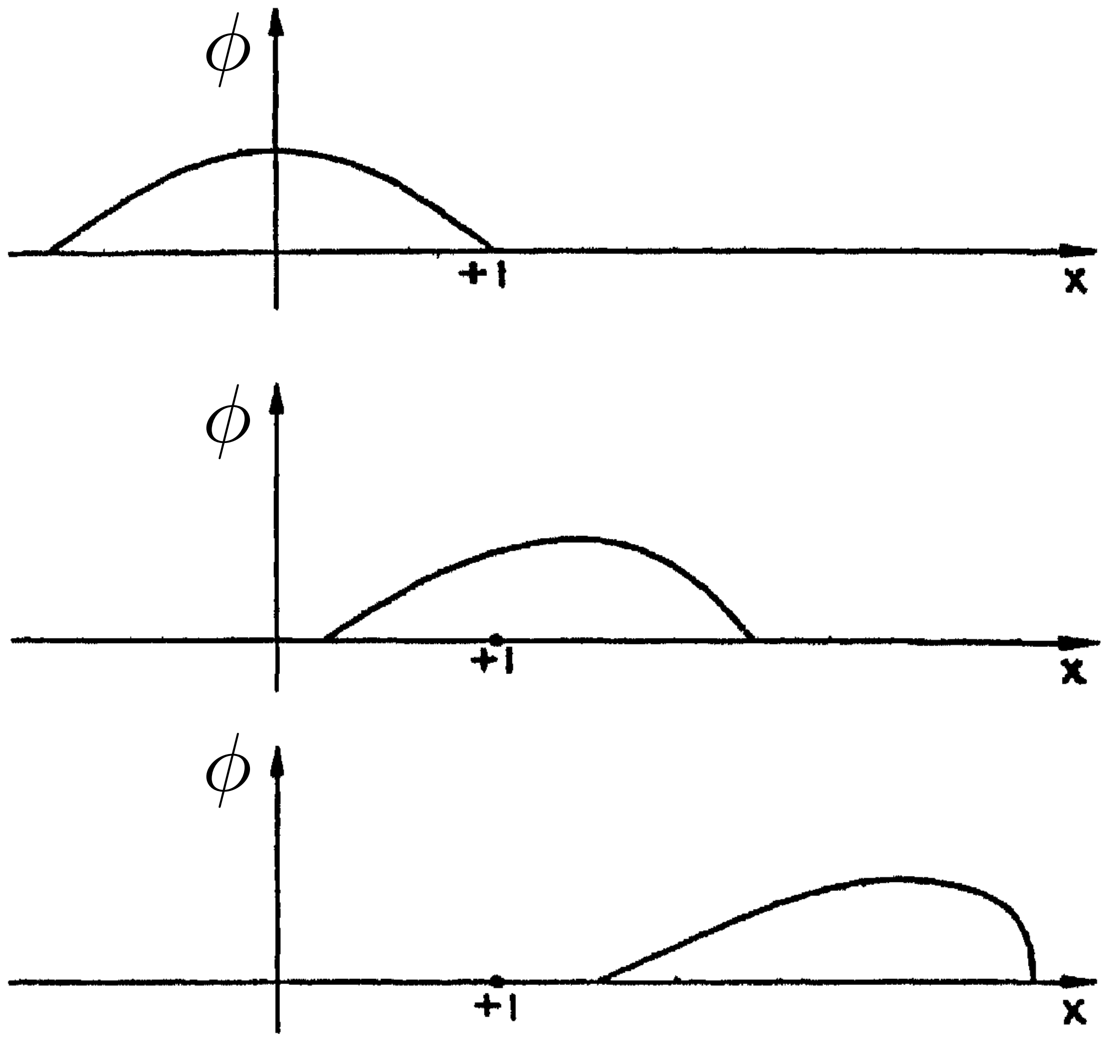


$$\tilde{G}^{\mu\nu}\partial_\mu\partial_\nu\varphi = 0$$

Cones of influence for the scalar field do not coincide with those of the photons and gravitons.

non-linear example

hydrodynamics



Are there caustics in k-essence?

K-essence: EOM

Action:

$$S = \int d^4x\sqrt{-g}\;\mathcal{L}(X) \qquad X \equiv \tfrac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi)$$

$$c_s^2 = \frac{1}{1 + 2X\frac{\mathcal{L}_{XX}}{\mathcal{L}_X}}$$

Sub/Superluminality

EOM:

$$(\mathcal{L}_X g^{\mu\nu} + \mathcal{L}_{XX}\nabla^\mu\varphi\nabla^\nu\varphi)\nabla_\mu\nabla_\nu\varphi = 0$$

$$\| $$

$$\tilde{G}^{\mu\nu}$$

K-essence: characteristics

1+1 dimensions

New variables: $\tau = \dot{\varphi}, \quad \chi = \varphi'$

EOM: $A\dot{\tau} + 2B\tau' + C\chi' = 0$

$$A = \mathcal{L}_X + \tau^2 \mathcal{L}_{XX}, \;\; B = -\tau\chi\mathcal{L}_{XX}, \;\; C = -\mathcal{L}_X + \chi^2 \mathcal{L}_{XX}$$

Introduce two families of curves in (t,x) plane, with parameters $\sigma_{\pm}$ along the curves.

$$A\xi^2 - 2B\xi + C = 0$$

Characteristic equation

$$\xi = \frac{x_\sigma}{t_\sigma}$$

Signals propagate along characteristics (small perturbations on top of a particular solution in the eikonal approximation, $\omega \to \infty$).

K-essence: characteristics

We can rewrite the original PDE as a set of ODEs:

$$\frac{dx}{d\sigma_+} = \xi_+ \frac{dt}{d\sigma_+}, \quad \frac{dx}{d\sigma_-} = \xi_- \frac{dt}{d\sigma_-},$$

$$(\xi_+ A) \frac{d\tau}{d\sigma_+} + C \frac{d\chi}{d\sigma_+} = 0, \quad (\xi_- A) \frac{d\tau}{d\sigma_-} + C \frac{d\chi}{d\sigma_-} = 0$$

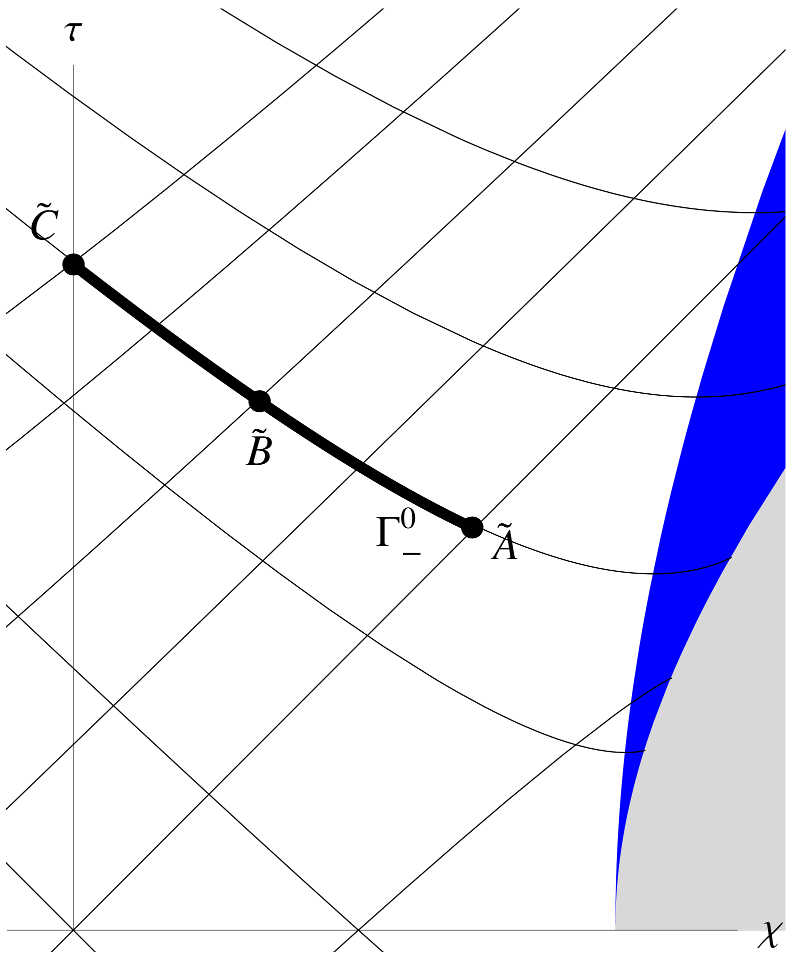
The equations are decoupled.

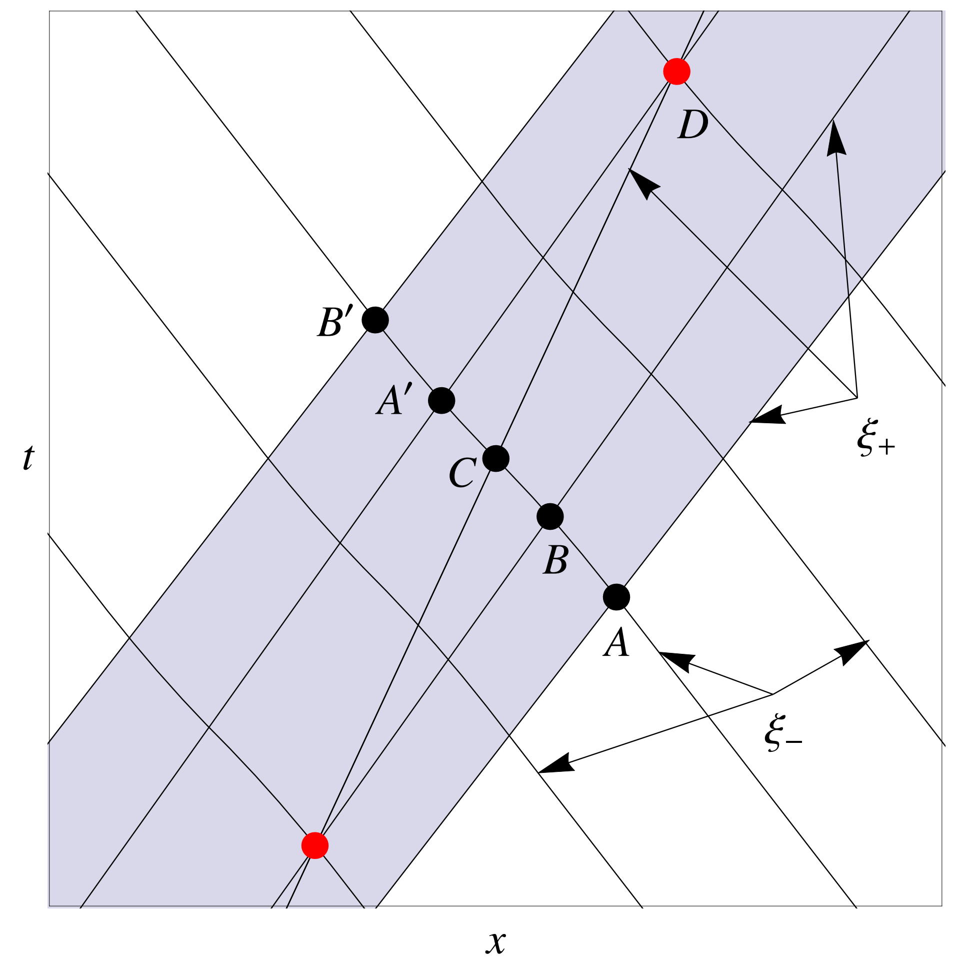
$$\xi_\pm = \frac{v \pm c_s}{1 \pm v c_s} \qquad v = -\frac{\chi}{\tau}$$

- Pressureless perfect fluid: $c_s = 0 \Rightarrow \xi_\pm = v$
- Canonical kinetic term: $\mathcal{L}_X = X \Rightarrow c_s = 1 \Rightarrow \xi_\pm = \pm 1$
- Travelling wave $\varphi = \varphi(t \pm x), \;\; \tau = \pm\chi \Rightarrow \xi_\pm = \pm 1, \;\; X = 0$

[EB, Mukhanov, Vikman'07, Evslin'11, EB'12]

K-essence: generic wave



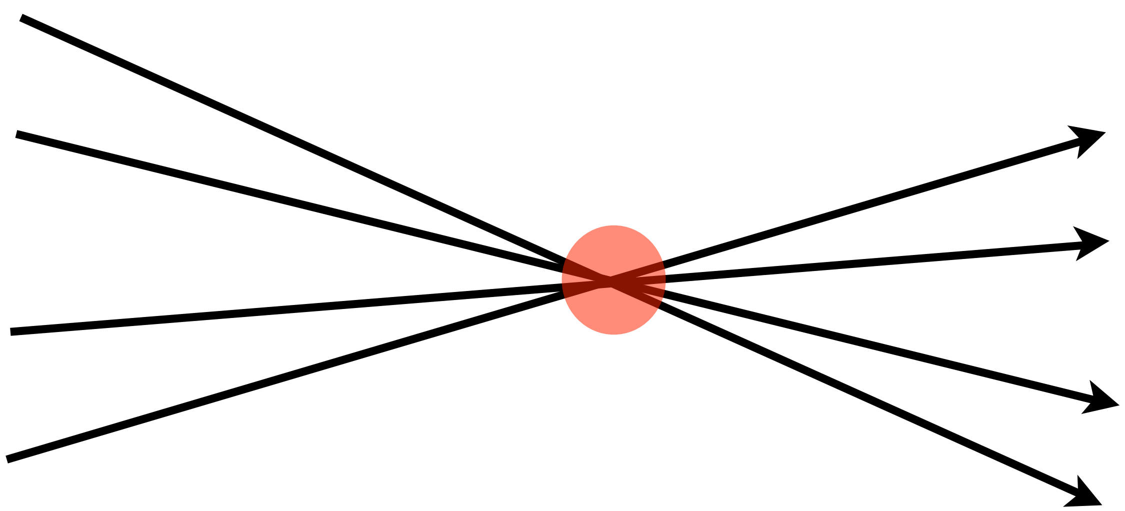


Caustics form

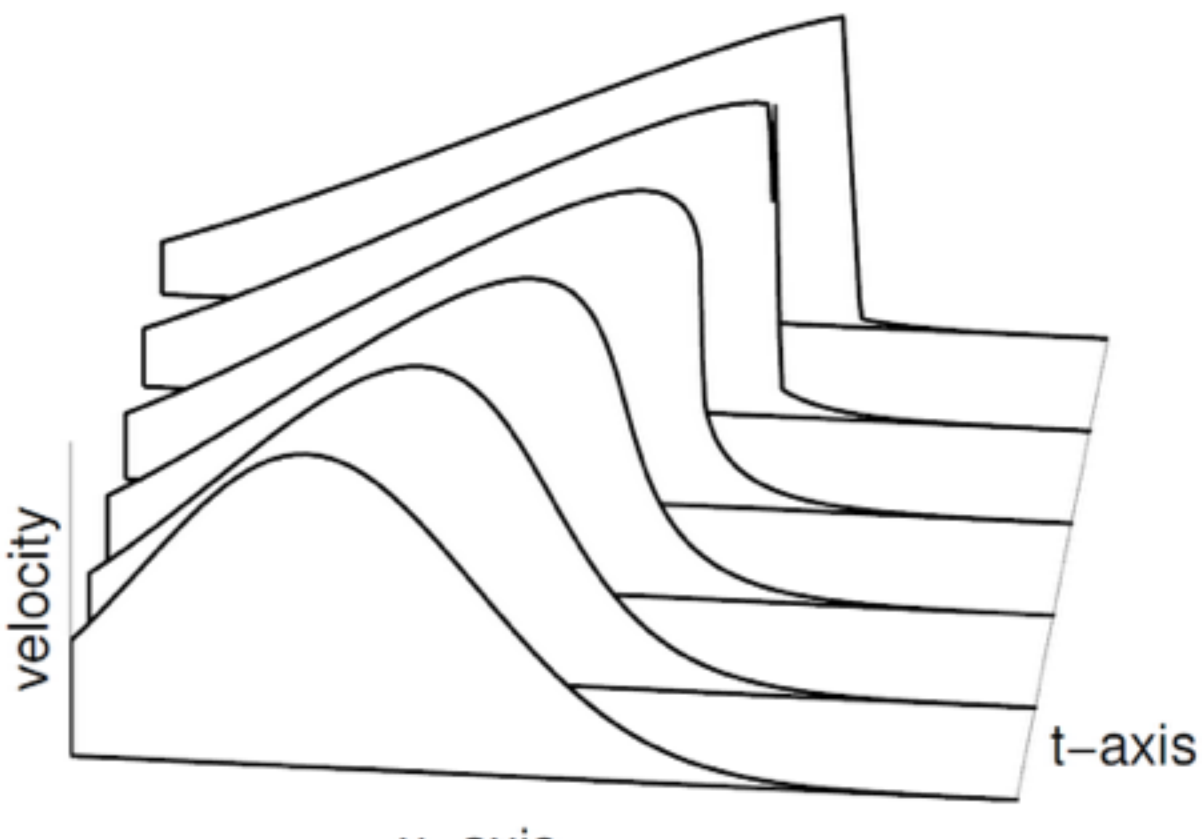
Other galileon terms do not influence the solution =>
singularity is not cured when other galileons are included

Are caustics dangerous?

Optical caustics:
Light rays form caustics, but the underlying theory - EM - resolves the problem (e.g. infinite radiation power)



Are caustics dangerous?



Fluids: Explosion shock wave.

k-essence as a model of 2 scalar fileds

$$S = \int d^4x \left[\frac{\lambda^2}{2} (\partial_\mu \varphi)^2 - V(\lambda) \right]$$

For generic $V(\lambda)$ the new ingredient λ is an auxiliary field

$$\frac{V'(\lambda)}{\lambda} = (\partial_\mu \varphi)^2 = X$$

- $V(\lambda) \propto -\lambda^2 + \lambda^4 \Rightarrow \mathcal{L}(X) \propto X + X^2$ subluminal
- $V(\lambda) \propto \lambda^2 - \lambda^4 \Rightarrow \mathcal{L}(X) \propto X - X^2$ superluminal

Other models as well

$$S = \int d^4x \left[\frac{\lambda^2}{2} \left(\partial_\mu \varphi\right)^2 - V(\lambda) \right]$$

- $V(\lambda) = \frac{\lambda^2}{2} \Rightarrow \lambda$ is a Lagrange multiplier.

  $$S = \int d^4x \frac{\lambda^2}{2} \left[\left(\partial_\mu \varphi\right)^2 - 1 \right] \quad \Rightarrow \quad T_{\mu\nu} = \lambda^2 \partial_\mu \varphi \partial_\nu \varphi$$

  Pressureless perfect fluid

- $V(\lambda) = \frac{\lambda^2}{2} + \frac{\lambda^4}{4} \Rightarrow \mathcal{L}(X) = (X-1)^2$ Ghost condensate

Idea : Promote λ to a dynamical degree of freedom

$$S = \int d^4x \left[\frac{(\partial_\mu \lambda)^2}{2M^2} + \frac{\lambda^2}{2} (\partial_\mu \varphi)^2 - V(\lambda) \right] ,$$ M is large

Field transformation,

$$\tilde{\lambda} = \frac{\lambda}{M}, \quad \tilde{\varphi} = M\varphi$$

$$\Psi = \tilde{\lambda} e^{i\tilde{\phi}}$$

$$S = \int d^4x \left[\frac{1}{2} \partial_\mu \Psi^* \partial^\mu \Psi - V(|\Psi|) \right]$$

The model is manifestly caustic free and signals propagate with the speed of light

Let us take the simplest renormalizable potential

$$V(\Psi) = \frac{\alpha M^2 |\Psi|^2}{2} + \frac{\beta M^4 |\Psi|^4}{4\Lambda^4}$$

- $\alpha > 0,\ \beta = 0$: pressureless perfect fluid
- $\alpha < 0,\ \beta > 0$: subluminal k-essence
- $\alpha > 0,\ \beta < 0$: superluminal k-essence
- $\alpha > 0,\ \beta > 0$: ghost condensate

Pressureless fluid: general case

2 vs 1 degrees of freedom

General solution

$$\Psi = \int dk\alpha(k)e^{ikx+i\sqrt{k^2+M^2}t} + \int dk\beta(k)e^{-ikx-i\sqrt{k^2+M^2}t}$$

One branch of the solution should be killed to reproduce pressureless perfect fluid:

$$\boxed{\beta(k)=0} \quad \Rightarrow \quad \Psi = \int dk\alpha(k)e^{ikx+i\sqrt{k^2+M^2}t}$$

Provided that $L^{-1} \ll Mv$

one can show that pressureless fluid dynamics is reconstructed

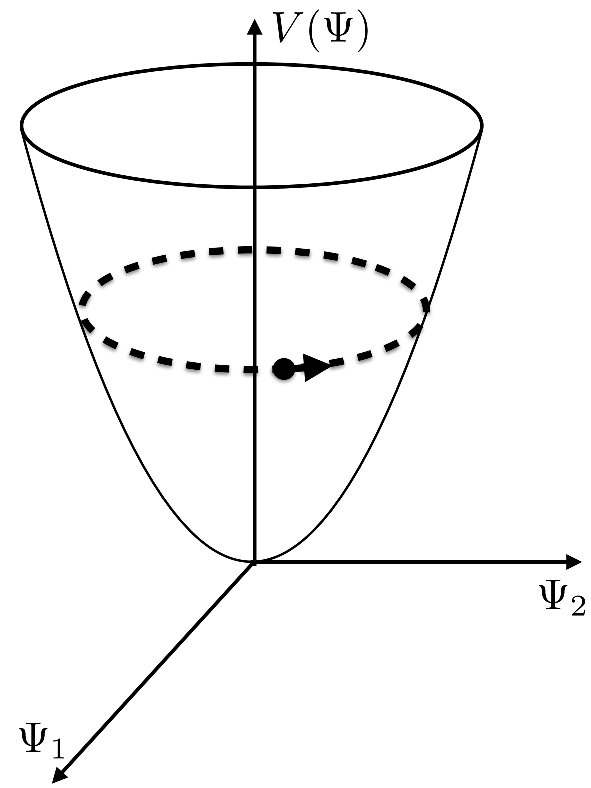
$$\partial_\mu\varphi\partial^\mu\varphi = 1$$

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta, \quad \partial_\mu\phi = u_\mu$$

Pressureless fluid: homogeneous case

2 vs 1 degrees of freedom

Again, special initial conditions are required



$$\Psi = Ae^{iMt} + Be^{-iMt} \;\rightarrow\; B = 0 \;\rightarrow\; \Psi = Ae^{iMt}$$

Cosmic drag $$\Psi = \frac{A}{a^{3/2}} e^{iMt} \;\Rightarrow\; P = 0,\, \rho \propto a^{-3} \qquad (M \gg H)$$

Pressureless fluid: non-relativistic case

2 vs 1 degrees of freedom

$$\Psi = \int dk \alpha(k) e^{ikx + i\sqrt{k^2+M^2}t}$$

Consider non-relativistic limit $k^2 \ll M^2$

$$\Psi = \tilde{\Psi} e^{iMt}$$

Schroedinger equation $$i\frac{\partial \tilde{\Psi}}{\partial t} = -\frac{1}{2M}\Delta\tilde{\Psi}$$

Madelung transformation $$\tilde{\Psi} = \sqrt{\rho}\, e^{i\tilde{\phi} - iMt}$$

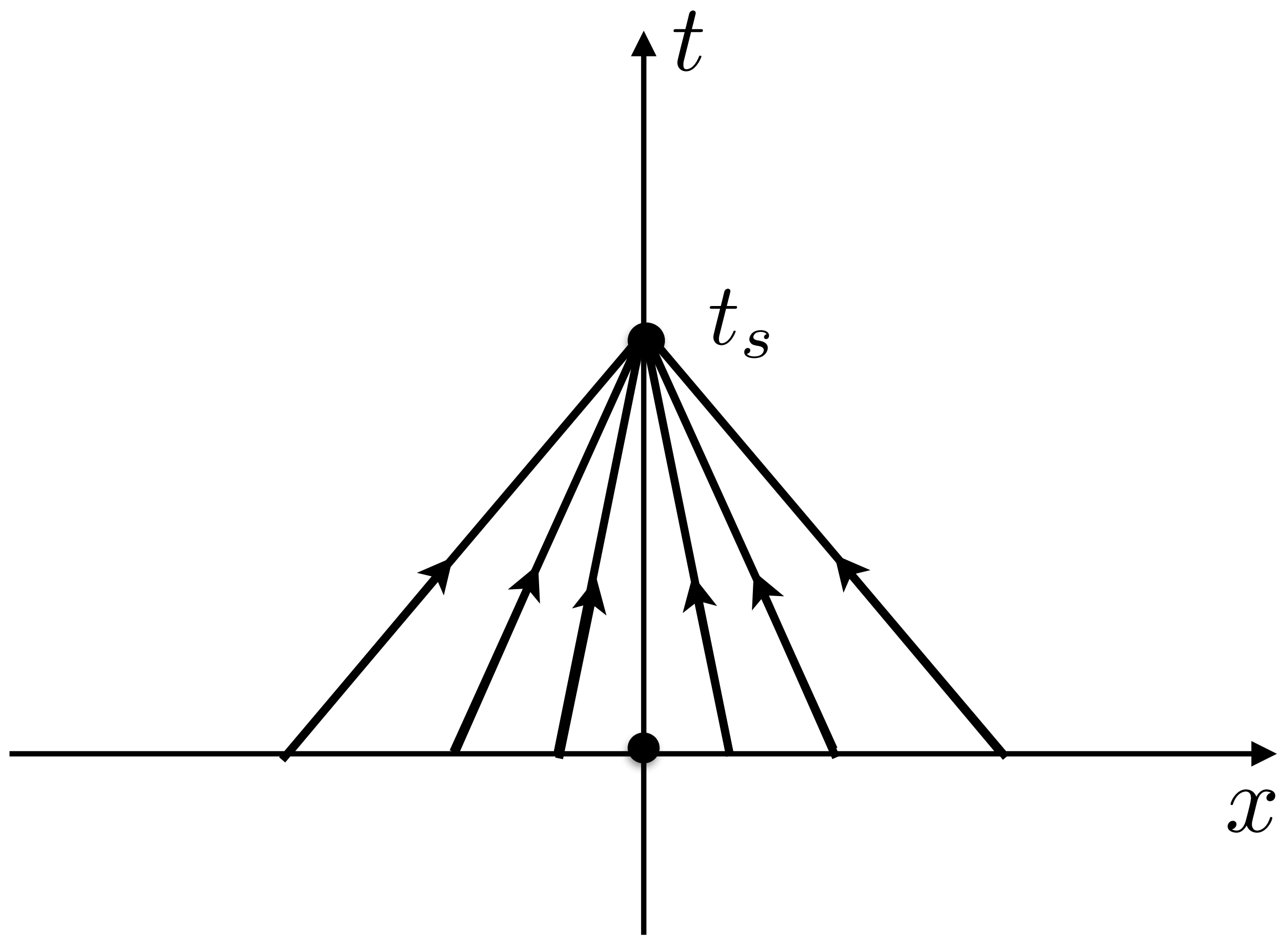
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{2M^2}\cdot\nabla\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$$

"Quantum pressure"

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\mathbf{v}) = 0$$

Caustic formation

Characteristics cross



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Resolving caustic singularity

$$\rho(t=0) = A^2 \exp\left(-\frac{x^2}{L^2}\right), \quad v(t=0) = -\beta x$$

$$\Rightarrow \ v = -\frac{x}{t_s - t}, \ t_s = \frac{1}{\beta}$$

In pressureless fluid case there is a singularity

$$\Psi = \frac{A \cdot e^{iMt}}{\sqrt{1 - \frac{t}{T}}} \cdot \exp\left(-\frac{iMx^2}{2(t-T)}\right)$$

$$\boxed{t_s \to T = T_1 - iT_2} \qquad \frac{T_2}{T_1} \propto \frac{1}{M}$$

Collapse time is promoted to a complex number, i.e. the singularity is not developed in real time!

K-essence case

In the homogeneous case we were able to show that the cosmology for k-essence is reproduced in the 2-field model.

$$V(\Psi) \propto |\Psi|^4 \;\Rightarrow\; P = \frac{1}{3}\rho \quad (\mathcal{L} \propto X^2)$$

Conclusions

- Evolution of a generic simple wave leads to formation of caustics in k-essence and Horndeski (and beyond)
- Exceptional cases are the standard kinetic term and pure k-essence DBI.
- K-essence, pressureless perfect fluid and ghost condensate can be seen as the the same two scalar field model with different potential.
- Simple caustic free completion by means of the complex scalar field.
- Specific initial conditions are required to reproduce pressureless perfect fluid or k-essence
- Mechanism of resolving caustics: the collapse time is promoted to a complex number in the complete picture