Stochastic backgrounds

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Stochastic backgrounds of GWs can be a vast subject

- Experimental searches
  - GW interferometers
    - *data analysis techniques. Wiener filtering, two-detector correlations*
    - *existing limits and sensitivities of adv. LIGO/Virgo, eLISA*
  - CMB
    - *temperature anisotropies, polarization (E- and B-modes)*
    - *Planck, Bicep2*
  - pulsar timing arrays
- Cosmological production mechanisms
  - inflation
  - phase transition
  - cosmic strings
  - ...

we will develop only one topic:

Stochastic GWs and inflation

... but (hopefully) we will do it well !

Final aims:

- *compute the production of stochastic GWs during inflation*
- *understand the Planck results in the* (n_s, r) *plane*
- *compute the relic background today*

...still a long and interesting journey

- Cosmological perturbations
  - *Basics of FRW cosmology*
  - *Scalar-vector-tensor decomposition*
  - *Evolution of cosmological perturbations*
  - *Power spectra, the tensor-to-scalar ratio r, $\Omega_{\rm gw}$*
- Inflationary cosmology
  - *Large- and small-field models*
  - *Starobinsky inflation*
- Computation of the stochastic background
  - *Quantum fields in de Sitter space*
  - *Computation of power spectra, r and n_s*
  - *Comparison of inflationary models with Planck data*
  - *The stochastic background at the present epoch*

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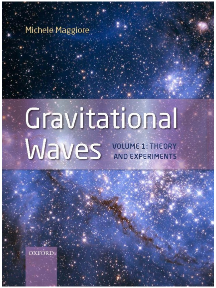
References

we will follow:

MM, Phys. Rept. 331 (2000) 283-367

MM, Gravitational Waves. Vol. 1. Theory and Experiments, Oxford University Press, 574 p., 2007.



...and, yes, Vol. 2 is almost finished! (>500 pages ready)

...where full references to the original literature can be found

Cosmological perturbations

Basics of FRW

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2\,, \qquad H = \dot{a}/a \qquad \text{(spatially flat)}$$

Horizon scale $\lambda_H \equiv H^{-1}$. Today $H_0^{-1} \simeq 4.3\,\text{Gpc}$

Physical coordinates $\mathbf{x}_{\text{ph}} = a(t)\mathbf{x}$

$$ds^2 = -\left(1 - H^2\mathbf{x}_{\text{ph}}^2\right)dt^2 + d\mathbf{x}_{\text{ph}}^2 - 2H\mathbf{x}_{\text{ph}}\cdot d\mathbf{x}_{\text{ph}}dt$$

Newtonian intuition applies at $x_{\text{ph}} \ll \lambda_H$

compare TT gauge and proper detector frame for the metric of a GW!

Correspondingly, we have comoving momentum $\mathbf{k}$ and physical momentum $\mathbf{k}_{\text{ph}} = \mathbf{x}/a(t)$

Dynamics: $G^{\mu}_{\nu} = 8\pi G\, T^{\mu}_{\nu}$

perfect fluid: $T^0_0 = -\rho(t)\,, \qquad T^i_0 = 0\,, \qquad T^i_j = p(t)\,\delta^i_j$

eq. of state: $p = w\rho$

$\nabla_\mu T^\mu_\nu = 0 \quad \Rightarrow \qquad \dot\rho + 3H(\rho + p) = 0$

$\mu = \nu = 0 \quad \Rightarrow \qquad H^2 = \frac{8\pi G}{3}\rho$

	w	$\rho(a)$	$a(t)$
radiation	$1/3$	a^{-4}	$t^{1/2}$
matter	0	a^{-3}	$t^{2/3}$
vacuum energy	-1	const.	e^{Ht}

Conformal time η:

$$d\eta = \frac{dt}{a(t)}$$

FRW metric:

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\mathbf{x}^2\right)$$

When using conformal time, one also introduces

$$\mathcal{H} = \frac{a'}{a}$$

notation: $f' = df/d\eta, \qquad \dot{f} = df/dt$

Scalar-vector-tensor decomposition

Helicity decomposition in flat space: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$h_{00} = 2\psi\,, \qquad h_{0i} = \beta_i + \partial_i\gamma$$

$$h_{ij} = -2\phi\delta_{ij} + [\partial_i\partial_j - (1/3)\delta_{ij}\boldsymbol{\nabla}^2]\lambda + (1/2)(\partial_i\epsilon_j + \partial_j\epsilon_i) + h_{ij}^{\rm TT}$$

$$\partial_i\beta^i = 0\,, \quad \partial_i\epsilon^i = 0\,, \quad \partial^j h_{ij}^{\rm TT} = 0\,, \quad \delta^{ij}h_{ij}^{\rm TT} = 0$$

In Fourier space $\partial_i \leftrightarrow ik_i$, e. g.

$$\tilde{h}_{0i}(\mathbf{k}) = \tilde{\beta}_i(\mathbf{k}) + ik_i\tilde{\gamma}(\mathbf{k})\,,$$

where $\mathbf{k}\cdot\tilde{\beta}(\mathbf{k}) = 0$. Is a decomposition in helicity eigenstates

Gauge transformation:

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

$$\xi_0 = A\,, \qquad \xi_i = B_i + \partial_i C\,, \qquad (\partial_i B^i = 0)$$

$$\begin{aligned} \psi &\to \psi - \dot{A} \\ \phi &\to \phi + (1/3)\boldsymbol{\nabla}^2 C \\ \gamma &\to \gamma - A - \dot{C} \\ \lambda &\to \lambda - 2C \end{aligned}$$

$$\beta_i \to \beta_i - \dot{B}_i\,, \qquad \epsilon_i \to \epsilon_i - 2B_i\,,$$

$$h_{ij}^{\rm TT} \to h_{ij}^{\rm TT}$$

$h_{ij}^{\rm TT}$ is gauge-inv for trivial reasons: no tensor part in ξ_μ

- *Synchronous gauge:*

$$h_{00} = h_{0i} = 0, \qquad \text{i.e.} \qquad \psi = \gamma = \beta_i = 0$$

Residual gauge transformation:

$$A = -f(\mathbf{x}), \quad C = f(\mathbf{x})t, \quad B_i = B_i(\mathbf{x})$$

$\Rightarrow$ spurious gauge modes

- *Newtonian gauge:* $\lambda = \gamma = \beta_i = 0$

no residual gauge freedom

scalar pert: $ds^2 = -(1 - 2\psi)dt^2 + (1 - 2\phi)d\mathbf{x}^2$

Better approach: gauge-inv variables (Bardeen variables)

$$\begin{aligned} \psi &\to \psi - \dot{A} \\ \phi &\to \phi + (1/3)\nabla^2 C \\ \gamma &\to \gamma - A - \dot{C} \\ \lambda &\to \lambda - 2C \\ \beta_i &\to \beta_i - \dot{B}_i\,, \\ \epsilon_i &\to \epsilon_i - 2B_i \end{aligned}$$

$$\begin{aligned} \Phi &= -\phi - \frac{1}{6}\nabla^2\lambda\,, \\ \Psi &= -\psi + \dot{\gamma} - \frac{1}{2}\ddot{\lambda} \\ \Xi_i &= \beta_i - \frac{1}{2}\dot{\epsilon}_i\,, \\ &h_{ij}^{TT} \end{aligned}$$

In the Newtonian gauge: $\Psi = -\psi, \quad \Phi = -\phi$

so, in the scalar sector: $ds^2 = -(1+2\Psi)dt^2 + (1+2\Phi)d\mathbf{x}^2$

Is quite convenient to work in the Newtonian gauge:

- *no residual gauge freedom, so no spurious gauge mode*
- *the equations that one obtains for Φ and Ψ are valid in any gauge*

Linearized Einstein eqs

$$T_{00} = \rho\,, \qquad T_{0i} = S_i + \partial_i S$$
$$T_{ij} = p\delta_{ij} + [\partial_i\partial_j - (1/3)\delta_{ij}\nabla^2]\sigma + (1/2)(\partial_i\sigma_j + \partial_j\sigma_i) + \sigma_{ij}^{TT}$$

$$\partial_i\sigma^i = 0\,, \quad \partial_i S^i = 0\,, \quad \partial^i\sigma_{ij}^{TT} = 0\,, \quad \delta^{ij}\sigma_{ij}^{TT} = 0$$

$$\begin{aligned} \nabla^2\Phi &= -4\pi G\rho \\ \nabla^2\Psi &= +4\pi G(\rho - 2\nabla^2\sigma) \\ \nabla^2\Xi_i &= -16\pi G S_i \\ \Box h_{ij}^{TT} &= -16\pi G\sigma_{ij}^{TT} \end{aligned}$$

- *4 pure gauge dof*
- *4 physical but non-propagating dof:* Φ, Ψ, Ξ_i
  note that $\Phi + \Psi = -8\pi G\sigma \quad \Rightarrow \quad \Psi = -\Phi \quad$ if $\sigma = 0$
- *2 physical propagating dof:* h_{ij}^{TT}

Similar construction in FRW: $ds^2 = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$

$$\Phi = -\phi - \frac{1}{6}\nabla^2\lambda + \mathcal{H}\left(\gamma - \frac{1}{2}\frac{d\lambda}{d\eta}\right)$$

$$\Psi = -\psi + \frac{1}{a}\frac{d}{d\eta}\left[a\left(\gamma - \frac{1}{2}\frac{d\lambda}{d\eta}\right)\right]$$

gauge-invariant Bardeen variables

conformal Newtonian gauge:

$$ds^2 = a^2(\eta)\left[-(1+2\Psi)d\eta^2 + (1+2\Phi)d\mathbf{x}^2 + h_{ij}^{\mathrm{TT}}dx^i dx^j\right]$$

- vector perts decay with time and are irrelevant
- scalar and tensor perts evolve separately but we must study both to extract the predictions from a model

  e.g. prediction in the (n_s, r) plane, see later

Evolution of cosmological perturbations

$$\delta G^{\mu}_{\nu} = 8\pi G\, \delta T^{\mu}_{\nu}$$

Scalar sector

- $\Psi = -\Phi$ if $\sigma = 0$
- Φ still a non-propagating dof

$$\nabla^2\Phi = -4\pi G a^2(\delta\rho - 3\mathcal{H}\delta S) \qquad (T_{0i} = \partial_i S)$$

- but time-evolving. Master eq for $\tilde{\Phi}(\eta, \mathbf{k})$:

$$\tilde{\Phi}'' + 3\mathcal{H}(1 + c_s^2)\tilde{\Phi}' + \left[3\mathcal{H}^2(c_s^2 - w) + c_s^2 k^2\right]\tilde{\Phi} = 0$$

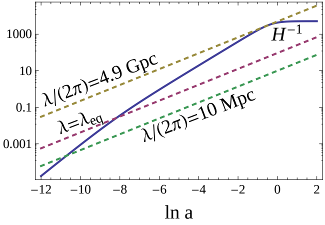
$$(p = w\rho, \delta p = c_s^2\delta\rho)$$

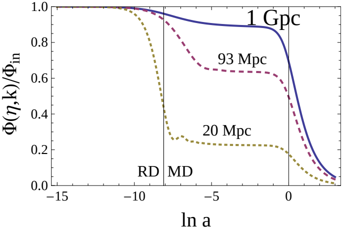
Results:

the evolution depends on whether $k \ll \mathcal{H}(\eta)$ or $k \gg \mathcal{H}(\eta)$
and on whether we are in RD, MD, or dark-energy dominance

H^{-1} and $a(t)\lambda/(2\pi)$

for different values of λ





Tensor perturbations

$$(h_{ij}^{\rm TT})'' + 2\mathcal{H}(h_{ij}^{\rm TT})' - \nabla^2 h_{ij}^{\rm TT} = 16\pi G a^2 \sigma_{ij}^{\rm TT}$$

note: perfect fluids are not sources for GWs!

go in momentum space and expand

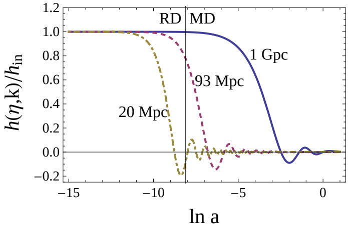
$$\tilde{h}_{ij}^{\rm TT}(\eta, \mathbf{k}) = \sum_{A=+,\times} e_{ij}^{A}(\hat{\mathbf{k}})\tilde{h}_A(\eta, \mathbf{k})$$

$$e_{ij}^{+}(\hat{\mathbf{k}}) = \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j\,, \qquad e_{ij}^{\times}(\hat{\mathbf{k}}) = \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j + \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j$$

when $\hat{\mathbf{k}} = (0, 0, 1)$, $\quad e_{ab}^{+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab}, \qquad e_{ab}^{\times} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ab}$

Then:

$$\tilde{h}_A'' + 2\mathcal{H}\tilde{h}_A' + k^2\tilde{h}_A = 16\pi G a^2 \tilde{\sigma}_A \qquad \text{set } \tilde{\sigma}_A = 0$$



$$\tilde{h}(\eta, k) = \tilde{h}_{\rm in}(k)\frac{\sin(k\eta)}{k\eta} \quad \text{(RD)}$$

$$\tilde{h}(\eta, k) = \tilde{h}_{\rm in}(k)\frac{3j_1(k\eta)}{k\eta} \quad \text{(MD)}$$

$$\Rightarrow \tilde{h}_k(\eta) \propto 1/a(\eta)$$

both in RD and MD, inside the horizon,

$$\rho_{\rm GW} \propto \langle \dot{h}^2 \rangle \propto a^{-4}$$

the notion of graviton only makes sense inside the horizon!

Effect of stochastic GWs on the CMB

- Only GW modes with $k\eta_{\rm dec} \lesssim 1$ contribute
- A GW with wave-number k imprints anisotropies on the CMB on a length-scale $\lambda = 2\pi/k \gtrsim 2\pi\eta_{\rm dec}$
- An observer at the Earth sees the corresponding perturbations under an angle

$$\theta \gtrsim \frac{(\lambda/2)}{\eta_0 - \eta_{\rm dec}} \simeq \frac{\lambda}{2\eta_0} \qquad (\eta_0 \simeq 13.7\,{\rm Gpc}, \eta_0/\eta_{\rm dec} \simeq 49.4)$$

Thus, GWs can only affect large-angle anisotropies. Numerically

$$\theta \gtrsim 3.6\,{\rm deg}$$

which corresponds to CMB multipoles with

$$l \simeq \frac{\pi}{\theta} \lesssim O(50)$$

Power spectra, r, and $\Omega_{\rm gw}$

Initial conditions in cosmology are stochastic

for gaussian initial conditions all information is contained in the 2-point correlator

$$\langle\tilde{\Phi}_{\rm in}(\mathbf{k})\tilde{\Phi}^*_{\rm in}(\mathbf{k}')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')P_{\Phi,{\rm in}}(k)$$

$P_{\Phi,{\rm in}}(k)$ is the (primordial) power spectrum

$$\begin{aligned}\langle\Phi_{\rm in}(\mathbf{x})\Phi_{\rm in}(\mathbf{x}')\rangle &= \int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\langle\tilde{\Phi}_{\rm in}(\mathbf{k})\tilde{\Phi}^*_{\rm in}(\mathbf{k}')\rangle e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}'}\\ &= \int\frac{d^3k}{(2\pi)^3}P_{\Phi,{\rm in}}(k)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\end{aligned}$$

$$\langle\Phi^2_{\rm in}(\mathbf{x})\rangle = \int\frac{d^3k}{(2\pi)^3}P_{\Phi,{\rm in}}(k) = \frac{1}{2\pi^2}\int_0^\infty\frac{dk}{k}\,k^3P_{\Phi,{\rm in}}(k)$$

Define

$$\mathcal{P}_{\Phi,{\rm in}}(k) = \frac{k^3}{2\pi^2}P_{\Phi,{\rm in}}(k)$$

The initial power spectrum is parametrized as

$$\mathcal{P}_{\Phi,\mathrm{in}}(k) = A_\Phi \left(\frac{k}{k_*}\right)^{n_s-1}$$

- $n_s = 1$ *Harrison-Zeldovich spectrum*
- $n_s > 1$ "blue", $n_s < 1$ "red" spectrum
- The pivot scale k_* is a parameter chosen by the experimentalist.
- running index: $\mathcal{P}_{\Phi,\mathrm{in}}(k) = A_\Phi \left(\frac{k}{k_*}\right)^{n_s-1+\frac{1}{2}\left(\frac{dn_s}{d\log k}\right)\log(k/k_s)}$
- *a technical point:* instead of Φ one uses the curvature perturbation $\mathcal{R} = \Phi + \mathcal{H}v$, where the velocity potential v is defined by $\delta T^i_0 = -(\bar{\rho}+\bar{p})\partial^i v$. For adiabatic initial conditions it is constant on super-horizon scales (even at the RD-MD transition). Deep in RD $\mathcal{R} = (3/2)\Phi$. Then

$$\mathcal{P}_{\mathcal{R},\mathrm{in}}(k) = A_\mathcal{R} \left(\frac{k}{k_*}\right)^{n_s-1}, \qquad A_\mathcal{R} = (9/4)A_\Phi$$

Tensor perturbations

$$\langle \tilde{h}_{A,\mathrm{in}}(\mathbf{k})\tilde{h}^{*}_{A',\mathrm{in}}(\mathbf{k}')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{AA'}\frac{1}{4}P_{T,\mathrm{in}}(k)$$

Then

$$\langle \tilde{h}^{\mathrm{TT}}_{ij}(\eta_{\mathrm{in}},\mathbf{k})(\tilde{h}^{\mathrm{TT}}_{ij})^{*}(\eta_{\mathrm{in}},\mathbf{k}')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')P_{T,\mathrm{in}}(k).$$

Define again

$$\mathcal{P}_{T,\mathrm{in}}(k) \equiv \frac{k^3}{2\pi^2}P_{T,\mathrm{in}}(k)$$

We parametrize it as

$$\mathcal{P}_{T,\mathrm{in}}(k) = A_T\left(\frac{k}{k_*}\right)^{n_T}$$

The tensor-to-scalar ratio r is

$$r(k) \equiv \frac{\mathcal{P}_{T,\mathrm{in}}(k)}{\mathcal{P}_{\mathcal{R},\mathrm{in}}(k)}$$

In particular, setting $k = k_*$

$$r \equiv r(k_*) = \frac{A_T}{A_{\mathcal{R}}}$$

Relation to the GW energy density

$$\rho_{\rm gw} = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle = \int_{f=0}^{f=\infty} d(\log f)\ \frac{d\rho_{\rm gw}}{d\log f}$$

The critical density of the Universe is

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

Define

$$\Omega_{\rm gw}(f) \equiv \frac{1}{\rho_c}\frac{d\rho_{\rm gw}}{d\log f}$$

Then

$$\Omega_{\rm gw}(f) = \frac{\pi^2}{3H_0^2} f^2 \mathcal{P}_{T,0}(f)$$

Summary

- degrees of freedom of the gravitational field
  - *4 pure gauge dof,*
  - *4 physical but non-propagating dof:* Φ, Ψ, Ξ_i
  - *2 physical propagating dof:* $h_{ij}^{\rm TT}$
- evolution eqs for Φ, $h_{ij}^{\rm TT}$ in FRW

  (while typically $\Psi = -\Phi$, and Ξ_i only has decreasing modes)
- stochasticity enters through the initial conditions
  - def of scalar and tensor power spectra; $P_{T,0}(f)$, $\mathcal{P}_{T,0}(f)$
  - amplitudes and tilts A_Φ (or $A_{\mathcal{R}}$), n_S; A_T, n_T; $r = A_T/A_{\mathcal{R}}$
  - $\Omega_{\rm gw}(f)$ and relation to $\mathcal{P}_{T,0}(f)$

Inflationary cosmology

Single-field slow-roll inflation

$$S_\phi = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned}$$

effectively acts as a fluid with

$$w = \frac{p_\phi}{\rho_\phi} = \frac{(1/2)\dot{\phi}^2 - V(\phi)}{(1/2)\dot{\phi}^2 + V(\phi)}$$

$$(1/2)\dot{\phi}^2 \ll V(\phi) \quad \Rightarrow \quad w \simeq -1 \quad \Rightarrow \quad \text{accelerated expansion}$$

eq. of motion

$$3H^2 = 8\pi G\,\rho_\phi$$
$$\ddot{\phi} + 3H\dot{\phi} + V' = 0$$

first slow-roll condition

$\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$

$$\varepsilon \equiv \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \ll 1$$

second slow-roll condition

$|\ddot{\phi}| \ll |V'(\phi)|$

$$\eta \equiv \frac{1}{8\pi G}\frac{V''}{V} \ll 1$$

Inflation ends when one of the two conditions is violated.

Number of e-folds:

$$N(t_2, t_1) \equiv \int_{t_1}^{t_2} H(t)dt \overset{\text{de Sitter}}{=} \log\frac{a(t_2)}{a(t_1)}$$

The number of e-folds $N_e(\phi)$ to the end of inflation is

$$N_e(\phi) = 8\pi G\int_{\phi_e}^{\phi} d\phi_1\,\frac{V(\phi_1)}{V'(\phi_1)}$$

Large- and small-field models

Large-field models: typical example: $V(\phi) = g_n \phi^n$

$$\varepsilon = \frac{n^2}{16\pi} \frac{M_{\rm Pl}^2}{\phi^2} \qquad \eta = \frac{n(n-1)}{8\pi} \frac{M_{\rm Pl}^2}{\phi^2}$$

Inflation takes place at $\phi \gtrsim M_{\rm Pl}$!

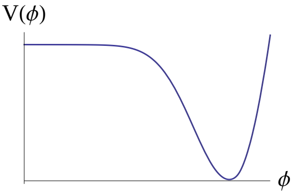
Consistency problems with quantum gravity ? Non-trivial to embed in a fundamental theory

Trading ϕ for N_e

$$\varepsilon = \frac{n}{4N_e}$$

$$\eta = \frac{n-1}{2N_e}$$

Small-field models: typical example: $V(\phi) \simeq V_0 - \frac{\lambda}{4}\phi^4$



As long as $(\lambda/4)\phi^4 \ll V_0$:

$$\eta = -\frac{3}{8\pi}\frac{\lambda M_{\rm Pl}^2\phi^2}{V_0}$$

$$\varepsilon = -\eta(\lambda\phi^4/V_0)$$

$$\Rightarrow \varepsilon \ll |\eta|$$

The slow-roll conditions are satisfied for sufficiently small ϕ, and inflation ends when

$$\phi_e^2 \simeq \frac{8\pi}{3}\frac{V_0}{\lambda M_{\rm Pl}^2}$$

Since we assumed $(\lambda/4)\phi^4 \ll V_0$, this is consistent as long as $(\lambda/4)\phi_e^4 \ll V_0$, i.e.

$$V_0 \ll \left(\frac{3}{4\pi}\right)^2 \lambda M_{\rm Pl}^4$$

In this model

$$N_e(\phi) \simeq \frac{4\pi V_0}{\lambda M_{\rm Pl}^2 \phi^2}$$

and we get

$$\varepsilon = \frac{27}{2N_e^3}\left[\left(\frac{4\pi}{3}\right)^2 \frac{V_0}{\lambda M_{\rm Pl}^4}\right] \ll \frac{27}{2N_e^3}$$

$$\eta = -\frac{3}{2N_e}$$

Starobinsky inflation

$$S = \frac{1}{16\pi G} \int d^4x\sqrt{-g}\left[R + \frac{R^2}{6M^2}\right]$$

where M is a new mass scale. It belongs to a more general class of theories

$$S = \frac{1}{16\pi G} \int d^4x\sqrt{-g}\, f(R)$$

Equivalence to scalar-tensor theories: see De Felice-Tsujikawa 2010

$$S[g_{\mu\nu}, \chi] = \frac{1}{16\pi G} \int d^4x\sqrt{-g}\,\left[f(\chi) + (R-\chi)f'(\chi)\right]$$

In fact, the variation with respect to χ gives

$$f''(\chi)(R-\chi) = 0$$

so, as long as $f''(\chi) \neq 0$, we have $\chi = R$.

Define $\varphi = f'(\chi)$, $\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu}$. Then

$$S[\tilde{g}_{\mu\nu}, \varphi] = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{16\pi G}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \right]$$

where

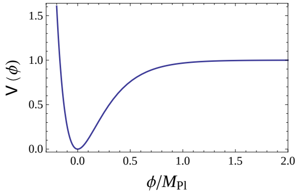
$$V(\phi) = \frac{M_{\rm Pl}^2}{16\pi} \frac{\chi(\varphi)\varphi - f[\chi(\varphi)]}{\varphi^2}$$

For Starobinsky model

$$V(\phi) = \frac{3M_{\rm Pl}^2 M^2}{32\pi} \left(1 - e^{-\sqrt{\frac{16\pi}{3}}\frac{\phi}{M_{\rm Pl}}}\right)^2$$

$\phi_e \simeq 0.19\, M_{\rm Pl}$

$\phi_i \simeq M_{\rm Pl}\left(1.07 + 0.24 \log \frac{\Delta N}{60}\right)$



In terms of N_e:

$$\varepsilon \simeq \frac{3}{4N_e^2}$$

$$\eta \simeq -\frac{1}{N_e}$$

Summary

- small-field and large-field inflation, super-Planckian problem
- definition of the slow-roll parameters ε, η
- relation of ε, η to N_e
  - $\varepsilon = \mathcal{O}(1/N_e)$ in large-field models of the form $V(\phi) \propto \phi^n$
  - $\varepsilon = \mathcal{O}(1/N_e^2)$ in Starobinsky model
  - and possibly even smaller in small-field models

Quantum fields in curved space

Consider a real massive scalar field ϕ in curved space

$$S = -\frac{1}{2}\int d^4x\,\sqrt{-g}\,\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]$$

The equation of motion is the KG equation

$$(\Box - m^2)\phi = 0$$

$$\Box\phi = \boldsymbol{\nabla}_\mu\boldsymbol{\nabla}^\mu\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\right)\phi$$

In flat space the solutions are the plane waves

$$u_{\mathbf{k}}(x) = \frac{1}{\sqrt{2\omega_k}}e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}}$$

and the quantum field is

$$\phi(x) = \int\frac{d^3k}{(2\pi)^3}\left[a_{\mathbf{k}}u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}}u^*_{\mathbf{k}}(x)\right]$$

In FRW, using conformal time

$$\phi''_{\mathbf{k}} + 2\mathcal{H}\phi'_{\mathbf{k}} + k^2\phi_{\mathbf{k}} = 0$$

Introduce

$$\chi(\eta, \mathbf{x}) = a(\eta)\phi(\eta, \mathbf{x})$$

$$\chi''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right)\chi_{\mathbf{k}} = 0$$

For de Sitter $a(\eta) = -1/(H\eta)$ (with $\eta < 0$):

$$\chi''_k + \left(k^2 - \frac{2}{\eta^2}\right)\chi_k = 0$$

Two independent solutions of this equations are $f_k(\eta)$ and $f_k^*(\eta)$, with

$$f_k(\eta) = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \left(1 - \frac{i}{k\eta}\right)$$

The quantum field is

$$\chi(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left(f_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} + f_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}}^\dagger\right)$$

with

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

$$a_{\mathbf{k}}|0\rangle_a = 0$$

This choice of modes defines the Bunch-Davies vacuum.

Bogolubov transformation

One could choose the modes

$$F_k(\eta) = \alpha_k f_k(\eta) + \beta_k f_k^*(\eta)$$

write

$$\chi(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left(F_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} A_{\mathbf{k}} + F_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} A_{\mathbf{k}}^\dagger \right)$$

and define a vacuum state from $A_{\mathbf{k}} |0\rangle_A = 0$

NB. The modes are chosen orthonormal wrt the scalar product

$$\langle u_1 | u_2 \rangle = i \int_\Sigma d\Sigma^\mu \left[u_1^* \partial_\mu u_2 - (\partial_\mu u_1^*) u_2 \right]$$

This requires

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

Expressing F_k in terms of f_k, f_k^* we get

$$a_{\mathbf{k}} = \int \frac{d^3k'}{(2\pi)^3} \left[\alpha_{\mathbf{k}'\mathbf{k}} A_{\mathbf{k}'} + \beta^*_{\mathbf{k}'\mathbf{k}} A^\dagger_{\mathbf{k}'}\right]$$

or

$$A_{\mathbf{k}} = \int \frac{d^3k'}{(2\pi)^3} \left[\alpha^*_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}'} - \beta^*_{\mathbf{k}\mathbf{k}'} a^\dagger_{\mathbf{k}'}\right]$$

Then

$$\langle 0_a | A^\dagger_{\mathbf{k}} A_{\mathbf{k}} | 0_a \rangle = \int \frac{d^3k'}{(2\pi)^3} \, |\beta_{\mathbf{k}\mathbf{k}'}|^2$$

The vacuum state wrt to $a_{\mathbf{k}}$ is not a vacuum state wrt $A_{\mathbf{k}}$

The *Bunch-Davies vacuum* is selected physically by the fact that, well inside the horizon, where $|k\eta| \gg 1$, it reduces to a positive frequency mode with respect to Minkowski space
Because of the cosmological red-shift, it is a very sensible choice for all momenta relevant today

Vacuum fluctuations of scalar fields

We now compute in FRW the quantum expectation value

$$\begin{aligned}\langle 0|\phi^2(\eta,\mathbf{x})|0\rangle &= \frac{1}{a^2(\eta)}\langle 0|\chi^2(x)|0\rangle \\ &= \frac{1}{a^2(\eta)}\int \frac{d^3k}{(2\pi)^3}\,|f_k(\eta)|^2 \\ &= \frac{1}{2\pi^2 a^2(\eta)}\int_0^\infty \frac{dk}{k}\,k^3|f_k(\eta)|^2\end{aligned}$$

Recalling the definition of the power spectrum

$$\langle 0|\phi^2(\eta,\mathbf{x})|0\rangle = \int_0^\infty \frac{dk}{k}\,\mathcal{P}_\phi(k)$$

we get

$$\mathcal{P}_\phi(k) = \frac{1}{2\pi^2 a^2(\eta)}k^3|f_k(\eta)|^2$$

In Minkowski space, $|f_k| = (2k)^{-1/2}$, so

$$\mathcal{P}_\phi(k) = \frac{k^2}{4\pi^2}$$

In de Sitter space, with a Bunch-Davies vacuum

$$\mathcal{P}_\phi(k) = \frac{H^2 k^2}{4\pi^2}\left(\eta^2 + \frac{1}{k^2}\right)$$

- at sufficiently early time $|k\eta| \gg 1$, the mode is well inside the horizon and fluctuations of field $\chi = a^{-1}\phi$ approaches the Minkowski value.
- when the mode is well outside the horizon,

$$\mathcal{P}_\phi(k) \simeq \frac{H^2}{4\pi^2}$$

so the power spectrum approaches a constant value.

Prediction for primordial scalar and tensor spectra

Scalar power spectrum

start from $\delta G^\mu_\nu = 8\pi G\, \delta T^\mu_\nu$

metric sector: $\Psi(\eta, \mathbf{x})$ $(\Phi = -\Psi)$

inflaton perturbations: $\phi(\eta, \mathbf{x}) = \phi_0(\eta) + \delta\phi(\eta, \mathbf{x})$

For the inflaton $T^\mu_\nu = g^{\mu\rho}\partial_\rho\phi\partial_\nu\phi - \delta^\mu_\nu \left[\frac{1}{2} g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right]$
so

$$\delta T^0_0 = -\frac{1}{a^2}\left[-(\phi_0')^2\Psi + \phi_0'\delta\phi'\right] - \frac{dV(\phi_0)}{d\phi_0}\delta\phi$$

$$\delta T^i_0 = \frac{1}{a^2}\phi_0'\partial_i\delta\phi$$

Using $\delta T^i_0 = -(\bar\rho + \bar p)\partial^i v$, we get the velocity potential $v_\phi = -\frac{\delta\phi}{\phi_0'}$

from the definition $\mathcal{R} = \Phi + \mathcal{H} v$ it follows that

$$\mathcal{R} = -\Psi - \frac{\mathcal{H}\delta\phi}{\phi_0'}$$

define:

$$z(\eta) = \frac{a\phi_0'}{\mathcal{H}} \qquad u(\eta, \mathbf{x}) = -z\mathcal{R}$$

Then

$$u'' - \frac{z''}{z} u - \nabla^2 u = 0$$

To leading order in the slow-roll expansion $\phi_0'/\mathcal{H}$ is constant. Then $z''/z \simeq a''/a$

$\Rightarrow$ u satisfies the same eq as the scalar field that we called χ !

(and its action has the canonical normalization) on super-horizon scales its power spectrum is then

$$\mathcal{P}_{u/a}(k) \simeq \frac{H^2}{4\pi^2}$$

We use

$$\frac{u(x)}{a} = -\frac{\phi_0'}{\mathcal{H}}\mathcal{R}$$

so, passing to cosmic time,

$$\mathcal{R} = -\frac{H}{\dot{\phi}_0}\frac{u(x)}{a}$$

Then, the primordial spectrum of $\mathcal{R}$ for super-horizon modes is given by

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\phi}_0}\right)^2 \mathcal{P}_{u/a}(k) \simeq \left(\frac{H^2}{2\pi\dot{\phi}_0}\right)^2$$

This result is valid to lowest order in the slow-roll expansion, i.e. assuming that H and $\dot{\phi}_0$ are exactly constant

In this approximation we find $n_s = 0$

To next order in the slow-roll expansion:

- as long as the mode is well inside the horizon, $z''/z \simeq a''/a$ is negligible and the solution reduces to the standard Minkowski modes $(2k)^{-1/2} e^{-ik\eta}$
- When the wavelength of the mode becomes of order of the horizon size we can no longer neglect the term a''/a. However, for a limited number of e-folds near horizon crossing, the solution that matches the mode $(2k)^{-1/2} e^{-ik\eta}$ inside the horizon is obtained replacing H is the value at time of horizon crossing, $H_k \equiv H(\eta_k)$, where the conformal time of horizon exit, η_k, is the solution of

$$H(\eta_k) = \frac{k}{a(\eta_k)}$$

To first order in the slow-roll expansion, the result for the power spectrum is obtained simply replacing H by H_k and similarly $\dot{\phi}_0$ by $(\dot{\phi}_0)_k = \dot{\phi}_0(\eta_k)$

The result for the primordial power spectrum of $\mathcal{R}$ on super-horizon scales is then

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi|\dot{\phi}_0|}\right)^2_k$$

using the slow-roll eqs.

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{128\pi}{3M_{\rm Pl}^6}\left(\frac{V^3}{V'^2}\right)_k$$

The amplitude at a pivot scale k_* is then

$$A_{\mathcal{R}} = \frac{128\pi}{3M_{\rm Pl}^6}\left(\frac{V^3}{V'^2}\right)_{k_*}$$

Writing

$$\log \mathcal{P}_{\mathcal{R}}(k) = \log A_{\mathcal{R}} + (n_s - 1)\log(k/k_*)$$

we get

$$n_s(k) - 1 = \frac{d\log \mathcal{P}_{\mathcal{R}}(k)}{d\log k} = \frac{d\log(V^3/V'^2)_k}{d\log k}$$

and then $n_s \equiv n_s(k_*)$

the dependence of V and V' on k follows from the fact that they are functions of $\phi(\eta)$, and $\phi(\eta)$ must be evaluated at the time η_k

Next transform the derivative with respect to k into a derivative with respect to $\phi_k \equiv \phi(\eta_k)$

Let t_k be the cosmic time corresponding to conformal time η_k, and $t_* = t_{k_*}$ be the cosmic time at which the mode with momentum k_* equal to the pivot scale crosses the horizon. Integrating the slow-roll eq from an initial time t_* to $t_k = t_* + dt_k$ we get

$$d\phi_k = -\frac{M_{\rm Pl}}{\sqrt{24\pi}} \left(\frac{V'}{V^{1/2}}\right)_{k_*} dt_k$$

where $d\phi_k = \phi(t_k) - \phi(t_*)$. In slow-roll inflation $H(t)$ is to first approximation constant, while $a(t)$ evolves exponentially with t. Therefore, to lowest order, t_k is approximately determined by the equation $a(t_k) \simeq k/H_*$, so

$$\left(\frac{da}{dt}\right)_{t_*} dt_k \simeq \frac{dk}{H_*}$$

Therefore

$$(aH)_{t_*} \, dt_k \simeq \frac{dk}{H_*}$$

which gives

$$dt_k = \frac{1}{H_*}\frac{dk}{k}$$

Finally

$$d\phi_k = -\frac{M_{\rm Pl}^2}{8\pi} \left(\frac{V'}{V}\right)_{k_*} d\log k$$

Then we get

$$\begin{aligned} n_s - 1 &= \frac{M_{\rm Pl}^2}{8\pi}\left[\left(\frac{V'}{V}\right)\left(\frac{2V''}{V'} - \frac{3V'}{V}\right)\right]_{k_*} \\ &= \frac{M_{\rm Pl}^2}{8\pi}\left[2\frac{V''}{V} - 3\left(\frac{V'}{V}\right)^2\right]_{k_*} \end{aligned}$$

Recalling the definition of the slow-roll parameters, we finally find

$$n_s - 1 = 2\eta - 6\epsilon$$

where it is understood that η and ϵ are evaluated at the time $\eta(k_*)$.

Tensor power spectrum

We already saw that, when the anisotropic stress $\sigma_A = 0$:

$$\tilde{h}_A'' + 2\mathcal{H}\tilde{h}_A' + k^2\tilde{h}_A = 0$$

This is exactly the same as the equation of motion of a scalar field ϕ!

Normalization determined looking at the action. Expanding to quadratic order

$$S_2[h] = -\frac{1}{32\pi G}\sum_A \int d^4x\sqrt{-\bar{g}}\,\bar{g}^{\mu\nu}\partial_\mu h_A \partial_\nu h_A$$

where $\bar{g}_{\mu\nu} = a^2\eta_{\mu\nu}$ is the background FRW metric. Then

$$\varphi_A(\eta, \mathbf{x}) = \frac{1}{\sqrt{16\pi G}}\, h_A(\eta, \mathbf{x})$$

$$\frac{1}{4}P_{T,\mathrm{in}}(k) = \langle|\tilde{h}_{A,\mathrm{in}}(\mathbf{k})|^2\rangle = 16\pi G\, P_{\varphi}(k)$$

Using the result for $\mathcal{P}_{\varphi}(k)$ we get

$$\boxed{\mathcal{P}_T(k) = \frac{16}{\pi}\,\frac{H_k^2}{M_{\mathrm{Pl}}^2}}$$

and, using the slow-roll eqs.

$$\mathcal{P}_T(k) = \frac{128}{3}\,\frac{V_k}{M_{\mathrm{Pl}}^4}$$

This gives the amplitude and tilt

$$\begin{aligned} A_T &= \frac{16}{\pi}\,\frac{H_*^2}{M_{\mathrm{Pl}}^2} \simeq \frac{128}{3}\,\frac{V_{k_*}}{M_{\mathrm{Pl}}^4} \\ n_T &= \left(\frac{d\log V}{d\log k}\right)_{k_*} = -2\varepsilon \end{aligned}$$

Summary

Single-field slow-roll inflation predicts

$$A_{\mathcal{R}} = \frac{128\pi}{3M_{\rm Pl}^6}\left(\frac{V^3}{V'^2}\right)_{k_*}$$
$$n_s - 1 = 2\eta - 6\epsilon$$

$$A_T = \frac{128}{3}\frac{V_{k_*}}{M_{\rm Pl}^4}$$
$$n_T = -2\varepsilon$$

Therefore

$$r = \frac{A_T}{A_R} = 16\varepsilon$$

Observe also that $r = -8n_T$ independently of the potential $V(\phi)$

Comparison of inflationary models with Planck data

$\varepsilon = \varepsilon(k_*)$ depends on $N_* = N_e(\phi_{k_*})$. We take N_* as the free parameter of the model (typically $N_* \sim 60$ for having sufficient inflation)

- For $V(\phi) \propto \phi^n$, we found $\varepsilon = n/(4N_*)$, $\eta = (n-1)/(2N_*)$

$$\begin{aligned} n_s - 1 &= -\frac{n+2}{2N_*} \\ r &= \frac{4n}{N_*} = \frac{n}{15}\left(\frac{60}{N_*}\right) \end{aligned}$$

  - $\lambda\phi^4$ ruled out by far (already by WMAP)
  - $m^2\phi^2$ almost ruled out by Planck

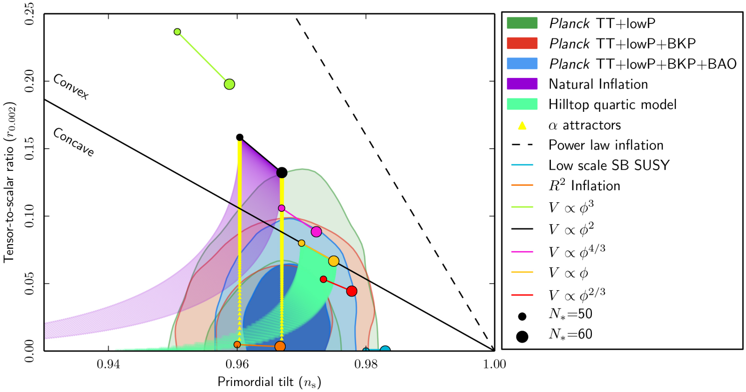
- For Starobinsky model,

$$\begin{aligned} n_s - 1 &\simeq -\frac{2}{N_*} \\ r &\simeq \frac{12}{N_*^2} \end{aligned}$$

Starobinsky (1980),
Mukhanov and Chibisov (1981)

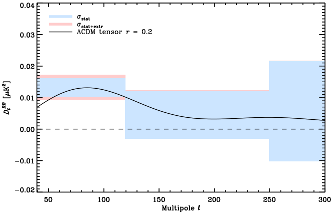
Consistent with the Planck 2015 data (at 95% c.l.) for $54 < N_* < 62$

Temperature anisotropies



Planck, paper XX (2015)

Polarization anisotropies



The extrapolation of the *Planck* measurement of dust polarization at the BICEP2 frequency and in the BICEP2 field of view (vertical bands) compared to a primordial GW signal with $r = 0.2$ (black line).

Planck Collaboration (2014)

Joint analysis of data from BICEP2/Keck Array and Planck:

- strong evidence for dust and no statistically significant evidence for tensor modes
- $r_{0.05} < 0.12$ at 95% c.l. $\qquad (k_* = 0.05\,\mathrm{Mpc}^{-1})$

The relic GW background today

We now evolve the relic inflationary background to the present epoch, to make contact with the searches at GW interferometers

As long as a GW is outside the horizon, its amplitude stays constant, inside the horizon it decays

Depending on its comoving momentum, the mode will then re-enter the horizon during RD or during MD.

Which mode re-enters at the RD-MD transition? Solving $k_{\rm eq} = \mathcal{H}(\eta_{\rm eq})$, we get $k_{\rm eq} \simeq 0.015\,(h_0/{\rm Mpc})$.

$$\Rightarrow \quad f_{\rm eq} = k_{\rm eq}/(2\pi) \simeq 1.62 \times 10^{-17}\,{\rm Hz}$$

(comoving quantities are the same as physical quantities today, since $a(t_0) = 1$)

LIGO/Virgo, eLISA, PTA, etc. are therefore potentially sensitive to primordial GWs with $f \gg f_{\rm eq}$, so $\lambda \ll \lambda_{\rm eq}$, which re-entered the horizon deep in the RD era.

Recall that, in RD,

$$\tilde{h}(\eta, k) = \tilde{h}_{\rm in}(k) \frac{\sin(k\eta)}{k\eta}$$

and $a(\eta) \propto \eta$, so in RD $\tilde{h}(\eta, k) \propto \sin(k\eta)/a(\eta)$

the scaling $\tilde{h}(\eta, k) \propto 1/a(\eta)$ persists in MD

$$\Rightarrow \qquad \tilde{h}_k^2(\eta_0) = \frac{1}{2} \tilde{h}_{k,\rm in}^2 \left(\frac{a_*(k)}{a_0} \right)^2$$

where

- $\tilde{h}_{k,\rm in}$ is the primordial value of the tensor perturbation, determined when it is well outside the horizon
- $a_*(k)$ is the value of the scale factor when the mode with comoving momentum k re-enters the horizon, and $a_0 = 1$
- the factor of $1/2$ comes from $\langle \sin^2(k\eta) \rangle$ (precise numerical factors require numerical integration)

The tensor power spectrum today $\mathcal{P}_{T,0}(k)$ is related to the primordial power spectrum $\mathcal{P}_{T,\text{in}}(k)$ by

$$\mathcal{P}_{T,0}(k) = \frac{1}{2}a_*^2(k)\mathcal{P}_{T,\text{in}}(k)$$

$a_*(k)$ is determined by

$$H[a_*(k)] = \frac{k}{a_*(k)}$$

$H(a)$ is determined by the Friedmann eq

$$H^2 = \frac{8\pi G}{3}\rho$$

define

$$\rho_0 = \frac{3H_0^2}{8\pi G} \qquad \Omega_R = \frac{\rho_{R,0}}{\rho_0} \qquad \Omega_M = \frac{\rho_{M,0}}{\rho_0}$$

Since $\rho_R(a) \propto a^{-4}$, $\rho_M(a) \propto a^{-3}$, we have

$$\rho_R(a) = \rho_0\Omega_R a^{-4}, \qquad \rho_M(a) = \rho_0\Omega_M a^{-3}$$

Then, in RD, $H(a) \simeq H_0 \Omega_R^{1/2}/a^2$

More precisely, taking into account the effective number of relativistic species, deep in RD

$$H(a) \simeq 0.62 \left(\frac{106.75}{g_*(T)}\right)^{1/6} \frac{H_0 \Omega_R^{1/2}}{a^2}$$

Then

$$a_*(k) \simeq 0.62 \left(\frac{106.75}{g_*(T_k)}\right)^{1/6} \frac{H_0 \Omega_R^{1/2}}{k}$$

($g_*(T_k)$ is the value of $g_*(T)$ when the mode with comoving momentum k re-enters the horizon)
Then

$$\mathcal{P}_{T,0}(k) \simeq 0.38 \left(\frac{106.75}{g_*(T_k)}\right)^{1/3} \frac{H_0^2 \Omega_R}{2k^2} A_T \left(\frac{k}{k_*}\right)^{n_T}, \qquad (k \gg k_{\rm eq})$$

Recalling that

$$\Omega_{gw}(f) = \frac{\pi^2}{3H_0^2} f^2 \mathcal{P}_{T,0}(f)$$

we get (setting henceforth $g_*(T_k) = 106.75$)

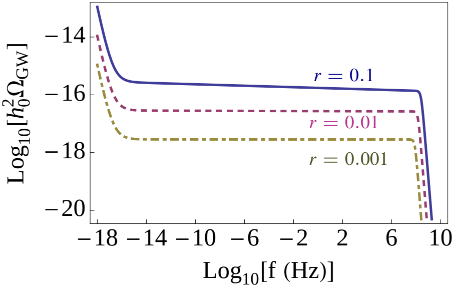
$$\Omega_{gw}(f) \simeq \frac{0.38}{24} \Omega_R A_T \left(\frac{f}{f_*}\right)^{n_T}, \qquad (f \gg f_{eq})$$

The factors f^2 canceled and, for $f \gg f_{eq}$, $\Omega_{gw}(f)$ is almost flat, with its only dependence on the frequency given n_T.

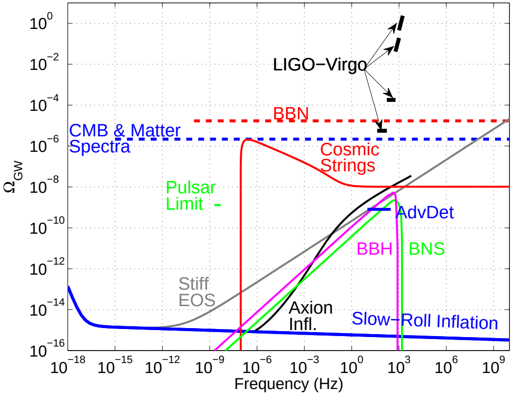
Write $A_T = rA_{\mathcal{R}}$ and use the value of $A_{\mathcal{R}}$ and the limit on r determined by Planck

$$h_0^2 \Omega_{gw}(f) \simeq 1.43 \times 10^{-16} \left(\frac{A_{\mathcal{R}}}{2.14 \times 10^{-9}}\right) \left(\frac{r}{0.1}\right) \left(\frac{f}{f_*}\right)^{-r/8}$$

For frequencies $f < f_{\rm eq}$ there is a further $1/f^2$ enhancement due to the fact that they enter the horizon later, in MD



The inflationary background that we have studied is just an element of a larger picture of theoretical predictions and existing bounds:



LIGO and Virgo coll., PRL 2014