On the saturation of rotational instabilities in neutron stars and the associated gravitational wave emission

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8th Aegean summer school
Gravitational waves: from theory to observations

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EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Outline

1. Oscillation modes
   - Fluid equations
   - Classes of modes
2. The CFS instability
   - The instability window
3. Mode coupling
   - Quadratic perturbation equations
   - Equations of motion
   - Parametric resonance instability
   - Saturation conditions
4. Results and remarks

Oscillation modes

- Fluid equations (in corotating frame):

$$\dot{\rho} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\dot{\boldsymbol{v}} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = -\frac{\nabla p}{\rho} - \nabla \phi$$

$$\nabla^2 \phi = 4\pi G \rho$$

$$p = p(\rho, \mu)$$

- *Linearly* perturbed fluid equations:

$$\delta\rho + \nabla \cdot (\rho \boldsymbol{\xi}) = 0$$

$$\ddot{\boldsymbol{\xi}} + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{\xi}} = -\frac{\nabla \delta p}{\rho} + \frac{\nabla p}{\rho^2}\delta\rho - \nabla \delta\phi$$

$$\nabla^2 \delta\phi = 4\pi G \delta\rho$$

$$\frac{\Delta p}{p} = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_\mu \frac{\Delta \rho}{\rho} + \left(\frac{\partial \ln p}{\partial \ln \mu}\right)_\rho \frac{\Delta \mu}{\mu}$$

Eulerian (δ) and *Lagrangian* (Δ) perturbations related via $\Delta f = \delta f + (\boldsymbol{\xi} \cdot \nabla) f$

Oscillation modes

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Assuming $\boldsymbol{\xi}(\boldsymbol{r}, t) = \boldsymbol{\xi}(\boldsymbol{r}) e^{i\omega t}$:

$$\boxed{-\omega^2 \boldsymbol{\xi} + i\omega \boldsymbol{\mathcal{B}}(\boldsymbol{\xi}) + \boldsymbol{\mathcal{C}}(\boldsymbol{\xi}) = \boldsymbol{0}}$$

+

boundary conditions

$\Downarrow$

$\omega, \boldsymbol{\xi}$

Eulerian (δ) and *Lagrangian* (Δ) perturbations related via $\Delta f = \delta f + (\boldsymbol{\xi} \cdot \nabla) f$

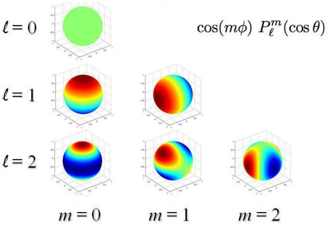
Oscillation modes

$$\boldsymbol{\xi}(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[U_l^m(r)Y_l^m(\theta,\phi)\hat{\boldsymbol{e}}_r + V_l^m(r)\nabla Y_l^m(\theta,\phi) + W_l^m(r)\hat{\boldsymbol{e}}_r \times \nabla Y_l^m(\theta,\phi)\right]$$

- Polar modes: $W_l^m = 0$
- Axial modes: $U_l^m = V_l^m = 0$

as $\Omega \to 0$

l :	degree
m :	order
n :	overtone



Mode name	Mode class	Mode type	Restoring force
p-mode	Polar	Sound wave ($\omega \to \infty$ as $n \to \infty$)	Pressure gradient
f-mode	Polar	Low-ω sound wave / High-ω gravity wave } $n = 0$	
g-mode	Polar	Gravity wave ($\omega \to 0$ as $n \to \infty$)	Buoyancy
r-mode	Axial	Inertial wave	Coriolis
Hybrid mode	Combination	Zero-buoyancy limit or r- and g-modes	

● Only for non-zero rotation ● Only for non-zero buoyancy

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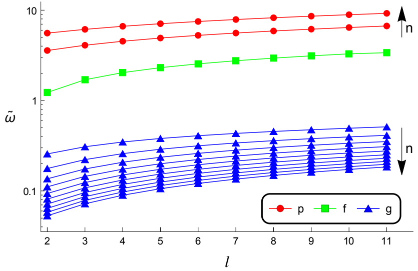

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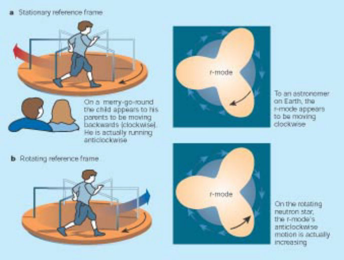
The CFS instability

Are the perturbations stable?

Rapidly rotating stars are prone to *secular instabilities*, i.e. instabilities related to dissipation mechanisms (viscosity, gravitational radiation).



$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{GW}} = -\sum_{l\geq 2}^{\infty} N_l\, \omega\, (\omega - m\Omega)^{\,2l+1} \left(|\delta D_l^m|^2 + |\delta J_l^m|^2\right)$$

- Polar (axial) modes emit through the mass (current) multipoles
- If $\omega(\omega - m\Omega) < 0$, then $\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{GW}} > 0$

CFS instability

For *any* rotation rate Ω there is always a mode driven unstable by gravitational radiation emission [Chandrasekhar, 1970, Friedman and Schutz, 1978].

- f-modes and r-modes are the most susceptible to GW-driven instabilities

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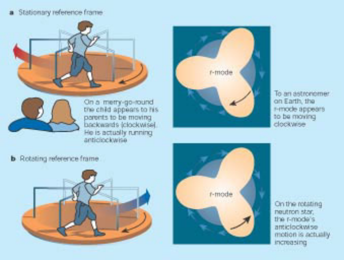

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  inertial-frame frequency

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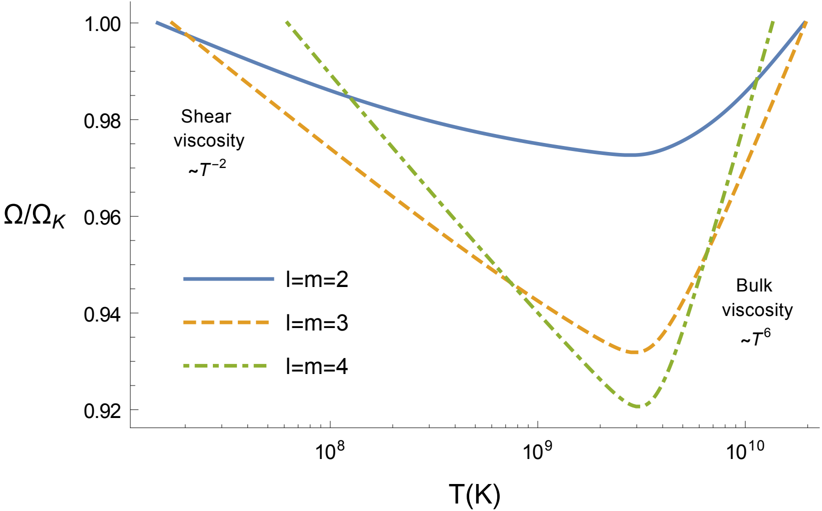
The instability window



Mode coupling

Do the unstable modes grow boundlessly?

Non-linear mode coupling inhibits the instability's growth [Dziembowski, 1982, Schenk et al., 2002, Arras et al., 2003, Brink et al., 2004]

- Quadratically perturbed fluid equations:

$$\delta\dot{\rho} + \nabla\cdot(\rho\boldsymbol{v}) + \nabla\cdot(\delta\rho\,\boldsymbol{v}) = 0$$

$$\ddot{\boldsymbol{\xi}} + \boldsymbol{\mathcal{B}}(\dot{\boldsymbol{\xi}}) + \boldsymbol{\mathcal{C}}(\boldsymbol{\xi}) + \boldsymbol{\mathcal{N}}(\boldsymbol{\xi},\boldsymbol{\xi}) = \mathbf{0}$$

$$\nabla^2\delta\phi = 4\pi G\delta\rho$$

$$\frac{\Delta p}{p} = \Gamma_1\frac{\Delta\rho}{\rho} + \frac{1}{2}\left[\Gamma_1(\Gamma_1-1) + \left(\frac{\partial\Gamma_1}{\partial\ln\rho}\right)_\mu\right]\left(\frac{\Delta\rho}{\rho}\right)^2, \qquad \boxed{\Gamma_1 = \left(\frac{\partial\ln p}{\partial\ln\rho}\right)_\mu}$$

Eulerian (δ) and *Lagrangian* (Δ) perturbations related via

$$\Delta f = \delta f + (\boldsymbol{\xi}\cdot\nabla)f + (\boldsymbol{\xi}\cdot\nabla)\delta f + \tfrac{1}{2}\boldsymbol{\xi}\cdot[\boldsymbol{\xi}\cdot\nabla(\nabla f)]$$

- Perturbation decomposition:

$$\boldsymbol{\xi}(\boldsymbol{r},t) = \sum_\alpha Q_\alpha(t)\,\boldsymbol{\xi}_\alpha(\boldsymbol{r})e^{i\omega_\alpha t}$$

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- Perturbation decomposition:

$$\boldsymbol{\xi}(\boldsymbol{r},t) = \sum_\alpha \underbrace{Q_\alpha(t)}_{\text{mode amplitude}}\,\boldsymbol{\xi}_\alpha(\boldsymbol{r})e^{i\omega_\alpha t}$$

Mode coupling

- Modes couple in *triplets*

$$\dot{Q}_\alpha = \gamma_\alpha Q_\alpha + i\omega_\alpha \mathcal{H}\, Q_\beta Q_\gamma\, e^{-i\Delta\omega t}$$

$$\dot{Q}_\beta = \gamma_\beta Q_\beta + i\omega_\beta \mathcal{H}\, Q_\gamma^* Q_\alpha\, e^{i\Delta\omega t}$$

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- Detuning $\Delta\omega \equiv \omega_\alpha - \omega_\beta - \omega_\gamma \approx 0$ resonance condition

  The system exhibits *internal resonances*

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- Detuning $\Delta\omega \equiv \omega_\alpha - \omega_\beta - \omega_\gamma \approx 0$ resonance condition
- Coupling coefficient $\mathcal{H} \neq 0$ if

$$\left.\begin{array}{l} m_\alpha = m_\beta + m_\gamma \\ l_\alpha + l_\beta + l_\gamma = \text{even number} \\ |l_\beta - l_\gamma| \leq l_\alpha \leq l_\beta + l_\gamma \end{array}\right\} \text{coupling selection rules}$$

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- Growth/damping rates $\gamma_i = \dfrac{1}{2E_i}\dfrac{\mathrm{d}E_i}{\mathrm{d}t} \gtrless 0$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{GW}} + \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{BV}} + \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{SV}} \gtrless 0$$

Parametric resonance instability

$$\dot{Q}_\alpha = \gamma_\alpha Q_\alpha + i\omega_\alpha \mathcal{H}\, Q_\beta Q_\gamma\, e^{-i\Delta\omega t}$$
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Detuning $\Delta\omega$

Coupling coefficient $\mathcal{H}$

Growth/damping rates γ_i

- *Parent mode:* unstable f-mode ($\gamma_\alpha > 0$)
- *Daughter modes:* other (stable) polar modes ($\gamma_{\beta,\gamma} < 0$)

Parametric resonance instability

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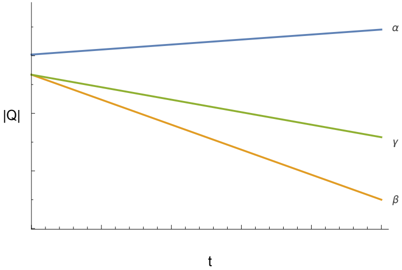
<u>No mode interaction:</u> $\mathcal{H} = 0$ or $\Delta\omega \gg 0$



- Modes evolve independently
- No non-linear interaction

$$\dot{Q}_\alpha = \gamma_\alpha Q_\alpha$$
$$\dot{Q}_\beta = \gamma_\beta Q_\beta$$
$$\dot{Q}_\gamma = \gamma_\gamma Q_\gamma$$

Parametric resonance instability

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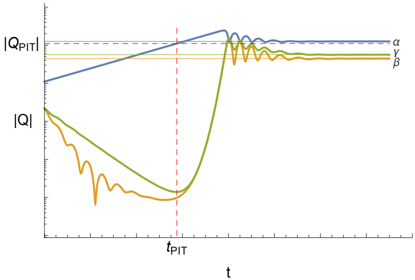
Detuning $\Delta\omega$
Coupling coefficient $\mathcal{H}$
Growth/damping rates γ_i

- *Parent mode:* unstable f-mode ($\gamma_\alpha > 0$)
- *Daughter modes:* other (stable) polar modes ($\gamma_{\beta,\gamma} < 0$)

Parametric resonance instability: $\mathcal{H} \neq 0$ and $\Delta\omega \approx 0$



- Parent feeds daughters and makes them grow
- *Parametric instability threshold:* daughters grow when

$$|Q_\alpha|^2 > |Q_{\rm PIT}|^2 \equiv \frac{\gamma_\beta \gamma_\gamma}{\omega_\beta \omega_\gamma \mathcal{H}^2}\left[1 + \left(\frac{\Delta\omega}{\gamma_\beta + \gamma_\gamma}\right)^2\right]$$

- Parent *saturation amplitude* ≈ parametric instability threshold

Saturation conditions

$$\dot{Q}_\alpha = \gamma_\alpha Q_\alpha + i\omega_\alpha \mathcal{H}\, Q_\beta Q_\gamma\, e^{-i\Delta\omega t}$$
$$\dot{Q}_\beta = \gamma_\beta Q_\beta + i\omega_\beta \mathcal{H}\, Q_\gamma^* Q_\alpha\, e^{i\Delta\omega t}$$
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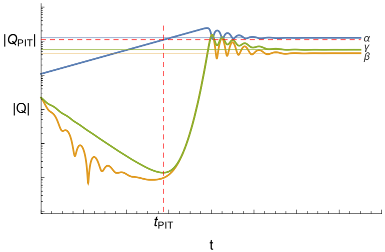
$$\gamma_\alpha > 0,\ \gamma_{\beta,\gamma} < 0$$
$$|Q_{\rm PIT}|^2 \equiv \frac{\gamma_\beta\gamma_\gamma}{\omega_\beta\omega_\gamma\mathcal{H}^2}\left[1 + \left(\frac{\Delta\omega}{\gamma_\beta + \gamma_\gamma}\right)^2\right]$$

Detuning $\Delta\omega$
Coupling coefficient $\mathcal{H}$
Growth/damping rates γ_i

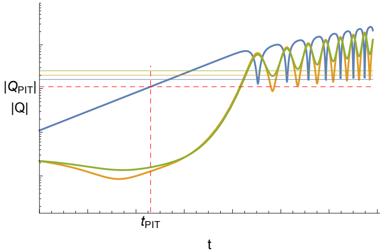
- Saturation successful if:

$$|\gamma_\beta + \gamma_\gamma| \gtrsim \gamma_\alpha \quad \text{and} \quad \Delta\omega \gtrsim |\gamma_\alpha + \gamma_\beta + \gamma_\gamma|$$

Saturation successful

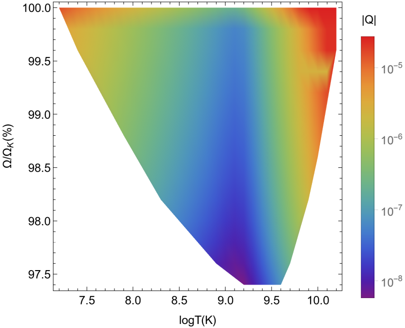


Saturation unsuccessful

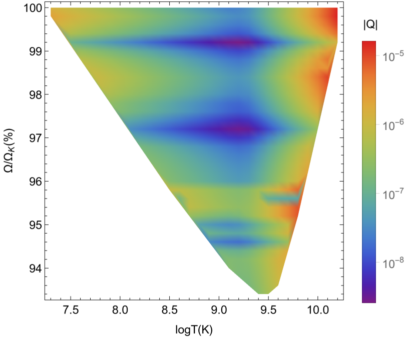


Results and remarks

Saturation amplitude
of the $l = m = 2$ f-mode



Saturation amplitude
of the $l = m = 3$ f-mode



Model: $M = 1.4\,M_\odot$, $R = 10\,\mathrm{km}$, $p \propto \rho^3$, $\Gamma_1 = 3.1$
Units: $E_{\rm mode} = |Q|^2 M c^2$

- GW signal from f-mode depends on saturation amplitude
- Neutron star *equation of state* probing: GW asteroseismology
- Promising sources: post-merger remnants

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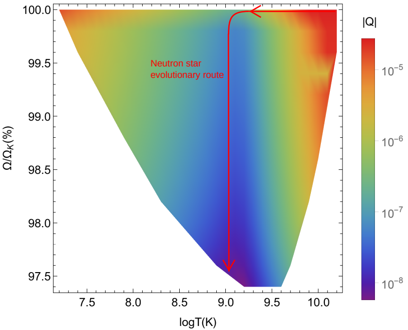

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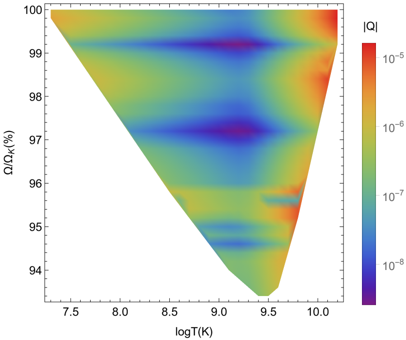

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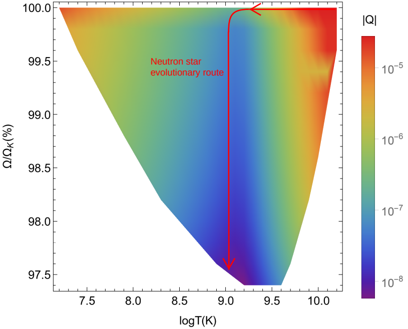
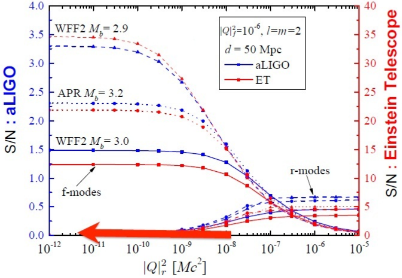


f-mode vs r-mode
[Doneva et al., 2015]



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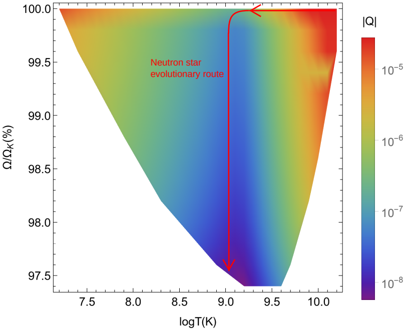
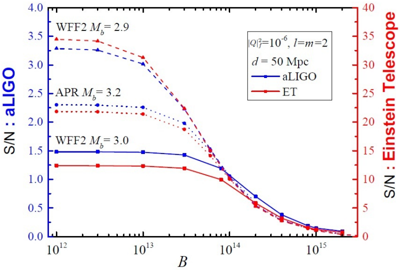


f-mode vs magnetic field
[Doneva et al., 2015]



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