Teleparallel gravity and modifications

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Plan

- Notion of teleparallelism
  - First version : express Einstein-Hilbert Lagrangian, Einstein equations,... in terms of the torsion of Weitzenböck connection $\omega(e)$
  - Second version : express in terms of e, ω with $Riem(\omega) = 0$

  Like Einstein-Cartan, but with the constraint $Riem(\omega) = 0$
- Teleparallel equivalent of Gauss-Bonnet
- Modified gravities, applications
- Non-minimal derivative coupling of scalar field with torsion

- Christoffel connection: $\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})$
- $g_{\mu\nu} = \eta_{ab} e^{a}{}_{\mu} e^{b}{}_{\nu}$ orthonormal vielbein
- algebraic substitution of the field variable
  provides new perspectives for defining a local energy-momentum tensor for the gravitational field, for regarding gravity as a gauge theory of local translations, for constructing (at least covariant under diffeomorphisms) modified gravity theories, maybe related to quantization issues, etc.
- $\Gamma^{\lambda}{}_{\mu\nu} = e_{a}{}^{\lambda} e^{a}{}_{\mu,\nu} - \mathcal{K}^{\lambda}{}_{\mu\nu}$
  $\mathcal{K}_{\lambda\nu\mu} = \frac{1}{2}(T_{\mu\lambda\nu} - T_{\nu\mu\lambda} - T_{\lambda\nu\mu})$
  $T^{\lambda}{}_{\mu\nu} = e_{a}{}^{\lambda}(e^{a}{}_{\nu,\mu} - e^{a}{}_{\mu,\nu})$ tensor under diffeomorphisms
  $\tilde{x}^{\mu} = \tilde{x}^{\mu}(x^{\nu})\,,\ \tilde{e}_{a}{}^{\mu} = \gamma^{\mu}{}_{\nu} e_{a}{}^{\nu}\,,\ \gamma^{\mu}{}_{\nu} = \frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}}$

Torsion

- Arbitrary connection $\omega^{\lambda}{}_{\mu\nu}$ of zero non-metricity $\nabla_{\omega} g = 0$:
  $\omega^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + \mathcal{K}^{\lambda}{}_{\mu\nu}$ (identity)
  $\mathcal{K}_{\lambda\nu\mu} = \frac{1}{2}(T_{\mu\lambda\nu} - T_{\nu\mu\lambda} - T_{\lambda\nu\mu})$ contorsion
  $T^{\lambda}{}_{\mu\nu} = \omega^{\lambda}{}_{\nu\mu} - \omega^{\lambda}{}_{\mu\nu}$ torsion of ω (tensor under diffeomorphisms)
- $\omega^{\lambda}{}_{\mu\nu}(e) = e_a{}^{\lambda} e^a{}_{\mu,\nu}$ Weitzenböck connection
  - $\omega^{\lambda}{}_{\mu\nu}$ metric compatible
  - $\tilde{x}^{\mu} = \tilde{x}^{\mu}(x^{\nu})$, $\tilde{e}_a{}^{\mu} = \gamma^{\mu}{}_{\nu} e_a{}^{\nu}$, $\gamma^{\mu}{}_{\nu} = \frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}}$:
    $\tilde{\omega}^{\lambda}{}_{\mu\nu} = \tilde{e}_a{}^{\lambda} \frac{\partial \tilde{e}^a{}_{\mu}}{\partial \tilde{x}^{\nu}} = \omega\gamma\gamma^{-1}\gamma^{-1} - \gamma^{-1}\gamma^{-1}\partial\gamma$
    in all coordinate systems ω has the same form (covariant)
  - $\omega^a{}_{bc} = \omega^{\lambda}{}_{\nu\mu} e^a{}_{\lambda} e_b{}^{\nu} e_c{}^{\mu} + e^a{}_{\nu} e_c{}^{\mu} e_b{}^{\nu}{}_{,\mu} = 0$ in the frame $e_a{}^{\mu}$ where it is defined
  - $\tilde{e}^a = \Lambda^a{}_b e^b$: $\tilde{\omega}^a{}_{bc} = -(\Lambda^{-1})^d{}_b (\Lambda^{-1})^e{}_c \Lambda^a{}_{d,e}$ in other frames (not covariant)
  - $\omega^{\lambda}{}_{\mu\nu}$ not Lorentz invariant (since a particular frame is used)

- Christoffel connection (vanishing torsion and non-metricity):
$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d}) + \frac{1}{2}(-C^a{}_{bc} + g_{bd} g^{ae} C^d{}_{ec} + g_{cd} g^{ae} C^d{}_{eb})$
$C^c{}_{ab} = e_a{}^\mu e_b{}^\nu (e^c{}_{\mu,\nu} - e^c{}_{\nu,\mu})$, $[e_a, e_b] = C^c{}_{ab} e_c$
$g_{ab} = g_{\mu\nu} e_a{}^\mu e_b{}^\nu$
  - Under difeomorphisms, $C^a{}_{bc}, g_{ab}, \Gamma^a{}_{bc}$, $\bar{R}^a{}_{bcd}$ invariants
  - Under $\tilde{e} = \gamma e$, $\tilde{C} = C\gamma\gamma^{-1}\gamma^{-1} + (\gamma^{-1}\gamma^{-1} - \gamma^{-1}\gamma^{-1})\partial\gamma$
  $\tilde{g} = \gamma^{-1}\gamma^{-1} g$
  $\tilde{\Gamma} = ... = \gamma\gamma^{-1}\gamma^{-1}\Gamma - \gamma^{-1}\gamma^{-1}\partial\gamma$ ($\bar{R}^a{}_{bcd}$ Lorentz tensor)
  - Under frame changes, $\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})$,
  $\bar{R}^\lambda{}_{\mu\nu\kappa}$ invariant ($g_{\mu\nu} = g_{ab} e^a{}_\mu e^b{}_\nu$ invariant)

Weitzenböck ...

- $T^{\lambda}{}_{\mu\nu} = e_a{}^{\lambda}(e^a{}_{\nu,\mu} - e^a{}_{\mu,\nu}) = e_a{}^{\lambda}(e^a{}_{\nu;\mu} - e^a{}_{\mu;\nu}) =$
  $-C^a{}_{bc} e_a{}^{\lambda} e^b{}_{\mu} e^c{}_{\nu} \quad (; \to \Gamma)$ not Lorentz invariant
  Everything will be expressed in terms of $T^{\lambda}{}_{\mu\nu}$, so we will have proper behaviour under diffeos, but not under Lorentz rotations
  ($T^a{}_{bc} = e_a{}^{\mu} e^b{}_{\nu} e^c{}_{\lambda} T^{\mu}{}_{\nu\lambda}$ not very useful - not Lorentz tensor)
  $\mathcal{K}^{\lambda}{}_{\nu\mu} = e_a{}^{\lambda} e^a{}_{\nu;\mu}$
  $R^{\lambda}{}_{\mu\nu\kappa} = R^a{}_{bcd} = 0$ (while still $\bar{R}^{\lambda}{}_{\mu\nu\kappa} \neq 0$)
  $e_a{}^{\mu}{}_{|\nu} = 0 \quad (| \to \omega)$, e_a autoparallel wrt $\omega^{\lambda}{}_{\mu\nu}$

Weitzenböck ...

- $\Gamma^\lambda{}_{\mu\nu} = \omega^\lambda{}_{\mu\nu} - \mathcal{K}^\lambda{}_{\mu\nu}$
  $\bar{R}^\mu{}_{\nu\kappa\lambda} = \cancel{R^\mu{}_{\nu\kappa\lambda}} - \mathcal{K}^\mu{}_{\nu\lambda;\kappa} + \mathcal{K}^\mu{}_{\nu\kappa;\lambda} - \mathcal{K}^\mu{}_{\rho\kappa}\mathcal{K}^\rho{}_{\nu\lambda} + \mathcal{K}^\mu{}_{\rho\lambda}\mathcal{K}^\rho{}_{\nu\kappa}$

  $$\begin{aligned} \bar{R}^\mu{}_\nu &= \cancel{R^\mu{}_\nu} - \mathcal{K}^{\mu\lambda}{}_{\lambda;\nu} + \mathcal{K}^{\mu\lambda}{}_{\nu;\lambda} - \mathcal{K}^{\mu\lambda\kappa}\mathcal{K}_{\kappa\lambda\nu} + \mathcal{K}^{\lambda\mu}{}_\nu\mathcal{K}_{\lambda\kappa}{}^\kappa \\ &= \cancel{R^\mu{}_\nu} + 2S_\nu{}^{\mu\lambda}{}_{;\lambda} + \delta^\mu_\nu T_\kappa{}^{\kappa\lambda}{}_{;\lambda} + 2S^{\kappa\lambda\mu}\mathcal{K}_{\kappa\lambda\nu} \end{aligned}$$

  $S^{\mu\nu\lambda} = \frac{1}{2}\mathcal{K}^{\nu\lambda\mu} + \frac{1}{2}(g^{\mu\lambda}T_\kappa{}^{\kappa\nu} - g^{\mu\nu}T_\kappa{}^{\kappa\lambda}) = -S^{\mu\lambda\nu}$
  $\bar{R} = \cancel{R} - T + 2T_\nu{}^{\nu\mu}{}_{;\mu}$
  $T = S^{\mu\nu\lambda}T_{\mu\nu\lambda} = \frac{1}{4}T^{\mu\nu\lambda}T_{\mu\nu\lambda} + \frac{1}{2}T^{\mu\nu\lambda}T_{\lambda\nu\mu} - T_\nu{}^{\nu\mu}T^\lambda{}_{\lambda\mu}$
  "*torsion scalar*"
  T scalar under diffeos, not Lorentz scalar
  Actually, under Lorentz rotation, $\tilde{T} = T + \partial(\ )$:
  T "quasi-invariant". Thus the eqm of T, $\tilde{T}$ are Einstein's which are indeed Lorentz invariant since they only contain $g_{\mu\nu}$

Weitzenböck ...

- Lagrangian : $e\bar{R} = e\cancel{R} - eT + 2(eT_\nu{}^{\nu\mu})_{,\mu}$
  Equivalent first order Lagrangian (up to boundary issues) : eT
  "Teleparallel equivalent Lagrangian of Einstein gravity"
  $L_{tel} = -eT$
  A splitting into diffeo invariant terms, but not Lorentz
- $L_{Einstein} = \sqrt{|g|}\bar{R} - \partial_\lambda(\sqrt{|g|}g^{\nu\rho}\Gamma^\mu{}_{\nu\kappa}\delta^{\lambda\kappa}_{\mu\rho})$
  $L_{Moller} = e\bar{R} - \partial_\lambda(ee_a{}^\mu e_b{}^\nu \Gamma^{ab}{}_\kappa \delta^{\lambda\kappa}_{\mu\nu})$
  the subtraction of the second derivatives terms is not covariant (energy-momentum pseudotensors are defined through Noether)

Weitzenböck ...

- $\bar{G}^{\mu}_{\nu} = \cancel{G^{\mu}_{\nu}} + 2S_{\nu}{}^{\mu\lambda}{}_{;\lambda} + 2S^{\kappa\lambda\mu}\mathcal{K}_{\kappa\lambda\nu} + \frac{1}{2}\delta^{\mu}_{\nu}S^{\kappa\lambda\rho}T_{\kappa\lambda\rho}$
  $\bar{G}^{\mu}_{\nu} = 0$ (in vacuum):
  $2S_{\nu}{}^{\mu\lambda}{}_{;\lambda} + 2S^{\kappa\lambda\mu}\mathcal{K}_{\kappa\lambda\nu} + \frac{1}{2}\delta^{\mu}_{\nu}S^{\kappa\lambda\rho}T_{\kappa\lambda\rho} = 0$ tensorial equation under diffeos; also Lorentz tensor, while the separate terms are not
  $2S_{\nu}{}^{\mu\lambda}{}_{;\lambda\mu} = S_{\nu}{}^{\mu\lambda}{}_{;\lambda\mu} - S_{\nu}{}^{\mu\lambda}{}_{;\mu\lambda} = \bar{R}... \neq 0$
- $\bar{G}^{\mu}_{\nu} = \cancel{G^{\mu}_{\nu}} + \frac{2}{e}(eS_{\nu}{}^{\mu\lambda})_{,\lambda} - 2\tau_{\nu}{}^{\mu}$
  $\tau_{\mu}{}^{\nu} = S^{\rho\lambda\nu}T_{\rho\lambda\mu} - \frac{1}{4}S^{\sigma\lambda\rho}T_{\sigma\lambda\rho}\delta^{\nu}_{\mu} + S^{\rho\nu\lambda}\omega_{\rho\mu\lambda}$ pseudotensor
  $\bar{G}^{\mu}_{\nu} = 0$ (in vacuum): $(eS_{\nu}{}^{\mu\lambda})_{,\lambda} - e\tau_{\nu}{}^{\mu} = 0$
  $(e\tau_{\nu}{}^{\mu})_{,\mu} = 0$
- $(ee_a{}^{\rho}S_{\rho}{}^{\nu\lambda})_{,\lambda} = ee_a{}^{\mu}(S^{\rho\lambda\nu}T_{\rho\lambda\mu} - \frac{1}{4}S^{\sigma\lambda\rho}T_{\sigma\lambda\rho}\delta^{\nu}_{\mu})$ tensor in ν

Modified gravities

- $e^{-1}L = f(T)$
  Eqm : ...$f(T)$... (tensor under diffeos, not Lorentz tensor)
  2nd order eqm, contrary to $f(R)$ theories
- $e^{-1}L = c_1 T^{\mu\nu\lambda} T_{\mu\nu\lambda} + c_2 T^{\mu\nu\lambda} T_{\lambda\nu\mu} + c_3 T_\nu{}^{\nu\mu} T^\lambda{}_{\lambda\mu}$
  2nd order eqm : tensor under diffeos, not Lorentz tensor
- Other constructions
- Under Lorentz rotations the equations for e are form-invariant, but not Lorentz covariant

Possible Deficits of the single field "e" formulation

- Under Lorentz transformations, the eqm are not transformed covariantly, so, e.g. probably you cannot exploit the Lorentz freedom to simplify the equations or to find the true degrees of freedom
- If you perform an energy calculation in the preferred frame, you cannot perform the calculation in another frame (because the zero Weitzenböck connection should transform to non zero value, but you do not have a covariant energy formula containing the connection)

Covariant Teleparallelism

- Diffeo+Lorentz covariant quantities, e.g. $T^a{}_{bc}(e,\omega)$:
  $R^a{}_{bcd}(\omega) = 0$
- $T^\lambda{}_{\mu\nu} = T^a{}_{bc} e_a{}^\lambda e^b{}_\mu e^c{}_\nu$
- $S_{Tel}(e,\omega,\lambda) = \int eT + \int \lambda^{abcd} R_{abcd}$
  δ_e : $\partial(eS) + \omega S + T^2 = 0$ (Einstein or modified)
  δ_λ : $R_{abcd} = \partial\omega - \partial\omega + \omega^2 - \omega^2 - C\omega = 0$
  δ_ω : $\partial\lambda + ... = 0$
  • one solution $\omega^a{}_{bc} = 0$
  • still there is a machinery to change frames (by transforming the zero connection to non-zero values) in a Lorentz covariant way, e.g. black hole energy

- Dynamical variables: $e_a = e_a{}^\mu \partial_\mu$, $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu = \omega^a{}_{bc} e^c$
  ω independent field, not necessarily expressed in terms of e
  Commutation relations $[e_a, e_b] = C^c{}_{ab} e_c \Leftrightarrow de^a = -\frac{1}{2} C^a{}_{bc} e^b \wedge e^c$
  $C^c{}_{ab} = e_a{}^\mu e_b{}^\nu (e^c{}_{\mu,\nu} - e^c{}_{\nu,\mu})$
- Torsion 2-form: $T^a = de^a + \omega^a{}_b \wedge e^b = \frac{1}{2} T^a{}_{bc} e^b \wedge e^c$
  $T^a{}_{bc} = \omega^a{}_{cb} - \omega^a{}_{bc} - C^a{}_{bc} = e^a{}_\mu e_b{}^\nu e_c{}^\lambda T^\mu{}_{\nu\lambda}$
  $T^a{}_{\mu\nu} = \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu + e^a{}_{\nu,\mu} - e^a{}_{\mu,\nu}$
  $T^\lambda{}_{\mu\nu} = \omega^\lambda{}_{\nu\mu} - \omega^\lambda{}_{\mu\nu} = e_a{}^\lambda T^a{}_{\mu\nu}$
  E.g. Weitzenböck $\omega^a{}_{bc} = 0$:
  $T^\lambda_{\mu\nu} = -e^a{}_\lambda e_b{}^\mu e^c{}_\nu C^a{}_{bc} = \omega^\lambda{}_{\nu\mu} - \omega^\lambda{}_{\mu\nu}$
- Curvature 2-form: $\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d$
  $R^a{}_{bcd} = \omega^a{}_{bd,c} - \omega^a{}_{bc,d} + \omega^e{}_{bd}\omega^a{}_{ec} - \omega^e{}_{bc}\omega^a{}_{ed} - C^e{}_{cd}\omega^a{}_{be}$
  $R^a{}_{b\mu\nu} = \omega^a{}_{b\nu,\mu} - \omega^a{}_{b\mu,\nu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu}$
  $R^\kappa{}_{\lambda\mu\nu} = e_a{}^\kappa e^b{}_\lambda R^a{}_{b\mu\nu}$
- metric g: $g(e_a, e_b) = g_{ab}$, $g_{\mu\nu} = g_{ab}\, e^a{}_\mu\, e^b{}_\nu$
  Any index behaves properly under coordinate, Lorentz transformations. In particular, $T^a{}_{bc}$ scalar under diffeos, Lorentz tensor

- Christoffel connection: $\Gamma^a{}_b$
  $\Gamma_{abc} = \frac{1}{2}(g_{ab,c} + g_{ca,b} - g_{bc,a}) + \frac{1}{2}(C_{cab} - C_{bca} - C_{abc})$
  $\bar{\mathcal{R}}^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b = \frac{1}{2}\bar{R}^a{}_{bcd} e^c \wedge e^d$
  $\bar{R}^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed} - C^e{}_{cd}\Gamma^a{}_{be}$
  $\bar{R}^a{}_{b\mu\nu} = \Gamma^a{}_{b\nu,\mu} - \Gamma^a{}_{b\mu,\nu} + \Gamma^a{}_{c\mu}\Gamma^c{}_{b\nu} - \Gamma^a{}_{c\nu}\Gamma^c{}_{b\mu}$
  $\bar{R}^\kappa{}_{\lambda\mu\nu} = e_a{}^\kappa e^b{}_\lambda \bar{R}^a{}_{b\mu\nu}$
- $\mathcal{K}_{ab} = -\mathcal{K}_{ba} = \omega_{ab} - \Gamma_{ab} = \mathcal{K}_{abc} e^c \Leftrightarrow \Gamma_{abc} = \omega_{abc} - \mathcal{K}_{abc}$
  $\mathcal{K}_{abc} = \frac{1}{2}(T_{cab} - T_{bca} - T_{abc}) = -\mathcal{K}_{bac}$ contorsion
  $T^a = \mathcal{K}^a{}_b \wedge e^b \Leftrightarrow T_{abc} = \mathcal{K}_{acb} - \mathcal{K}_{abc}$
- metric g: $g(e_a, e_b) = \eta_{ab} = \mathrm{diag}(-1, 1, \ldots 1)$, $g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu$
  η : simplifies calculations $+$ it is more natural
- zero non-metricity: $\eta_{ab|c} = 0 \Leftrightarrow \omega_{abc} = -\omega_{bac} \Leftrightarrow \omega_{ab} = -\omega_{ba}$
- Teleparallel condition: $R^a{}_{bcd} = 0$ set as a constraint in the action, $\lambda^{abcd} R_{abcd}$

- $\bar{R} = \cancel{R} - T + 2T_b{}^{ba}{}_{;a}$
  $T = S^{abc}T_{abc} = \frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{cba} - T_a{}^{ab}T^c{}_{cb}$
  $S^{abc} = \frac{1}{2}\mathcal{K}^{bca} + \frac{1}{2}(\eta^{ac}T_d{}^{db} - \eta^{ab}T_d{}^{dc}) = -S^{acb}$
  T scalar under diffeos and Lorentz scalar (splitting: the same)
- $\bar{G}_a{}^\mu = \cancel{G_a{}^\mu} + \frac{2}{e}(eS_a{}^{\mu\lambda})_{:\lambda} - 2t_a{}^\mu = \cancel{G_a{}^\mu} + \frac{2}{e}(eS_a{}^{\mu\lambda})_{,\lambda} - 2j_a{}^\mu$
  $t_\mu{}^a = (S^{cba}T_{cbd} - \frac{1}{4}S^{ebc}T_{ebc}\delta^a_d)e^d{}_\mu$
  $t_\mu{}^\nu = S^{\rho\lambda\nu}T_{\rho\lambda\mu} - \frac{1}{4}S^{\sigma\lambda\rho}T_{\sigma\lambda\rho}\delta^\nu_\mu$ GR energy-momentum tensor
  $j_a{}^\mu = t_a{}^\mu + S^{bdc}\omega_{bac}e_d{}^\mu$
  $(e\Phi^a_b)_{:\mu} \equiv (e\Phi^a_b)_{,\mu} + e\omega^a{}_{c\mu}\Phi^c_b - e\omega^c{}_{b\mu}\Phi^a_c$
  $(\tilde{e}\tilde{\Phi}^a_b)_{:\mu} = \Lambda^a{}_c(\Lambda^{-1})^d{}_b(e\Phi^c_d)_{:\mu}$ Fock-Ivanenko covariant derivative
  $()_{:\mu\nu} - ()_{:\nu\mu} = R... = 0$
- $\bar{G}^\mu_\nu = 0$ (in vacuum): $(et_a{}^\mu)_{:\mu} = 0$, $(ej_a{}^\mu)_{,\mu} = 0$
- Eqm of $f(T)$,... also Lorentz tensors
- ω_{ab} : easier calculations, Lorentz invariance, energy issues

Gauge theory (of translations)

- $\cancel{x}^a$ (tangent Minkowski) , $\cancel{e}^a{}_\mu = \cancel{x}^a{}_{,\mu}$, $x^a = \Lambda^a{}_b \cancel{x}^b$
  $\cancel{\omega}^a{}_{b\mu} = 0$, $e^a = \Lambda^a{}_b(\cancel{x}) d\cancel{x}^b \Leftrightarrow e^a{}_\mu = \Lambda^a{}_b \cancel{x}^b{}_{,\mu}$:
  $e^a{}_\mu = \mathfrak{D}_\mu x^a = x^a{}_{,\mu} + \omega^a{}_{b\mu} x^b$, $T^a{}_{\mu\nu} = 0$, $R^a{}_{b\mu\nu} = 0$
  $\omega^\lambda{}_{\nu\mu} = e_a{}^\lambda \mathfrak{D}_\mu e^a{}_\nu$, $\omega^a{}_{b\mu} = \Lambda^a{}_c \Lambda_b{}^c{}_{,\mu} = \Gamma^a{}_{b\mu}$
- $\mathtt{E}^a{}_\mu = e^a{}_\mu + A^a{}_\mu$
- $\delta_\varepsilon x^a = \varepsilon^b P_b x^a = \varepsilon^a$, $\delta_\varepsilon A^a{}_\mu = -\mathfrak{D}_\mu \varepsilon^a$: $\delta_\varepsilon \mathtt{E}^a{}_\mu = 0$
  (spacetime metric gauge independent)
- $$\begin{aligned} \mathtt{T}^a{}_{b\mu} &= \mathtt{E}^a{}_{\nu,\mu} - \mathtt{E}^a{}_{\mu,\nu} + \omega^a{}_{b\mu} \mathtt{E}^b{}_\nu - \omega^a{}_{b\nu} \mathtt{E}^b{}_\mu \\ &= \mathfrak{D}_\mu \mathtt{E}^a{}_\nu - \mathfrak{D}_\nu \mathtt{E}^a{}_\mu \\ &= A^a{}_{\nu,\mu} - A^a{}_{\mu,\nu} + \omega^a{}_{b\mu} A^b{}_\nu - \omega^a{}_{b\nu} A^b{}_\mu \\ &= \mathfrak{D}_\mu A^a{}_\nu - \mathfrak{D}_\nu A^a{}_\mu = F^a{}_{\mu\nu} \end{aligned}$$
  (torsion is field strength of gauge field)
- Lagrangian $\mathtt{T}^a \wedge * \mathtt{T}_a$

Teleparallel equivalent of Einstein gravity

- $S_{EH} = \frac{1}{2\kappa_D^2} \int_M \bar{\mathcal{L}}_1\,,$
  $\bar{\mathcal{L}}_1 = \frac{1}{(D-2)!} \epsilon_{a_1 \dots a_D} \bar{\mathcal{R}}^{a_1 a_2} \wedge e^{a_3} \wedge \dots \wedge e^{a_D} = \bar{R} * 1$

$$
\begin{aligned}
(D-2)!\,\mathcal{L}_1 = (D-2)!\,\bar{\mathcal{L}}_1 &+ d(\epsilon_{a_1 \dots a_D} \mathcal{K}^{a_1 a_2} \wedge e^{a_3} \wedge \dots \wedge e^{a_D}) \\
&+ \epsilon_{a_1 \dots a_D} \mathcal{K}^{a_1 a_2} \wedge d(e^{a_3} \wedge \dots \wedge e^{a_D}) \\
&+ \epsilon_{a_1 \dots a_D} (\Gamma^{a_1}{}_c \wedge \mathcal{K}^{c a_2} + \mathcal{K}^{a_1}{}_c \wedge \Gamma^{c a_2} \\
&+ \mathcal{K}^{a_1}{}_c \wedge \mathcal{K}^{c a_2}) \wedge e^{a_3} \wedge \dots \wedge e^{a_D}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_1 = \bar{\mathcal{L}}_1 &+ \frac{1}{(D-2)!} \epsilon_{a_1 \dots a_D} \mathcal{K}^{a_1}{}_c \wedge \mathcal{K}^{c a_2} \wedge e^{a_3} \wedge \dots \wedge e^{a_D} \\
&+ \frac{1}{(D-2)!} d(\epsilon_{a_1 \dots a_D} \mathcal{K}^{a_1 a_2} \wedge e^{a_3} \wedge \dots \wedge e^{a_D})
\end{aligned}
$$

- Teleparallel condition $\mathcal{R}^{ab} = 0$
  $\bar{\mathcal{L}}_1 = -\mathcal{T} - \frac{1}{(D-2)!} d(\epsilon_{a_1 \dots a_D} \mathcal{K}^{a_1 a_2} \wedge e^{a_3} \wedge \dots \wedge e^{a_D})$

$$\mathcal{T} = \frac{1}{(D-2)!} \epsilon_{a_1 \dots a_D} \mathcal{K}^{a_1}{}_c \wedge \mathcal{K}^{c a_2} \wedge e^{a_3} \wedge \dots \wedge e^{a_D}$$
$$= T \; e^1 \wedge \dots \wedge e^D$$

$$T = \mathcal{K}^{abc} \mathcal{K}_{cba} - \mathcal{K}^{ca}{}_a \mathcal{K}_{cb}{}^b$$

- $S^{(1)}_{Tel} = -\frac{1}{2\kappa_D^2} \int_M \mathcal{T} = -\frac{1}{2\kappa_D^2} \int_M d^D x \, e \, T$
- $D\Phi^a_b = d\Phi^a_b + \omega^a{}_c \wedge \Phi^c_b - (-1)^p \Phi^a_c \wedge \omega^c{}_b$
  $\mathcal{R}^{ab} = \bar{\mathcal{R}}^{ab} + \bar{D}\mathcal{K}^{ab} + \mathcal{K}^a{}_c \wedge \mathcal{K}^{cb}$
  $T^a = De^a, \; DT^a = \mathcal{R}^a{}_b \wedge e^b$
  $D\mathcal{R}^a{}_b = 0$
  $D^2 \Phi^a_b = \mathcal{R}^a{}_c \wedge \Phi^c_b - \Phi^a_c \wedge \mathcal{R}^c{}_b$
  $\bar{D} e^a = 0$

Teleparallel equivalent of Gauss-Bonnet gravity

- $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$
  $S_{GB} = \frac{1}{2\kappa_D^2}\int_M \bar{\mathcal{L}}_2$
  $\bar{\mathcal{L}}_2 = \frac{1}{(D-4)!}\epsilon_{a_1\ldots a_D}\bar{\mathcal{R}}^{a_1a_2}\wedge\bar{\mathcal{R}}^{a_3a_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D} = \bar{G}*1$
- $\bar{\mathcal{L}}_2 = \mathcal{T}_G - \frac{1}{(D-4)!}dB$

$$\begin{aligned}\mathcal{T}_G = \frac{1}{(D-4)!}\epsilon_{a_1\ldots a_D}\Big(&\mathcal{K}^{a_1}{}_c\wedge\mathcal{K}^{ca_2}\wedge\mathcal{K}^{a_3}{}_d\wedge\mathcal{K}^{da_4}\\ &-2\mathcal{K}^{a_1a_2}\wedge\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^c{}_d\wedge\mathcal{K}^{da_4}\\ &+2\mathcal{K}^{a_1a_2}\wedge D\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^{ca_4}\Big)\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}\\ = T_G\ e^1\wedge\ldots\wedge e^D&\end{aligned}$$

$$\begin{aligned}T_G = \Big(&\mathcal{K}^{a_1}{}_{ea}\mathcal{K}^{ea_2}{}_b\mathcal{K}^{a_3}{}_{fc}\mathcal{K}^{fa_4}{}_d - 2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^e{}_{fc}\mathcal{K}^{fa_4}{}_d\\ &+2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^{ea_4}{}_f\mathcal{K}^f{}_{cd}\\ &+2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^{ea_4}{}_{c|d}\Big)\delta^{a\,b\,c\,d}_{a_1a_2a_3a_4}\end{aligned}$$

$$e(\bar{R}^2 - 4\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \bar{R}_{\mu\nu\kappa\lambda}\bar{R}^{\mu\nu\kappa\lambda}) = eT_G + \text{total diverg.}$$

- $\bar{\mathcal{L}}_2^{(D=4)}$ topological invariant $\Rightarrow \mathcal{T}_G^{(D=4)}$ topological invariant
  $\mathcal{T}_G^{(D=4)} = d\big(32\pi^2\,\Pi_2 + B\big)$
  $\Pi_2 = -\frac{1}{8\pi^2}\epsilon_{abcd}\,n^a(\varepsilon\bar{\mathcal{R}}^{bc} \wedge \bar{D}n^d + \frac{2}{3}\bar{D}n^b \wedge \bar{D}n^c \wedge \bar{D}n^d)$ second Chern form, $n^a n_a = \varepsilon = \pm 1$, $\bar{\mathcal{L}}_2^{(D=4)} = 32\pi^2 d\Pi_2$
- $S_{Tel}^{(2)}[e^a, \omega^a{}_b] = \frac{1}{2\kappa_D^2}\int_M \mathcal{T}_G = \frac{1}{2\kappa_D^2}\int_M d^Dx\, e\, T_G$
  $S_{Tel}^{(2)}[e^a{}_\mu, \omega^a{}_{b\mu}]$ diffeomorphism and Lorentz invariant
- Weitzenböck connection $\omega^a{}_{bc} = 0$:

$$\begin{aligned} S_{tel}^{(2)} = \quad & \frac{1}{2(D-4)!\,\kappa_D^2}\int_M \epsilon_{a_1 \dots a_D}\big(\mathcal{K}^{a_1}{}_c \wedge \mathcal{K}^{ca_2} \wedge \mathcal{K}^{a_3}{}_d \wedge \mathcal{K}^{da_4} \\ & -2\mathcal{K}^{a_1 a_2} \wedge \mathcal{K}^{a_3}{}_c \wedge \mathcal{K}^c{}_d \wedge \mathcal{K}^{da_4} \\ & 2\mathcal{K}^{a_1 a_2} \wedge \mathcal{K}^{a_3}{}_c \wedge d\mathcal{K}^{ca_4}\big) \wedge e^{a_5} \wedge \dots \wedge e^{a_D} \end{aligned}$$

$S_{tel}^{(2)}$ diffeomorphism invariant

$F(T, T_G)$ gravity

- $S = \frac{1}{2\kappa_D^2} \int d^D x \, e \, F(T, T_G)$
  different than $F(T), F(R, G)$ gravities
  EGB : $F(T, T_G) = -T + \alpha T_G$
  $2\kappa_D^2 \delta_e S = \int d^D x \left(e F_T \delta_e T + e F_{T_G} \delta_e T_G + F \delta e \right)$
- $\delta_e S = 0$:

$$
\begin{aligned}
& 2 L_{e_b} H^{[ab]} - 2 i_{e_b} L_{e_c} (e^c H^{[ab]} + e^a H^{[cb]}) - C^d{}_{cb} \, i_{e_d} (e^a H^{cb}) \\
& \quad + 4 C_{(dc)}{}^a \, i_{e_b} (e^c H^{[db]}) + (T^a{}_{bc} + 2\omega^a{}_{[bc]}) H^{bc} - (-1)^D h^a \\
& \qquad + (F - T F_T - T_G F_{T_G}) \vartheta^a = 0
\end{aligned}
$$

$$(i_v \varphi)(v_1, ..., v_{p-1}) = \varphi(v, v_1, ..., v_{p-1}), \ \vartheta_a = i_{e_a}(e^1 \wedge ... \wedge e^D)$$

- $$H^{ab} = \frac{F_T}{(D-2)!}\epsilon^{a}{}_{a_1\dots a_{D-1}}\mathcal{K}^{ba_1}e^{a_2}\dots e^{a_{D-1}}$$
$$+\frac{F_{T_G}}{(D-4)!}\Big(2\epsilon^{a}{}_{a_1\dots a_{D-1}}\mathcal{K}^{ba_1}\mathcal{K}^{a_2}{}_c\mathcal{K}^{ca_3}e^{a_4}\dots e^{a_{D-1}}$$
$$+\epsilon_{a_1\dots a_D}\mathcal{K}^{aa_1}\mathcal{K}^{ba_2}\mathcal{K}^{a_3a_4}e^{a_5}\dots e^{a_D}$$
$$-\epsilon^{ab}{}_{a_1\dots a_{D-2}}\mathcal{K}^{a_1}{}_c\mathcal{K}^{c}{}_d\mathcal{K}^{da_2}e^{a_3}\dots e^{a_{D-2}}$$
$$+\epsilon^{ab}{}_{a_1\dots a_{D-2}}D\mathcal{K}^{a_1}{}_c\mathcal{K}^{ca_2}e^{a_3}\dots e^{a_{D-2}}$$
$$+\epsilon^{a}{}_{a_1\dots a_{D-1}}D\mathcal{K}^{ba_1}\mathcal{K}^{a_2a_3}e^{a_4}\dots e^{a_{D-1}}\Big)$$
$$-\frac{1}{(D-4)!}\epsilon^{a}{}_{a_1\dots a_{D-1}}D\big(F_{T_G}\mathcal{K}^{ba_1}\mathcal{K}^{a_2a_3}e^{a_4}\dots e^{a_{D-1}}\big)$$

- $$h_a = \frac{F_T}{(D-3)!}\epsilon_{a_1\dots a_{D-1}a}\mathcal{K}^{a_1}{}_c\mathcal{K}^{ca_2}e^{a_3}\dots e^{a_{D-1}}$$
$$+\frac{F_{T_G}}{(D-5)!}\epsilon_{a_1\dots a_{D-1}a}\Big(\mathcal{K}^{a_1}{}_c\mathcal{K}^{ca_2}\mathcal{K}^{a_3}{}_d\mathcal{K}^{da_4} - 2\mathcal{K}^{a_1a_2}\mathcal{K}^{a_3}{}_c\mathcal{K}^{c}{}_d\mathcal{K}^{da_4}$$
$$+2\mathcal{K}^{a_1a_2}D\mathcal{K}^{a_3}{}_c\mathcal{K}^{ca_4}\Big)e^{a_5}\dots e^{a_{D-1}}$$

$D = 4$, Weitzenböck : $H^{ab} = H^{abc}\vartheta_c\,, h^a = h^{ab}\vartheta_b$

- $$H^{abc} = F_T(\eta^{ac}\mathcal{K}^{bd}{}_d - \mathcal{K}^{bca}) + F_{T_G}\Big[$$
$$\epsilon^{cprt}\big(2\epsilon^a{}_{dkf}\mathcal{K}^{bk}{}_p\mathcal{K}^d{}_{qr} + \epsilon_{qdkf}\mathcal{K}^{ak}{}_p\mathcal{K}^{bd}{}_r + \epsilon^{ab}{}_{kf}\mathcal{K}^k{}_{dp}\mathcal{K}^d{}_{qr}\big)\mathcal{K}^{qf}{}_t$$
$$+ \epsilon^{cprt}\epsilon^{ab}{}_{kd}\mathcal{K}^{fd}{}_p\big(\mathcal{K}^k{}_{fr,t} - \frac{1}{2}\mathcal{K}^k{}_{fq}C^q{}_{tr}\big)$$
$$+ \epsilon^{cprt}\epsilon^{ak}{}_{df}\mathcal{K}^{df}{}_p\big(\mathcal{K}^b{}_{kr,t} - \frac{1}{2}\mathcal{K}^b{}_{kq}C^q{}_{tr}\big)\Big]$$
$$+ \epsilon^{cprt}\epsilon^a{}_{kdf}\Big[\big(F_{T_G}\mathcal{K}^{bk}{}_p\mathcal{K}^{df}{}_r\big)_{,t} + F_{T_G}C^q{}_{pt}\mathcal{K}^{bk}{}_{[q}\mathcal{K}^{df}{}_{r]}\Big]$$

- $$h^{ab} = F_T\epsilon^a{}_{kcd}\epsilon^{bpqd}\mathcal{K}^k{}_{fp}\mathcal{K}^{fc}{}_q$$

$$2(H^{[ac]b} + H^{[ba]c} - H^{[cb]a})_{,c} + 2(H^{[ac]b} + H^{[ba]c} - H^{[cb]a})C^d{}_{dc}$$
$$+ (2H^{[ac]d} + H^{dca})C^b{}_{cd} + 4H^{[db]c}C_{(dc)}{}^a + T^a{}_{cd}H^{cdb} - h^{ab}$$
$$+ (F - TF_T - T_G F_{T_G})\eta^{ab} = 0$$

$F(T, T_G)$ cosmology

- $S_{tot} = \frac{1}{2\kappa^2}\int d^4x\, e\, F(T, T_G) + S_m$
- $ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{\hat{i}\hat{j}}dx^{\hat{i}}dx^{\hat{j}}$
  $e^a{}_\mu = \mathrm{diag}(N(t), a(t), a(t), a(t))$
- $F - 12H^2 F_T - T_G F_{T_G} + 24H^3 \dot{F}_{T_G} = 2\kappa^2\rho$

$$F - 4(\dot{H} + 3H^2)F_T - 4H\dot{F}_T - T_G F_{T_G} + \frac{2}{3H} T_G \dot{F}_{T_G} + 8H^2 \ddot{F}_{T_G} = -2\kappa^2 p$$

  Same equations with variation of the minisuperspace Lagrangian w.r.t. a, N
- For $F(T, T_G) = F(T)$:
  $F - 12H^2 F_T = 2\kappa^2\rho$
  $F - 4(\dot{H} + 3H^2)F_T - 4H\dot{F}_T = -2\kappa^2 p$
- $F(T, T_G) = -T + f(T, T_G)$:
  $6H^2 + f - 12H^2 f_T - T_G f_{T_G} + 24H^3 \dot{f}_{T_G} = 2\kappa^2\rho$

$$2(2\dot{H} + 3H^2) + f - 4(\dot{H} + 3H^2)f_T - 4H\dot{f}_T - T_G f_{T_G} + \frac{2}{3H} T_G \dot{f}_{T_G} + 8H^2 \ddot{f}_{T_G} = -2\kappa^2 p$$

- $H^2 = \frac{\kappa^2}{3}(\rho + \rho_{DE})$
  $\dot{H} = -\frac{\kappa^2}{2}(\rho + p + \rho_{DE} + p_{DE})$
-

$$\rho_{DE} = -\frac{1}{2\kappa^2}(f - 12H^2 f_T - T_G f_{T_G} + 24H^3 \dot{f_{T_G}})$$
$$p_{DE} = \frac{1}{2\kappa^2}\Big[f - 4(\dot{H} + 3H^2)f_T - 4H\dot{f_T} - T_G f_{T_G} + \frac{2}{3H}T_G \dot{f_{T_G}} + 8H^2 \ddot{f_{T_G}}\Big]$$

$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0$
$w_{DE} = \frac{p_{DE}}{\rho_{DE}}$

Specific cases

- $T_G \sim T^2$, $T \sim \sqrt{T^2 + \beta_2 T_G}$
  $F(T, T_G) = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} + \alpha_1 T^2 + \alpha_2 T \sqrt{|T_G|}$
  - β_1, β_2 dimensionless (no new mass scale at late times)
  - F can describe in unified way both inflation and late-times acceleration

- Early-times inflationary (de-Sitter exponential) solutions for various parameter choices, without explicit cosmological constant term. Friedmann equations accept analytic solutions with $H \approx$ constant for T, $T_G \approx$ const.

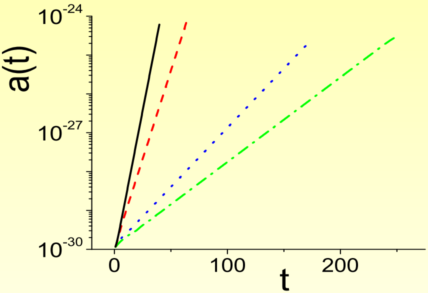


Figure: *Four inflationary solutions corresponding to a)* $\alpha_1 = -2.8$, $\alpha_2 = 8$, $\beta_1 = 0.001$, $\beta_2 = 1$ *(black-solid), b)* $\alpha_1 = -2$, $\alpha_2 = 8$, $\beta_1 = 0.001$, $\beta_2 = 1$ *(red-dashed), c)* $\alpha_1 = 8$, $\alpha_2 = 8$, $\beta_1 = 0.001$, $\beta_2 = 1$ *(blue-dotted), d)* $\alpha_1 = 20$, $\alpha_2 = 5$, $\beta_1 = 0.001$, $\beta_2 = 1$ *(green-dashed-dotted).*

- Late-times evolution with Ω_m decreasing with $\Omega_{m0} \approx 0.3$, and $\Omega_{DE} = 1 - \Omega_m$ increasing. w_{DE} in the quintessence regime.

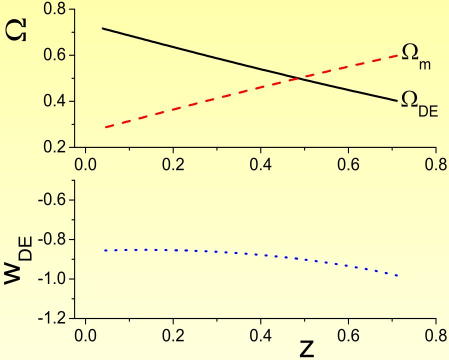


Figure: *Upper graph: The evolution of the dark energy density parameter Ω_{DE} (black-solid) and the matter density parameter Ω_m (red-dashed), as a function of the redshift z, with* $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $\beta_1 = 2.5$, $\beta_2 = 1.5$. *Lower graph: The evolution of the corresponding dark energy equation-of-state parameter w_{DE}. ($H_0 = 1$, and we have imposed $\Omega_{m0} \approx 0.3$, $\Omega_{DE0} \approx 0.7$ at present.)*

- Late-times evolution with Ω_m decreasing with $\Omega_{m0} \approx 0.3$, and $\Omega_{DE} = 1 - \Omega_m$ increasing. w_{DE} in the phantom regime or exhibits the phantom-divide crossing.

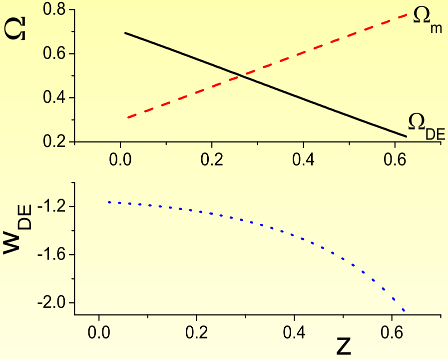


Figure: *Upper graph: The evolution of the dark energy density parameter Ω_{DE} (black-solid) and the matter density parameter Ω_m (red-dashed), as a function of the redshift z, with $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $\beta_1 = 2.6$, $\beta_2 = 2$. Lower graph: The evolution of the corresponding dark energy equation-of-state parameter w_{DE}.*

- Other more general forms
  $F(T, T_G) = -T + f(T^2 + \beta_2 T_G)$

$$6H^2 + f - (24H^2 T + \beta_2 T_G)f' + 24\beta_2 H^3 (2T\dot{T} + \beta_2 \dot{T}_G)f'' = 2\kappa^2 \rho$$

$$\begin{aligned} 2(2\dot{H} + 3H^2) + f - [8(\dot{H} + 3H^2)T + 8H\dot{T} + \beta_2 T_G]f' \\ + \left\{\left[\frac{2\beta_2 T_G}{3H} - 8HT\right](2T\dot{T} + \beta_2 \dot{T}_G) + 8\beta_2 H^2 (2T\dot{T} + \beta_2 \dot{T}_G)^{\cdot}\right\} f'' \\ + 8\beta_2 H^2 (2T\dot{T} + \beta_2 \dot{T}_G)^2 f''' = -2\kappa^2 p \end{aligned}$$

- $F(T, T_G) = -T + \beta_1(T^2 + \beta_2 T_G) + \beta_3(T^2 + \beta_4 T_G)^2$
  fourth-order torsion terms for early times

- Early-times inflationary solutions. More efficient inflation than before (more e-foldings in less time) due to the higher-order terms.

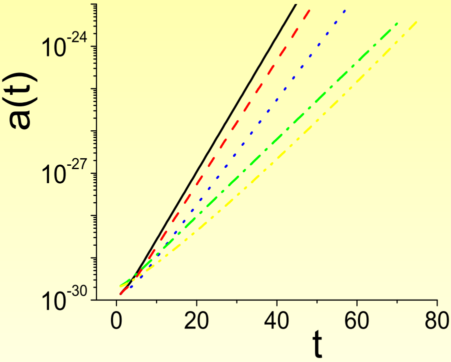


Figure: *Five inflationary solutions corresponding to a)* $\beta_1 = -0.01$, $\beta_2 = 1$, $\beta_3 = -1$, $\beta_4 = -2$ *(black-solid), b)* $\beta_1 = -0.1$, $\beta_2 = 1$, $\beta_3 = -2$, $\beta_4 = -2$ *(red-dashed), c)* $\beta_1 = -0.01$, $\beta_2 = 1$, $\beta_3 = -1$, $\beta_4 = -5$ *(blue-dotted), d)* $\beta_1 = -0.01$, $\beta_2 = 1$, $\beta_3 = -6$, $\beta_4 = -6$ *(green-dashed-dotted), e)* $\beta_1 = -0.01$, $\beta_2 = 1$, $\beta_3 = -10$, $\beta_4 = -10$ *(yellow-dashed-dotted-dotted).*

Dynamical systems analysis

- $\mathbf{X}' = \mathbf{f(X)}$, $\mathbf{X}$ column vector of auxiliary variables, $N = \ln a$
  $\mathbf{X}' = 0 \Leftrightarrow \mathbf{X}_c$ critical points
  $\mathbf{X} = \mathbf{X}_c + \mathbf{U}$, $\mathbf{U}' = \mathbf{Q} \cdot \mathbf{U}$ to first order
  eigenvalues of $\mathbf{Q}$ determine the type and stability of $\mathbf{X}_c$
- $F(T, T_G) = -T + \alpha_1 \sqrt{T^2 + \alpha_2 T_G}$ late-times modification

$$\kappa^2 \rho_{DE} = \frac{\sqrt{3}\alpha_1 H^2 \left\{\alpha_2^2 \ddot{H} + 9\alpha_2 H\dot{H} + \left[(3-2\alpha_2)\alpha_2 + 9\right] H^3\right\}}{D^{3/2}}$$

$$\frac{p_{DE}}{\kappa^{-2}} = \frac{\alpha_1 \left\{(2\alpha_2+3)\left[\alpha_2(10\alpha_2-51)-18\right]H^4 + \alpha_2\left[4\alpha_2(5\alpha_2-21)-90\right]H^2\dot{H} - 54\alpha_2^2\dot{H}^2\right\}H\dot{H}}{\sqrt{3}D^{5/2}}$$

$$-\frac{\alpha_1\alpha_2^2 H\dddot{H}}{\sqrt{3}D^{3/2}} + \frac{\sqrt{3}\alpha_1\alpha_2^3 H\ddot{H}^2}{D^{5/2}} - \frac{2\alpha_1\alpha_2^2\ddot{H}\left[2(\alpha_2-3)H^2\dot{H} + 2\alpha_2\dot{H}^2 + (6\alpha_2+9)H^4\right]}{\sqrt{3}D^{5/2}}$$

$$+\frac{\sqrt{3}\alpha_1(\alpha_2-3)(2\alpha_2+3)^2 H^7}{D^{5/2}}$$

$$D = 3H^2 + 2\alpha_2(\dot{H} + H^2)$$

- auxiliary variables

$$x = \sqrt{\frac{D}{3H^2}} = \sqrt{1 + \frac{2\alpha_2}{3}\left(1 + \frac{\dot{H}}{H^2}\right)}$$

$$\Omega_m = \frac{\kappa^2 \rho_m}{3H^2}$$

- autonomous system

$$x' = -\frac{x\left[3\alpha_1 x^2 - 6(1-\Omega_m)x + \alpha_1(3-4\alpha_2)\right]}{2\alpha_1\alpha_2}$$

$$\Omega_m' = -\frac{\Omega_m\left(3x^2 + \alpha_2 + 3\alpha_2 w_m - 3\right)}{\alpha_2}$$

phase space $\{(x, \Omega_m) | x \in [0, \infty), \Omega_m \in [0, \infty]\}$

- $q \equiv -1 - \frac{\dot{H}}{H^2} = \frac{3\left(1-x^2\right)}{2\alpha_2}$ deceleration parameter
  $\Omega_{DE} \equiv \frac{\kappa^2 \rho_{DE}}{3H^2} = 1 - \Omega_m$ dark energy density parameter
  $2q = 1 + 3(w_m\Omega_m + w_{DE}\Omega_{DE}) \Rightarrow w_{DE} = \frac{3x^2+\alpha_2+3\alpha_2 w_m\Omega_m-3}{3\alpha_2(\Omega_m-1)}$
  dark energy equation-of-state parameter
  dust matter, $w_m = 0$

Finite phase space analysis

Cr. P.	x	Ω_m	Existence	Stability
P_1	$\sqrt{1-\frac{\alpha_2}{3}}$	Ω_{m1}	$\frac{6}{5}<\alpha_2<3, \alpha_1\geq -2\sqrt{\frac{3(3-\alpha_2)}{(-6+5\alpha_2)^2}}$ or $\alpha_2=\frac{6}{5}$ or $\alpha_2<\frac{6}{5},\ \alpha_1\leq 2\sqrt{\frac{3(3-\alpha_2)}{(-6+5\alpha_2)^2}}$	Stable spiral for $\alpha_2<3$ and $-32\sqrt{3}\sqrt{\frac{(3-\alpha_2)^3}{\left(71\alpha_2^2-336\alpha_2+288\right)^2}}<\alpha_1<0$ or $\alpha_1<0, \alpha_2\leq\frac{1}{71}\left(168-36\sqrt{6}\right)\approx 1.124$. Saddle otherwise (hyperbolic cases).
P_2	x_2	0	$\alpha_2<\frac{3}{4},\ 0<\alpha_1\leq\sqrt{\frac{3}{3-4\alpha_2}}$ or $\alpha_1\neq 0,\ \alpha_2=\frac{3}{4}$ or $\alpha_2>\frac{3}{4},\ \alpha_1<0$	Stable node for $\alpha_2<0,\ 0<\alpha_1<2\sqrt{\frac{3(3-\alpha_2)}{(5\alpha_2-6)^2}}$ or $\frac{6}{5}<\alpha_2\leq 3,\ \alpha_1<-2\sqrt{\frac{3(3-\alpha_2)}{(5\alpha_2-6)^2}}$ or $\alpha_2>3,\ \alpha_1<0$. Unstable node for $0<\alpha_2<\frac{3}{4},\ 0<\alpha_1<\sqrt{\frac{3}{3-4\alpha_2}}$. Saddle otherwise (hyperbolic cases).
P_3	x_3	0	$\alpha_2<\frac{3}{4},\ 0<\alpha_1\leq\sqrt{\frac{3}{3-4\alpha_2}}$ or $\alpha_2\geq\frac{3}{4},\ \alpha_1>0$	Stable node for $\alpha_1>0, \alpha_2\geq\frac{6}{5}$. Unstable node for $\alpha_2<0,\ 0<\alpha_1<\frac{\sqrt{3}}{\sqrt{3-4\alpha_2}}$. Saddle otherwise (hyperbolic cases).
P_4	0	0	Always	Unstable node for $\frac{3}{4}<\alpha_2<3$. Saddle otherwise (hyperbolic cases).

Table: 1 . The critical points of the autonomous system. Existence and stability conditions.

physical characteristics of critical points

Cr. P.	Ω_{DE}	q	w_{DE}	Properties of solutions
P_1	$1-\Omega_{m1}$	$\frac{1}{2}$	0	Dark Energy - Dark Matter scaling solution
P_2	1	q_2	w_{DE2}	Decelerating solution for $\alpha_2<0,\ \frac{3}{3-2\alpha_2}<\alpha_1\leq\sqrt{\frac{3}{3-4\alpha_2}}$ or $0<\alpha_2<\frac{3}{4},\ 0<\alpha_1\leq\sqrt{\frac{3}{3-4\alpha_2}}$ or $\alpha_1\neq 0,\ \alpha_2=\frac{3}{4}$ or $\frac{3}{4}<\alpha_2\leq\frac{3}{2},\ \alpha_1<0$ or $\alpha_2>\frac{3}{2},\ \frac{3}{3-2\alpha_2}<\alpha_1<0.$ Quintessence solution for $\alpha_2\leq-\frac{3}{2},\ 0<\alpha_1<\frac{3}{3-2\alpha_2}$ or $-\frac{3}{2}<\alpha_2<0,\ -\sqrt{\frac{3(2\alpha_2+3)}{(\alpha_2-3)^2}}<\alpha_1<\frac{3}{3-2\alpha_2}$ or $\frac{3}{2}<\alpha_2\leq 3,\ \alpha_1<\frac{3}{3-2\alpha_2}$ or $\alpha_2>3,\ -\sqrt{\frac{3(2\alpha_2+3)}{(\alpha_2-3)^2}}<\alpha_1<\frac{3}{3-2\alpha_2}.$ De Sitter solution for $-\frac{3}{2}<\alpha_2<0,\ \alpha_1=\sqrt{\frac{3(2\alpha_2+3)}{(\alpha_2-3)^2}}$ or $\alpha_2>3,\ \alpha_1=-\sqrt{\frac{3(2\alpha_2+3)}{(\alpha_2-3)^2}}.$ Phantom solution for $-\frac{3}{2}<\alpha_2<0,\ 0<\alpha_1<\sqrt{\frac{3(2\alpha_2+3)}{(\alpha_2-3)^2}}$ or $\alpha_2>3,\ \alpha_1<-\sqrt{\frac{3(2\alpha_2+3)}{(\alpha_2-3)^2}}.$

Table: 2. The critical points of the autonomous system and the corresponding values of Ω_{DE}, q and w_{DE}. In the last column we summarize their physical description.

Cr. P.	Ω_{DE}	q	w_{DE}	Properties of solutions
P_3	1	q_3	w_{DE3}	Decelerating solution for $\alpha_2 < 0,\ 0 < \alpha_1 \leq \sqrt{\frac{3}{3-4\alpha_2}}$ or $0 < \alpha_2 < \frac{3}{4},\ \frac{3}{3-2\alpha_2} < \alpha_1 \leq \sqrt{\frac{3}{3-4\alpha_2}}$ or $\frac{3}{4} \leq \alpha_2 < \frac{3}{2},\ \alpha_1 > \frac{3}{3-2\alpha_2}$. Quintessence solution for $0 < \alpha_2 < \frac{3}{2},\ \sqrt{\frac{3(2\alpha_2+3)}{(\alpha_2-3)^2}} < \alpha_1 < -\frac{3}{2\alpha_2-3}$ or $\frac{3}{2} \leq \alpha_2 < 3,\ \alpha_1 > \sqrt{\frac{3(2\alpha_2+3)}{(\alpha_2-3)^2}}$. De Sitter solution for $0 < \alpha_2 < 3,\ \alpha_1 = \sqrt{\frac{3(2\alpha_2+3)}{(\alpha_2-3)^2}}$. Phantom solution for $0 < \alpha_2 < 3,\ 0 < \alpha_1 < \sqrt{\frac{3(2\alpha_2+3)}{(\alpha_2-3)^2}}$ or $\alpha_2 \geq 3,\ \alpha_1 > 0$.
P_4	1	$\frac{3}{2\alpha_2}$	$\frac{1}{\alpha_2} - \frac{1}{3}$	Decelerating solution for $\alpha_2 > 0$. Quintessence DE dominated solution for $\alpha_2 < -\frac{3}{2}$. De Sitter solution for $\alpha_2 = -\frac{3}{2}$. Phantom solution for $-\frac{3}{2} < \alpha_2 < 0$.

Table: 2. The critical points of the autonomous system and the corresponding values of Ω_{DE}, q and w_{DE}. In the last column we summarize their physical description.

- $q < 0$ acceleration , $q >$ deceleration , $q = -1$ de Sitter solution
- $w_{DE} > -1$ quintessence-like , $w_{DE} < -1$ phantom-like
- $\Omega_{DE} = 1$ dark-energy dominated universe, $\Omega_{DE} < 1$ scaling

  - Point P_1 : Stable (attractor)
    $\Omega_{DE} \sim \Omega_m \Rightarrow$ DE/DM scaling solution (alleviates coincidence problem)
    Disadvantage: $w_{DE} = 0$, no acceleration (maybe the today universe has not yet reached the asymptotic regime)
  - Point P_2 : Stable
    Dark energy dominated universe, can be accelerating
    w_{DE} quintessence/cosmological constant/phantom regime
    Good candidate for the description of universe as its future attractor
  - Point P_3 : Similar to P_2
  - Point P_4 : Unstable, similar characteristics to P_2

Phase space analysis at infinity

Dynamical system non-compact $\Rightarrow$ Fixed points at infinity

- Poincare projection method: $x = \frac{r}{1-r}\cos\theta\ ,\ \Omega_m = \frac{r}{1-r}\sin\theta$

$\theta \in \left[0, \frac{\pi}{2}\right], r \in [0,1)$

- Critical points at infinity ($x \to +\infty$ or $\Omega_m \to +\infty$) $\rightsquigarrow r \to 1^-$
- $(r' = ..., \theta' = ...)$ for $r \to 1^-$, set $\theta' = 0 \Rightarrow \theta = ...$

$$q = \frac{3\left(1 - 2r + r^2\sin^2\theta\right)}{2\alpha_2(1-r)^2}$$

$$w_{DE} = \frac{\alpha_2(1-r)^2 - 3\left(1 - 2r + r^2\sin^2\theta\right)}{3\alpha_2(1-r)\left[r(\sin\theta + 1) - 1\right]}$$

$$\Omega_{DE} = \frac{1 - r(1+\sin\theta)}{1-r}$$

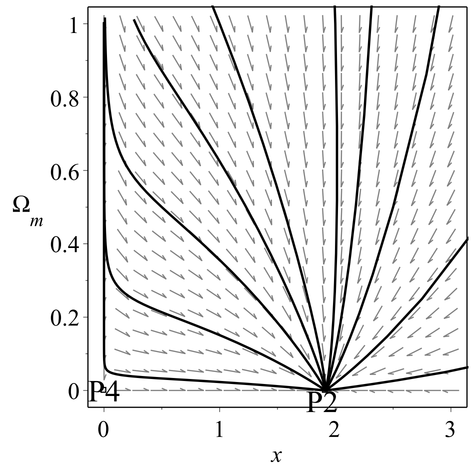
Cr. P.	θ	Stability	Ω_{DE}	q	w_{DE}
Q_1	0	saddle point	1	$-\mathrm{sgn}(\alpha_2)\infty$	$-\mathrm{sgn}(\alpha_2)\infty$
Q_2	$\arctan\left(\frac{\alpha_1}{2}\right)$	unstable $\alpha_2 > 0$ stable $\alpha_2 < 0$	$-\infty$	$-\mathrm{sgn}(\alpha_2)\infty$	$\mathrm{sgn}(\alpha_2)\infty$
Q_3	$\frac{\pi}{2}$	numerical elabor	$-\infty$	$\frac{3}{2\alpha_2}$	0

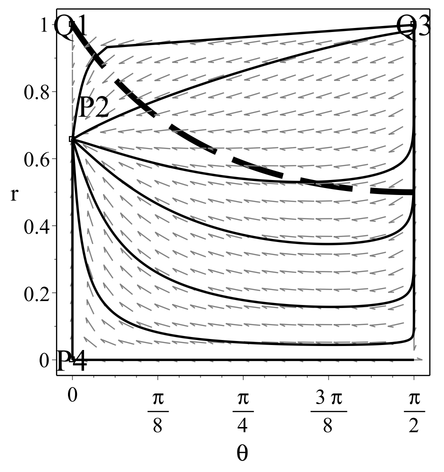
Table: 3. The critical points of the autonomous system at infinity, stability conditions, and the corresponding values of Ω_{DE}, q, and w_{DE}. All points correspond to a form of future, past, or intermediate singularity, depending on the parameters

- 3 critical points
- Q_2, Q_3 can be stable. Close to them $\Omega_m > 1$ (comparison with growth-index observations?)
- Cr. P. correspond to Big Rip, sudden, or other forms of singularities, depending on whether the singularity is reached at finite or infinite time, and on their observable features.

Examples

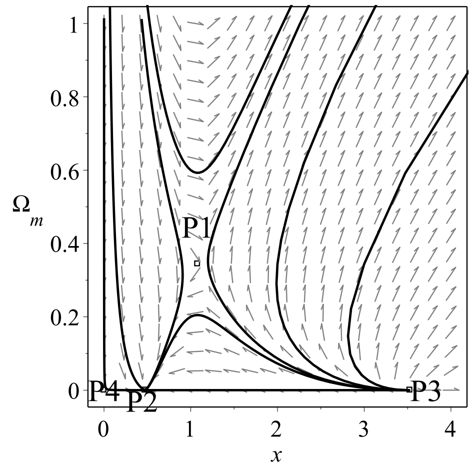
- $\alpha_1 = -\sqrt{33}$, $\alpha_2 = 4$

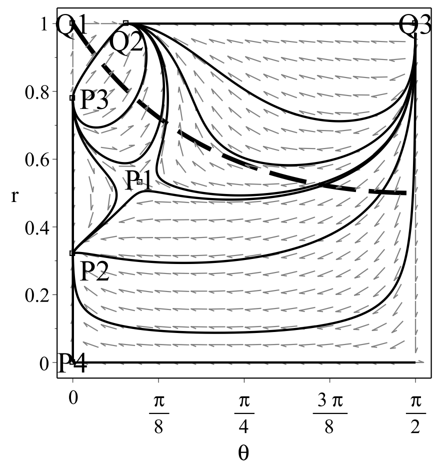




P_2 dark-energy dominated dS attractor, $w_{DE} = -1$ (P_4 saddle)
No stable point at infinity, so no form of singularity (Q_1 saddle point, Q_3 is unstable)
The dashed curve marks the region above which $\Omega_m > 1$ (and universe might result to future singularities)

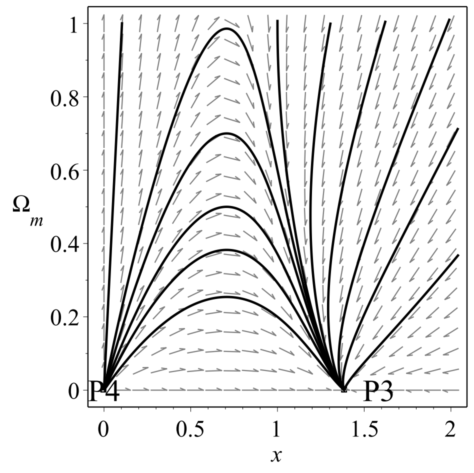
- $\alpha_1 = \frac{1}{2}$, $\alpha_2 = -\frac{1}{2}$

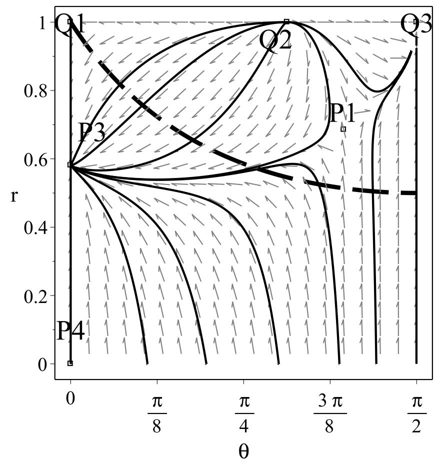




P_2 attractor, phantom solution
Q_2 attractor, future singularity

- $\alpha_1 = 3$, $\alpha_2 = -\frac{3}{2}$





P_3 attractor, quintessence solution

Q_3 attractor, future singularity

Non-minimal scalar field

- $$S = -\frac{1}{2\kappa_D^2}\int d^D x\, e\, T - \int d^D x\, e \left[\left(\frac{1}{2} - \xi T\right) g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + V\right]$$

• Corresponds to $R g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$ (ghosts)
• $f(\phi)T$ does not work due to presence of $r\theta$ equation
• Here, 2nd order eqm. Somehow, corresponds to
$G^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$ (for background cosmology the same)
• Other couplings of $\partial_\mu\phi\,\partial_\nu\phi$ with quadratics (or higher) $T_{\mu\nu\lambda}$
• Find spherically symmetric solutions in which the presence
of the torsion could leave a signature on the solution
• $\sqrt{|\xi|}$ introduces a new length scale

- Eqm for vielbein

$$\left(\frac{2}{\kappa_D^2}-4\xi\phi_{,\rho}\phi^{,\rho}\right)\left[(eS_\kappa{}^{\lambda\nu}e_b{}^\kappa)_{,\nu}e^b{}_\mu+e\left(\frac{1}{4}T\delta^\lambda_\mu-S^{\nu\kappa\lambda}T_{\nu\kappa\mu}\right)\right]$$
$$+4\xi\left[\frac{1}{2}eT\phi_{,\mu}\phi^{,\lambda}+eS_\mu{}^{\nu\lambda}(\phi_{,\kappa}\phi^{,\kappa})_{,\nu}\right]+e\left(\frac{1}{2}\phi_{,\rho}\phi^{,\rho}\delta^\lambda_\mu-\phi_{,\mu}\phi^{,\lambda}+V\delta^\lambda_\mu\right)$$
$$-\left(\frac{2}{\kappa_D^2}-4\xi\phi_{,\rho}\phi^{,\rho}\right)eS^{dca}\omega_{bdc}e_a{}^\lambda e^b{}_\mu=0$$

Adopt $\omega_{abc}=0$

- Eqm for the scalar field

$$\left[e(1-2\xi T)\phi^{,\mu}\right]_{,\mu} - e\frac{dV}{d\phi} = 0$$

- Spherical symmetry

$$ds^2 = -N(r)^2 dt^2 + K(r)^{-2} dr^2 + R(r)^2 d\Omega^2$$

Realize through

$$e^a{}_\mu = \mathrm{diag}\left(N(r), K(r)^{-1}, R(r), R(r)\sin\theta\right)$$

▶

- $$\frac{1}{K^2}\left(\phi'^2+\frac{2V}{K^2}\right)+2\left(\frac{1}{\kappa^2K^2}-2\xi\phi'^2\right)\left(\frac{R'^2}{R^2}+\frac{2R''}{R}+2\frac{R'}{R}\frac{K'}{K}-\frac{1}{K^2R^2}\right)=0$$
- $$8\xi\phi'^2\frac{R'}{R}\left(\frac{R'}{R}+\frac{2N'}{N}\right)+\frac{1}{K^2}\left(\phi'^2-\frac{2V}{K^2}\right)+2\left(2\xi\phi'^2-\frac{1}{\kappa^2K^2}\right)\left[\frac{R'}{R}\left(\frac{R'}{R}+\frac{2N'}{N}\right)-\frac{1}{K^2R^2}\right]=0$$
- $$\frac{1}{K^2}\left(\phi'^2+\frac{2V}{K^2}\right)+2\left(\frac{1}{\kappa^2K^2}-2\xi\phi'^2\right)\left[\frac{N'}{N}\left(\frac{R'}{R}+\frac{K'}{K}\right)+\frac{R'}{R}\frac{K'}{K}+\frac{N''}{N}+\frac{R''}{R}\right]=0$$
- $\frac{K'}{K}\phi'+\phi''=0$ *New feature not present in curvature – based theories*
- $$\left\{KNR^2\phi'\left[1+4\xi K^2\frac{R'}{R}\left(\frac{R'}{R}+\frac{2N'}{N}\right)\right]\right\}'-\frac{NR^2}{K}\frac{dV}{d\phi}=0$$

$tt, rr, \theta\theta, \theta r$, scalar

• Invariance under r-reparametrizations, i.e. $r \to \tilde{r}(r)$, $K \to K\frac{d\tilde{r}}{dr}$, $N \to N$, $R \to R$, $\phi \to \phi$, so one constraint is expected

• 4 functions $N(r), K(r), \phi(r), V(r)$ [$R(r)$ not counted: choice of radial gauge]; 5 eqm - 1 constraint = 4 eqm $\Rightarrow$ unique solution

• On the contrary, in the curvature-based theories, the absence of $r\theta$ eqm leads to 3 eqm for 4 unknowns

- Integro-differential system
- Master Equation

$$2\frac{d^2Y}{dx^2} - \frac{1}{Y}\Big(\frac{dY}{dx}\Big)^2 + 2\Big(2 - \frac{\eta\nu^2}{Y}\Big)\frac{dY}{dx} + 3Y + 2\eta\nu^2 \\ - 12\frac{\eta^2}{\tilde{\eta}^2}\,Y\,\frac{\frac{dY}{dx} + 3Y - \frac{2}{3\nu^2}e^{-2x}}{\frac{dY}{dx} + 3Y + 2\eta\nu^2} = 0$$

- $Y = \Big(\frac{1}{R}\frac{dR}{d\phi}\Big)^2$, $x = \ln R$, $\frac{d\phi}{dr} = \frac{\nu}{K}$ ν: hair of ϕ
  $\eta = \frac{\kappa^2}{2\nu^2(1-2\xi\kappa^2\nu^2)}$, $\tilde{\eta} = \frac{\kappa^2}{\nu^2(1-6\xi\kappa^2\nu^2)}$

A wormhole-like special solution

$$ds^2 = -\mathcal{C}^2 R^{3\epsilon-1} dt^2 + \frac{6\xi\, dR^2}{\xi - R^2} + R^2 d\Omega^2 \ , \ \epsilon = \pm 1$$

$$\tilde{\phi}(R) = \sqrt{\frac{6}{5}} \frac{1}{\kappa} \arctan\left(\sqrt{\frac{\xi}{R^2} - 1}\right)$$

$$V(\tilde{\phi}) = \frac{1}{2\xi\kappa^2} \tan^2\left(\sqrt{\frac{5}{6}}\, \kappa\, \tilde{\phi}\right) + \frac{7}{10\xi\kappa^2}$$

$\tilde{\phi}(R) = \epsilon_1[\phi(R) - \phi_1]$
Here, ν depends on ξ

- Scalar field finite everywhere from $R=0$ to $R^2=\xi$
- The potential becomes infinite at the origin $R=0$
- Ricci and Kretchmann scalars diverge at the origin and are finite elsewhere
- For $R^2=\xi$, it is N non-vanishing $\rightsquigarrow$ wormhole ($R^2=\xi$ "throat")
- Here "interior" wormhole from origin to "throat"; standard wormhole from "throat" to "mouth" or infinity

Asymptotically AdS linearized solution

$$ds^2 = -N^2 dt^2 + \frac{dR^2}{\frac{2|\eta|\nu^4}{3(1-2\sigma)} R^2 + \mathsf{c}_m R^{2-m} + \frac{2\sigma}{3(1-2\sigma)} + \frac{\mathsf{c}_\ell}{R^{\ell-2}}} + R^2 d\Omega^2$$

$$N^2(R) = \frac{c}{R} \left[R^3\, Y(R)\, e^{2\eta\nu^2 J(R)} \right]^{\frac{\zeta}{2\sigma}}$$

$$Y = \frac{2|\eta|\nu^2}{3(1-2\sigma)} + c_m R^{-m} + \frac{2\sigma}{3\nu^2(1-2\sigma)} R^{-2} + c_\ell R^{-\ell}$$

$$J(R) = \int \frac{dR}{R\, Y(R)} = \frac{\nu^2}{2} \int \frac{du}{\frac{2|\eta|\nu^4}{3(1-2\sigma)} u + \mathsf{c}_m u^{1-\frac{m}{2}} + \frac{2\sigma}{3(1-2\sigma)} + \frac{\mathsf{c}_\ell}{u^{\frac{\ell}{2}-1}}} \Bigg|_{u=R^2}$$

$$V=\frac{(1+2\sigma)\nu^2}{2(1-2\sigma)}-\frac{3-m}{2\eta\nu^2}\frac{\mathsf{c}_m}{R^m}+\frac{3-8\sigma}{6(1-2\sigma)\eta\nu^2}\frac{1}{R^2}+\frac{3-\ell}{2\eta\nu^2}\frac{\mathsf{c}_\ell}{R^\ell}$$

$$\phi=\phi_1+\epsilon_1\nu\int\frac{dR}{\sqrt{\frac{2|\eta|\nu^4}{3(1-2\sigma)}R^2+\mathsf{c}_m R^{2-m}+\frac{2\sigma}{3(1-2\sigma)}+\frac{\mathsf{c}_\ell}{R^{\ell-2}}}}$$

Asymptotic behaviour

$$\phi \approx \phi_1 + \frac{\epsilon_1}{\nu}\sqrt{\frac{3(1-2\sigma)}{2|\eta|}}\ \ln R$$

$$V(\tilde{\phi}) \approx \frac{(1+2\sigma)\nu^2}{2(1-2\sigma)} - \frac{(3-m)c_m}{2\eta\nu^2} e^{-m\nu\sqrt{\frac{2|\eta|}{3(1-2\sigma)}}\,\tilde{\phi}}$$

$\tilde{\phi} = \epsilon_1(\phi - \phi_1)$

$$ds_\infty^2 = -\frac{2|\eta|\nu^4}{3(1-2\sigma)}R^2 d\tilde{t}^2 + \frac{3(1-2\sigma)}{2|\eta|\nu^4}\frac{dR^2}{R^2} + R^2 d\Omega^2$$

• Different $|\Lambda| = \kappa^2 |V(R \to \infty)|$, $|\Lambda_{\rm eff}| = \frac{2|\eta|\nu^4}{1-2\sigma}$
$\frac{|\Lambda_{\rm eff}|}{|\Lambda|} = \frac{1}{2\xi\kappa^2\nu^2}$
• ξ, ν here are unrelated since the asymptotic solution is general (not special)
• ξ only appears through the combination $\xi\kappa^2\nu^2$, remnant of $Tg^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$
• AdS behaviour here (instead of dS) creates extra attraction force at large distances
• next term R^{2-m} $(1/2 < 2 - m < 2)$ expresses another large distance scale where gravity is modified (e.g. linear potential)

Conclusions

- The teleparallel approach to gravity may be useful from various aspects.
- We have constructed the teleparallel equivalent T_G of Gauss-Bonnet gravity in arbitrary dimensions.
- New classes of gravities $F(T, T_G)$ can be defined.
- We performed for such a modification a cosmological analysis, which can provide in principle the today acceleration and the inflation.
- A dynamical systems analysis has revealed various kinds of finite attractors or future singularities.
- For a non-minimally derivative of a scalar field with the torsion scalar, spherically symmetric solutions have been found.