Holographic Chern–Simons Theories

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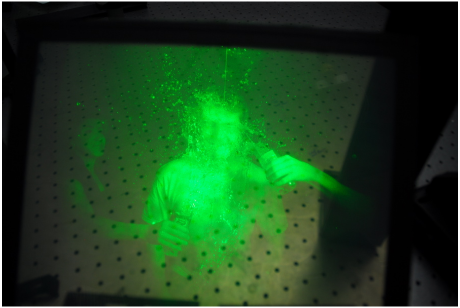
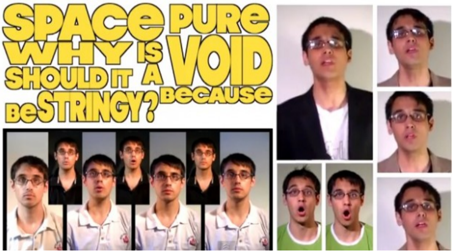
Chern-Simons Hall

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WHY IS VOID
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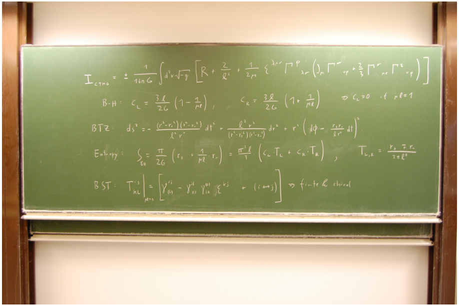

Motivation for studying gravity in 2 and 3 dimensions

- Quantum gravity
  - Address conceptual issues of quantum gravity
  - Black hole evaporation, information loss, black hole microstate counting, virtual black hole production, ...
  - Technically much simpler than 4D or higher D gravity
  - Integrable models: powerful tools in physics
  - Models should be as simple as possible, but not simpler
- Gauge/gravity duality + indirect physics applications
  - Deeper understanding of black hole holography
  - $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence best understood
  - Quantum gravity via AdS/CFT
  - Applications to 2D condensed matter systems
  - Gauge gravity duality beyond standard AdS/CFT: warped AdS, Lifshitz, Schrödinger, non-relativistic or log CFTs, higher spin holography ...
  - Flat space holography
- Direct physics applications
  - Cosmic strings
  - Black hole analog systems in condensed matter physics
  - Effective theory for gravity at large distances

Why particularly three dimensions?

- Simple
  - Riemann $\sim$ Ricci, only 6 independent components
  - can linearly combine Cartan variables,
    $A^a = \alpha\epsilon^{abc}\omega_{bc} + \beta e^a = \alpha\omega^a + \beta e^a$
  - Einstein–Hilbert gravity is topological field theory

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  - Einstein–Hilbert gravity is topological field theory
- Not too simple
  - One dimension: no interesting geometry
  - Two dimensions: no gravitons (not even off-shell), no sizable black hole horizons, dual field theories typically quantum mechanics
  - Three dimensions: gravitons (off-shell, in some models on-shell), black holes with S^1 topology, dual field theories typically QFT in two dimensions, not all metrics locally conformally flat

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- Can build non-topological models (massive gravity, see lectures by de Rahm, Townsend, Tolley, Volkov, Babichev and Spindel)
- Some recent breakthroughs: chiral gravity, AdS/log CFT correspondence (see talk by Zojer), new massive gravity theories (see lectures by Townsend, Bergshoeff and Merbis), higher spin gravity (see lectures by Vasiliev and Troncoso), non-AdS holography, flat space holography (see talk by Matulich), to be discovered...

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Three dimensional gravity models continue to surprise and teach us!

Chern–Simons and gravity in 3 dimensions

Chern–Simons summary (see lectures by Jorge Zanelli!)

Reminder: A is some connection 1-form, e.g. $sl(N)_k \oplus sl(N)_{-k}$ connection
Chern–Simons (CS) 3-form $\mathrm{CS}(A) = A\,\mathrm{d}A + \frac{2}{3}A \wedge A \wedge A$
Variation: $\delta\mathrm{CS}(A) = 2F + \mathrm{d}(A \wedge \delta A)$

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Action ($\mathcal{M}$ is topologically a cylinder):

$$I[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\,\mathrm{CS}(A) + \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{Tr}\left(A_+\,\mathrm{d}x^+ A_-\,\mathrm{d}x^-\right)$$

EOM:

$$F = 0$$

Boundary conditions:

$$\delta A_-|_{\partial\mathcal{M}} = 0 \qquad \text{or} \qquad A_+|_{\partial\mathcal{M}} = 0$$

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Gauge transformations:

$$\delta_\varepsilon A = \mathrm{d}\varepsilon + [A,\,\varepsilon] \qquad \leftrightarrow \qquad \delta_\varepsilon A^a_\mu = \partial_\mu \varepsilon^a + f^a{}_{bc} A^b_\mu \varepsilon^c$$

Number of local physical degrees of freedom: 0

Einstein–Hilbert action:

$$I_{\mathrm{EH}} = -\frac{1}{16\pi G_N}\int \mathrm{d}^3x\sqrt{|g|}\,\big(R+\frac{2}{\ell^2}\big)$$

Negative cosmological constant: $\Lambda = -\frac{1}{\ell^2}$ where $\ell =$ AdS radius

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Cartan formulation:

$$I_{\mathrm{EH}} \sim \int \big(e^a\wedge R_a + \frac{1}{\ell^2}\,\epsilon_{abc}e^a\wedge e^b\wedge e^c\big)$$

with

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Achucarro, Townsend '86, Witten '88: define

$$A = \omega + \tfrac{1}{\ell} e \qquad \bar{A} = \omega - \tfrac{1}{\ell} e \qquad \Rightarrow \qquad g_{\mu\nu} = \tfrac{1}{2}\,\mathrm{Tr}\big((A - \bar{A})_\mu (A - \bar{A})_\nu\big)$$

Einstein–Hilbert action in Cartan formulation equivalent to $[k = \ell/(8G_N)]$

$$I_{\mathrm{EH}} = \frac{k}{4\pi} \int \mathrm{CS}(A) - \frac{k}{4\pi} \int \mathrm{CS}(\bar{A}) = sl(2)_k \oplus sl(2)_{-k} \text{ CS theory}$$

Deser, Jackiw, Templeton '82: Topologically Massive Gravity

$$I_{\text{TMG}} = I_{\text{EH}} + \frac{1}{32\pi\, G\mu} \int \mathrm{d}^3x \sqrt{-g}\, \varepsilon^{\lambda\mu\nu}\, \Gamma^\rho{}_{\lambda\sigma}\big(\partial_\mu \Gamma^\sigma{}_{\nu\rho} + \frac{2}{3}\, \Gamma^\sigma{}_{\mu\tau}\Gamma^\tau{}_{\nu\rho}\big)$$

EOM:

$$R_{\alpha\beta} - \tfrac{1}{2}\, g_{\alpha\beta} R - \tfrac{1}{\ell^2}\, g_{\alpha\beta} + \frac{1}{\mu}\, C_{\alpha\beta} = 0$$

Chern–Simons and gravity in 3 dimensions

Topologically massive gravity (see lectures by Paul Townsend and Eric Bergshoeff)

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Some features:

- Local physical degree of freedom (massive graviton)
- AdS solutions like BTZ black holes
- non-AdS solutions (warped, Lifshitz, Schrödinger)
- chiral gravity (Li, Song, Strominger '08)
- AdS/log CFT correspondence (Grumiller, Johansson '08)
- interesting flat space holography (Bagchi et al. '12, '13)

Conformal Gravity:

$$I_{\text{CSG}} = \frac{k}{4\pi}\int \text{CS}(\Gamma) = \frac{k}{4\pi}\int \mathrm{d}^3x\sqrt{-g}\,\varepsilon^{\lambda\mu\nu}\,\Gamma^\rho{}_{\lambda\sigma}\big(\partial_\mu\Gamma^\sigma{}_{\nu\rho} + \frac{2}{3}\,\Gamma^\sigma{}_{\mu\tau}\Gamma^\tau{}_{\nu\rho}\big)$$

EOM:

$$C_{\mu\nu} = 0$$

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Conformal gravity (see talk by Xavier Bekaert)

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Some features:

- No local physical degree of freedom (partial masslessness)
- Weyl symmetry $g \to g e^{2\Omega}$
- AdS solutions like BTZ black holes
- rich AdS holography depending Weyl factor bc's (Afshar et al '11)
- non-AdS solutions (Lobachevsky, flat, dS)
- flat space chiral gravity (Bagchi, Detournay, Grumiller '12)

Holographic algorithm from gravity point of view

Universal recipe & Outline of lectures:

1. Identify bulk theory and variational principle

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Goal of these lectures:

Apply algorithm above to holographically describe CS theories

includes gravity and higher spin gravity; AdS, non-AdS and flat space holography

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Bulk theory and variational principle

CS theory with some gauge algebra; typically contains $sl(2) \times sl(2)$

$$I = I_{\text{CS}}[A] - I_{\text{CS}}[\bar{A}]$$

with

$$I_{\text{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \mathsf{Tr}(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A) + B[A]$$

and

$$B[A] = \frac{k}{4\pi} \int_{\partial\mathcal{M}} \mathsf{Tr}(A_+ \, \mathrm{d}x^+ \, A_- \, \mathrm{d}x^-)$$

Gauge invariant if infinitesimal gauge parameter obeys boundary condition

$$\partial_- \epsilon\big|_{\partial\mathcal{M}} = 0$$

Variational principle consistent for

$$\delta A_-\big|_{\partial\mathcal{M}} = 0 \qquad \text{or} \qquad A_+\big|_{\partial\mathcal{M}} = 0$$

Bar-sector works similarly, exchanging $\pm$

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Background and fluctuations

Take suitable group element b (often: $b = e^{\rho L_0}$) and make Ansatz for connection

$$A = b^{-1}\left(\hat{a}^{(0)} + a^{(0)} + a^{(1)}\right) b$$

- $\hat{a}^{(0)} \sim \mathcal{O}(1)$: determines asymptotic background
- $a^{(0)} \sim \mathcal{O}(1)$: determines state-dependent fluctuations
- $a^{(1)} \sim o(1)$: sub-leading fluctuations

Bar-sector is analog

Boundary-condition preserving gauge transformations generated by ϵ

$$\epsilon = b^{-1}\left(\epsilon^{(0)} + \epsilon^{(1)}\right) b$$

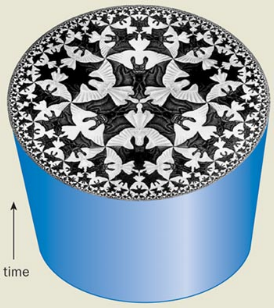
with $\epsilon^{(0)} \sim \mathcal{O}(1)$ (subject to constraints) and $\epsilon^{(1)} \sim o(1)$

Metric is then determined from

$$g_{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\left[(A-\bar{A})_\mu (A-\bar{A})_\nu\right]$$

Example: Lobachevsky holography in $sl(3)$ gravity

Lobachevsky plane times time:



Lobachevsky background ($x^+ = t$, $x^- = \varphi$):

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\!\rho\;\mathrm{d}\varphi^2$$

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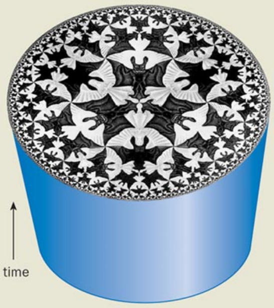

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$sl(3)$ with $sl(2)$ non-principally embedded:

$$L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; L_{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

singlet:

$$S = \frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

plus four more doublet generators $\psi^+_{\pm\frac{1}{2}}$, $\psi^-_{\pm\frac{1}{2}}$

$sl(2)$ weights:

$$[L_{\pm 1},\, L_0] = \pm L_{\pm 1} \qquad [S,\, L_0] = 0 \qquad [\psi_{\pm\frac{1}{2}},\, L_0] = \pm\tfrac{1}{2}\,\psi_{\pm\frac{1}{2}}$$

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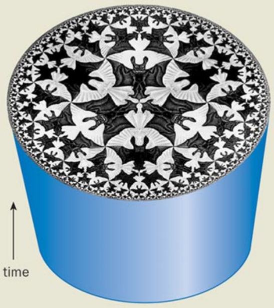

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Connections in n-p embedding of spin-3 gravity:

$$\begin{aligned} A_\rho &= L_0 & \bar{A}_\rho &= -\,L_0 \\ A_\varphi &= -\,\frac{1}{4}\,e^\rho L_1 & \bar{A}_\varphi &= -\,e^\rho L_{-1} \\ A_t &= 0 & \bar{A}_t &= \sqrt{3}S \end{aligned}$$

Indeed $\hat{a}^{(0)}$ is ρ-independent for $b = e^{\rho L_0}$

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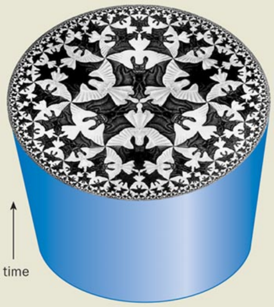

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$$A_t = 0 \qquad \bar{A}_t = \sqrt{3} S$$

Indeed $\hat{a}^{(0)}$ is ρ-independent for $b = e^{\rho L_0}$

Fluctuations:

$$a_\varphi^{(0)} = \frac{2\pi}{k} \left(\tfrac{3}{2} \mathcal{W}_0(\varphi) S + \mathcal{W}^+_{\frac{1}{2}}(\varphi)\psi^+_{-\frac{1}{2}} - \mathcal{W}^-_{\frac{1}{2}}(\varphi)\psi^-_{-\frac{1}{2}} - \mathcal{L}(\varphi) L_{-1} \right)$$

$$a_\mu^{(1)} = \mathcal{O}(e^{-2\rho})$$

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Canonical analysis and boundary charges

Story a la Brown–Henneaux: bulk generators of gauge transformations acquire boundary pieces, the canonical boundary charges $Q[\epsilon]$

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Background independent result:

$$\delta Q[\epsilon] = \frac{k}{2\pi} \oint \mathrm{Tr}\left(\epsilon^{(0)}\, \delta a_\varphi^{(0)}\, \mathrm{d}\varphi\right)$$

- Manifestly finite!
- Non-trivial?
- Integrable?
- Conserved?

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- Integrable?
- Conserved?

If any of these is answered with 'no' then back to square one in algorithm!

Example: Lobachevsky holography

Split boundary preserving gauge trafos into components:

$$\epsilon^{(0)} = \epsilon_1\, L_1 + \epsilon^+_{\frac{1}{2}}\, \psi^+_{\frac{1}{2}} + \epsilon^-_{\frac{1}{2}}\, \psi^-_{\frac{1}{2}} + \epsilon^L_0\, L_0 + \epsilon^S_0\, S + \epsilon^+_{-\frac{1}{2}}\, \psi^+_{-\frac{1}{2}} + \epsilon^-_{-\frac{1}{2}}\, \psi^-_{-\frac{1}{2}} + \epsilon_{-1}\, L_{-1}$$

Solving constraint that gauge trafos generated by $\epsilon^{(0)}$ preserve boundary conditions

$$\partial_\mu \epsilon^{(0)\,a} + f^a{}_{bc} \left(\hat{a}^{(0)}_\mu + a^{(0)}_\mu\right)^b \epsilon^{(0)\,c} = \mathcal{O}(a^{(0)}_\mu)^a$$

yields results for components of $\epsilon^{(0)}$

$$\epsilon_1 = \epsilon(\varphi) \qquad \epsilon^\pm_{\frac{1}{2}} = \epsilon^\pm_{\frac{1}{2}}(\varphi) \qquad \epsilon^L_0 = 4\epsilon'(\varphi) \qquad \epsilon^S_0 = \epsilon_0(\varphi)$$

$$\epsilon^\pm_{-\frac{1}{2}} = 4\epsilon^{\pm\,\prime}_{\frac{1}{2}}(\varphi) \mp \frac{4\pi}{k}\Big(2\mathcal{W}^\pm_{\frac{1}{2}}(\varphi)\epsilon(\varphi) - 3\mathcal{W}_0(\varphi)\epsilon^\pm_{\frac{1}{2}}(\varphi)\Big)$$

$$\epsilon_{-1} = 8\epsilon''(\varphi) + \frac{4\pi}{k}\Big(2\mathcal{L}(\varphi)\epsilon(\varphi) + \mathcal{W}^-_{\frac{1}{2}}(\varphi)\epsilon^+_{\frac{1}{2}}(\varphi) + \mathcal{W}^+_{\frac{1}{2}}(\varphi)\epsilon^-_{\frac{1}{2}}(\varphi)\Big)$$

Canonical charges:

$$Q[\epsilon^{(0)}] = \oint \mathrm{d}\varphi \left(\mathcal{L}\epsilon + \mathcal{W}_0\epsilon_0 + \mathcal{W}^+_{\frac{1}{2}}\epsilon^-_{\frac{1}{2}} + \mathcal{W}^-_{\frac{1}{2}}\epsilon^+_{\frac{1}{2}}\right)$$

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Classical asymptotic symmetry algebra

Dirac bracket algebra of canonical boundary charges:

$$\{Q[\epsilon_1],\, Q[\epsilon_2]\} = \delta_{\epsilon_2}\, Q[\epsilon_1]$$

- Either evaluate left hand side directly (Dirac brackets)
- Or evaluate right hand side (usually easier)

Exactly like in seminal Brown–Henneaux work!

Example: Lobachevsky holography

Dirac bracket algebra of canonical boundary charges:

$$\{\mathcal{L}(\varphi),\mathcal{L}(\bar\varphi)\} = -4\big(2\mathcal{L}\delta'(\varphi-\bar\varphi) - \mathcal{L}'\delta(\varphi-\bar\varphi)\big) - \frac{4k}{\pi}\delta'''(\varphi-\bar\varphi)$$

$$\{\mathcal{L}(\varphi),\mathcal{W}_0(\bar\varphi)\} = 0$$

$$\{\mathcal{L}(\varphi),\mathcal{W}^{\pm}_{\frac12}(\bar\varphi)\} = -4\Big(\frac{3}{2}\mathcal{W}^{\pm}_{\frac12}\delta'(\varphi-\bar\varphi) - \big(\mathcal{W}^{\pm}_{\frac12}{}' \pm \frac{3\pi}{k}\mathcal{W}^{\pm}_{\frac12}\mathcal{W}_0\big)\delta(\varphi-\bar\varphi)\Big)$$

$$\{\mathcal{W}_0(\varphi),\mathcal{W}_0(\bar\varphi)\} = \frac{k}{3\pi}\delta'(\varphi-\bar\varphi)$$

$$\{\mathcal{W}_0(\varphi),\mathcal{W}^{\pm}_{\frac12}(\bar\varphi)\} = \pm\mathcal{W}^{\pm}_{\frac12}\delta(\varphi-\bar\varphi)$$

$$\begin{aligned}\{\mathcal{W}^{+}_{\frac12}(\varphi),\mathcal{W}^{-}_{\frac12}(\bar\varphi)\} = \mathcal{L}\delta(\varphi-\bar\varphi) - 4\Big(&-3\mathcal{W}_0\delta'(\varphi-\bar\varphi) + \big(\frac{3}{2}\mathcal{W}_0{}' \\ &- \frac{9\pi}{2k}\mathcal{W}_0\mathcal{W}_0\big)\delta(\varphi-\bar\varphi) - \frac{k}{2\pi}\delta''(\varphi-\bar\varphi)\Big)\end{aligned}$$

Note: second and third line require Sugawara-shift

$$\mathcal{L} \to \mathcal{L} - \frac{6\pi}{k}\mathcal{W}_0\mathcal{W}_0 \equiv \hat{\mathcal{L}}$$

... continued

Replace Dirac brackets by commutators and make Fourier expansions

$$[J_n, J_m] = -\frac{2k}{3} n\delta_{n+m,0}$$
$$[J_n, \hat{L}_m] = nJ_{n+m}$$
$$[J_n, G^\pm_m] = \pm G^\pm_{m+n}$$
$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{m+n} + \frac{c}{12} n\big(n^2-1\big)\delta_{n+m,0}$$
$$[\hat{L}_n, G^\pm_m] = \big(\frac{n}{2} - m\big)G^\pm_{n+m}$$
$$[G^+_n, G^-_m] = \hat{L}_{m+n} + \frac{3}{2}(m-n)J_{m+n} + \frac{3}{k}\sum_{p\in\mathbb{Z}} J_{m+n-p}J_p + k\big(n^2 - \frac{1}{4}\big)\delta_{m+n,0}$$

Semi-classical (large k) Polyakov–Bershadsky algebra $W_3^{(2)}$

Note: resembles $N = 2$ superconformal algebra

Holographic algorithm from gravity point of view

1. Identify bulk theory and variational principle
2. Fix background and impose suitable boundary conditions
3. Perform canonical analysis and check consistency of bc's
4. Derive (classical) asymptotic symmetry algebra and central charges
5. Improve to quantum ASA
6. Study unitary representations of quantum ASA
7. Identify/constrain dual field theory

Quantum asymptotic symmetry algebra

Introducing normal ordering in expressions like

$$\sum_{p\in\mathbb{Z}} :J_{n-p}J_p: = \sum_{p\geq 0} J_{n-p}J_p + \sum_{p<0} J_pJ_{n-p}$$

can make semi-classical algebra inconsistent

First example I am aware of: Henneaux–Rey 2010 in spin-3 AdS gravity

Quantum violations of Jacobi-identities possible!

Resolution: deform suitable structure constants/functions and demand validity of Jacobi identities

Example: Lobachevsky holography

Five deformation parameters in $[J, J]$ and $[G^+, G^-]$
Solving Jacobi identities yields (quantum) Polyakov–Bershadsky algebra

$$[J_n, J_m] = \frac{2\hat{k}+3}{3} n\delta_{n+m,0}$$

$$[J_n, \hat{L}_m] = nJ_{n+m}$$

$$[J_n, \hat{G}^{\pm}_m] = \pm G^{\pm}_{m+n}$$

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{m+n} + \frac{\hat{c}}{12} n(n^2-1)\delta_{n+m,0}$$

$$[\hat{L}_n, \hat{G}^{\pm}_m] = \left(\frac{n}{2} - m\right)\hat{G}^{\pm}_{n+m}$$

$$[\hat{G}^+_n, \hat{G}^-_m] = -(\hat{k}+3)\hat{L}_{m+n} + \frac{3}{2}(\hat{k}+1)(n-m)J_{m+n} + 3\sum_{p\in\mathbb{Z}} : J_{m+n-p}J_p : + \frac{(\hat{k}+1)(2\hat{k}+3)}{2}(n^2 - \frac{1}{4})\delta_{m+n,0}$$

with central charge $\hat{c} = -(2\hat{k}+3)(3\hat{k}+1)/(\hat{k}+3) = -6\hat{k} + \mathcal{O}(1)$

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Standard questions:

- Is $\hat{u}(1)$ level non-negative?
- Is central charge non-negative?
- Are there any negative norm states?
- Are there null states?

To be decided on case-by-case basis!

Example: Lobachevsky holography

Non-negativity of $\hat{u}(1)$ level:

$$\hat{k} \geq -\frac{3}{2}$$

Non-negativity of central charge:

$$-\frac{1}{3} \geq \hat{k} \geq -\frac{3}{2}$$

Norm of vacuum descendants at level $\frac{3}{2}$:

$$K^{(\frac{3}{2})} = (\hat{k}+1)(2\hat{k}+3) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Positive and negative norm states, unless pre-factor vanishes

Only two possible values of level $\hat{k}$ compatible with unitarity:

$$\hat{k} = -1 \qquad \text{or} \qquad \hat{k} = -\frac{3}{2}$$

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Identify or at least constrain dual field theory

Collect all clues and make reasonable guess!

			1			7	4	
	5			9			3	2
		6	7			9		
4			8					
	2						1	
					9			5
		4			7	3		
7	3			2			6	
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Example: Lobachevsky holography

Unitary case $\hat{k} = -\frac{3}{2}$ has vanishing central charge, $\hat{c} = 0$:

Only state in theory is vacuum!

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- Dual CFT: free boson

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Chern–Simons holography provides many avenues for future research — also many projects for students...

Thanks for your attention!

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