Introduction to Chern-Simons forms in Physics - I

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__Lecture I:__

1. Topological invariants
2. Minimal coupling
3. Chern-Simons
4. Brane couplings
5. Quantization
6. Summary

__Lecture II:__

1. Spacetime geometry
2. Metric and affine structures: e^{a} , ω^{a}b
3. Building blocks
4. Gravity actions: L-L, T, CS
5. CS gravities, BHs, SUGRAs
6. Summary

Characteristic Forms and Geometric Invariants

Shiing-Shen Chern; James Simons

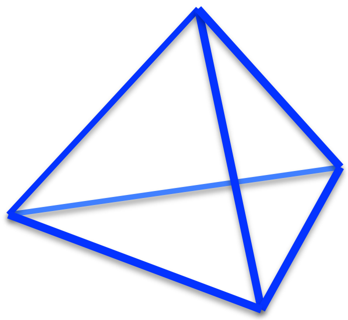
The Annals of Mathematics, 2nd Ser., Vol. 99, No. 1 (Jan., 1974), 48-69.

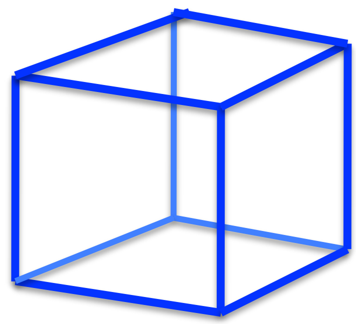
(Received June 13, 1972)
(Revised December 1, 1972)

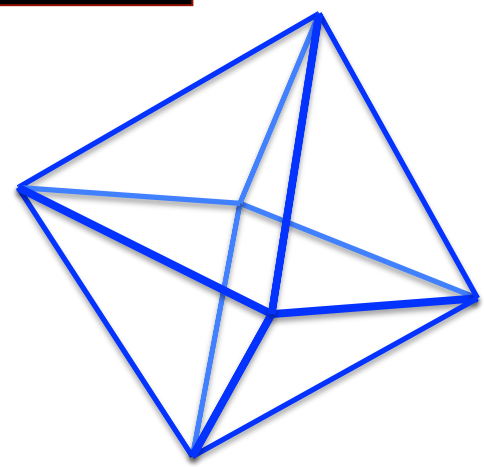
This work, ... grew out of an attempt to derive a purely combinatorial formula for the Pontrjagin number of a 4-manifold...
This process got stuck by the emergence of a boundary term which did not yield to a simple combinatorial analysis. The boundary term seemed interesting in its own right and it and its generalization are the subject of this paper.

1. Topological invariants

Euler characteristic







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V=4
E=6
F=4

V=8
E=12
F=6

V=6
E=12
F=8

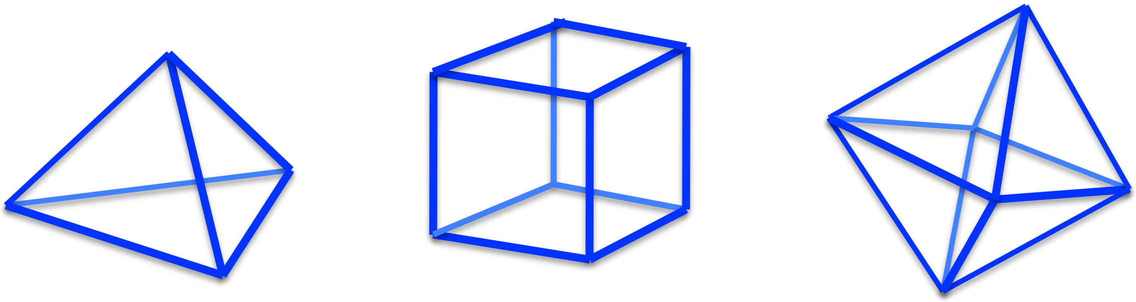
$$\chi = V - E + F = 2$$

- Independent of REGULARITY
- Independent of SIZE
- Independent of SHAPE
- Independent of NUMBER OF ELEMENTS

The Euler characteristic, χ, is unchanged under continuous deformations of the surface:

χ is a topological (homotopic) invariant.

Euler characteristic χ

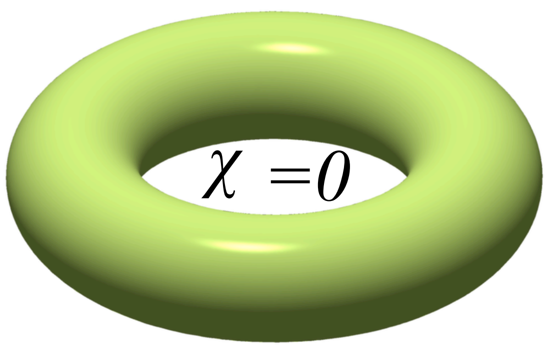


……

χ *remains unchanged so long as the topology remains the same.*
V, E, F *can be as large as we wish…*

In the continuum limit, χ *has an integral expression for any closed 2-dimensional* M*:*

$$\chi(M) = \frac{1}{2\pi}\int_M R\, d\Omega = 2 - 2g$$



The Euler characteristic belongs to a family of famous invariants:

- Sum of exterior angles of a polygon
- Residue theorem in complex analysis
- Winding number of a map
- Poincaré-Hopff theorem ("*one cannot comb a sphere*")
- Atiyah-Singer index theorem
- Witten index
- Dirac's monopole quantization
- Aharonov-Bohm effect
- Gauss' law
- Bohr-Sommerfeld quantization
- Soliton/Instanton topologically conserved charges
- etc…

All of these examples involve topological invariants called the Chern characteristic classes and Chern-Simons forms.

2. Minimal coupling

Electromagnetic coupling

$$I[A,z] = \int_M j^\mu A_\mu d^4 x$$

This coupling is invariant under gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Omega(x)$$

provided the current is conserved,

$$\partial_\mu j^\mu = 0.$$

Gauge invariance $\Longleftrightarrow$ Current conservation

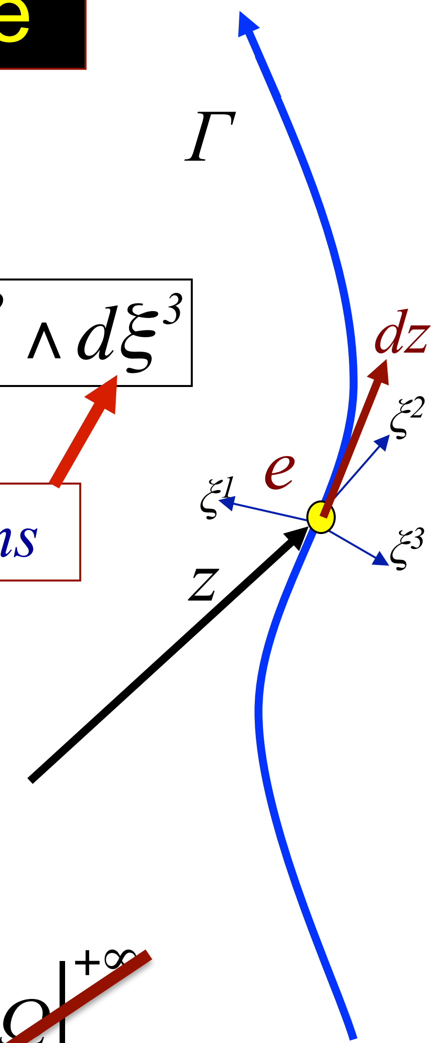
Lesson 1. ***Consistent (gauge invariant) minimal coupling between a gauge potential and a charged source is possible only if the source satisfies a conservation law.***

Minimal coupling for a point charge

$$I[A,z] = \int_M j^\mu A_\mu d^4x$$

$$= \int_M *j \wedge A \qquad \boxed{*j = e\delta(\Gamma)d\xi^1 \wedge d\xi^2 \wedge d\xi^3}$$

Transverse directions



$$I[A,z] = e\int_\Gamma A \,, \quad (A \equiv A_\mu(z)dz^\mu)$$

Is this gauge-invariant? (A *is not*)

$$A(x) \to A(x) + d\Omega(x) \Longleftrightarrow I[A] \to I[A] + \cancel{\Omega\Big|_{-\infty}^{+\infty}}$$

PBCs: $\Omega(+\infty)=\Omega(-\infty)$ *guarantee gauge invariance*

In other words,

$$\delta I = e\int_\Gamma \delta A = e\int_\Gamma d\Omega = 0$$

can vanish under physically reasonable boundary conditions.

Lesson 2. ***A gauge non-invariant integrand can define a gauge-invariant integral, provided the right boundary conditions are satisfied*** (*e.g., periodic b. c.*).

Generalization to higher-dimensional sources:

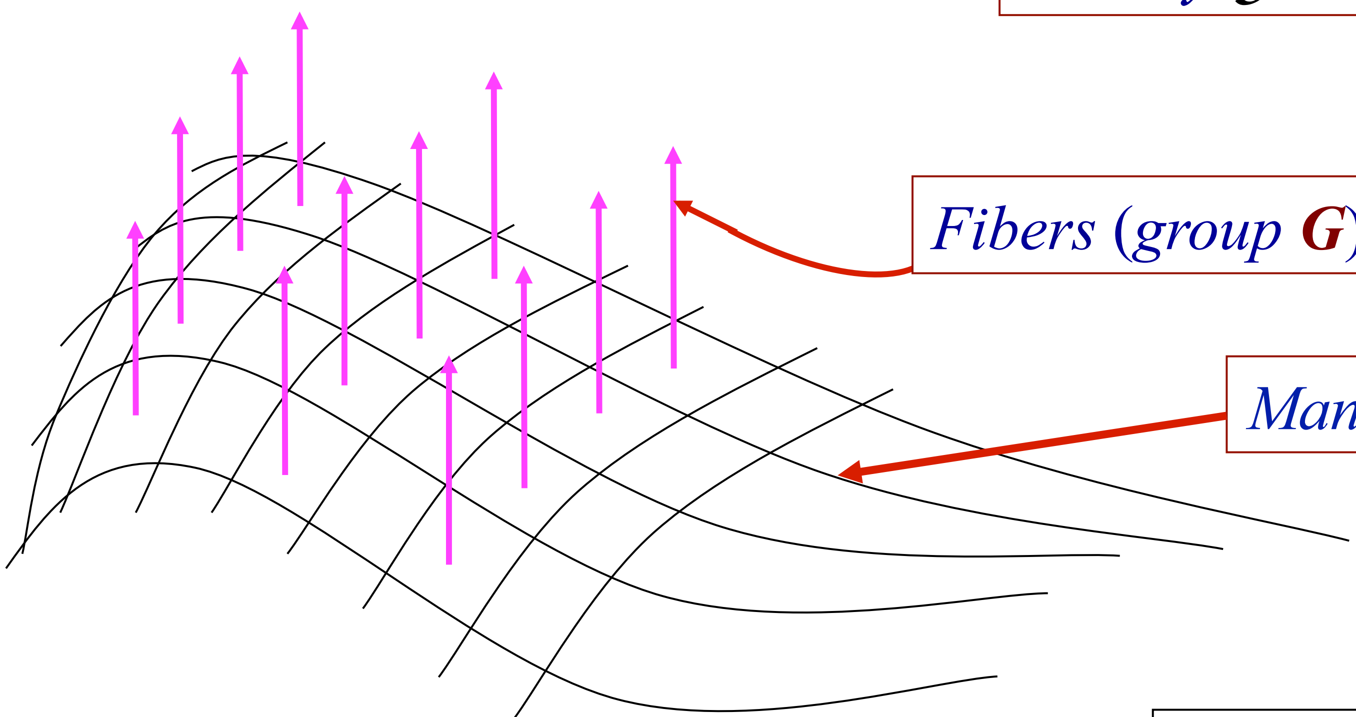
Chern-Simons forms

3. Chern-Simons forms

Gauge theories

All four basic forces of nature share the same mathematical structure

Fiber bundle ($\mathcal{F}$): locally $\mathcal{F} = M \times G$



*Fibers (group **G**)*

*Manifold **M***

Connection (Lie-algebra valued 1-form)

$$\boldsymbol{A}_\mu = A_\mu^a \mathbf{J}_a$$

Interaction Field

Lie alg. generator

Electrodynamics

Action functional

$$I[A] = \int_M \left(\tfrac{1}{2} F \wedge {}^*F - j \wedge A\right)$$

where

$$F = dA = \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu$$

Field strength (curvature)

Yang-Mills (electro-weak and strong interactions)

Action

$$I[A] = \int_M Tr[\tfrac{1}{2} F \wedge {}^*F - j \wedge A]$$

where A takes values in a nonabelian Lie algebra, and

$$F = dA + A \wedge A = [\partial_\mu A_\nu^a + f_{bc}^a A_\mu^b A_\nu^c] J_a dx^\mu \wedge dx^\nu = F^a J_a$$

What is a Chern-Simons action?

Y-M / EM action [*any D*]:

$$I[A]=\frac{1}{4\kappa}\int_{M^D}\sqrt{g}\,g^{\mu\alpha}g^{\nu\beta}\gamma_{ab}F^{a}{}_{\mu\nu}F^{b}{}_{\alpha\beta}d^{D}x$$

Chern-Simons action [3dim]:

$$I[A]=\kappa\int_{M^3}\left\langle A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right\rangle$$

and in general [(2n+1)-dim.]:

$$I[A]=\kappa\int_{M^{2n+1}}\left\langle A\wedge(dA)^{n}+\alpha_{1}A^{3}\wedge(dA)^{n-1}+\cdots+\alpha_{n}A^{2n+1}\right\rangle$$

Fixed rational numbers

What is a Chern-Simons action?

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$$I[A]=\frac{1}{4\kappa}\int_{M^D}\sqrt{g}g^{\mu\alpha}g^{\nu\beta}\gamma_{ab}F^a{}_{\mu\nu}F^b{}_{\alpha\beta}d^Dx$$

Chern-Simons action [3dim]:

$$I[A]=\kappa\int_{M^3}\left\langle A\wedge dA+\tfrac{2}{3}A\wedge A\wedge A\right\rangle$$

and in general [(2n+1)-dim.]:

Invariant symmetric trace in the Lie algebra

$$I[A]=\kappa\int_{M^{2n+1}}\left\langle A\wedge(dA)^n+\alpha_1 A^3\wedge(dA)^{n-1}+\cdots+\alpha_n A^{2n+1}\right\rangle$$

Chern-Simons forms

$$C_{2n+1} = \left\langle A \wedge (dA)^n + \alpha_1 A^3 \wedge (dA)^{n-1} + \cdots + \alpha_n A^{2n+1} \right\rangle \equiv \left\langle \tilde{C}_{2n+1} \right\rangle$$

- *No dimensionful constants:* α_k *= fixed rational numbers*
- *No adjustable coefficients:* α_k *cannot get renormalized*
- *No metric needed; scale invariant*
- *Entirely determined by the Lie algebra and the dimension*
- *Only defined in odd dimensions*
- *Unique gauge quasi-invariants:* $\delta C_{2p+1} = d\Omega_{2p}$ ← Physics
- *Related to the Chern characteristic classes* ← Mathematics

Their exterior derivatives are invariant polynomials

$$dC_{2n-1}(A) = P_{2n}(F)$$

(*characteristic classes*)

Characteristic classes: Given A = connection , for a certain Lie algebra. Then, under gauge transformations,

$$A \rightarrow A' = g^{-1}Ag + g^{-1}dg\,, \quad F \rightarrow F' = g^{-1}Fg$$

the polynomial $P_{2n}(F) = \langle F^n \rangle$ *is invariant*

$$\boxed{P_{2n}(F') = P_{2n}(F)}$$

and closed: $\boxed{dP_{2n}(F) = 0.}$ [*Chern-Weil Theorem*]

These invariants have an additional remarkable property. For a suitable normalization (#),

$$\# \int_{M^{2n}} P_{2n}(F) = \in Z$$

just like the Euler characteristic.

What makes the CS forms useful in physics is that under gauge transformations, they change like an abelian connection:

$$dC_{2n-1}(A) = P_{2n}(F) \Rightarrow \delta C_{2n+1}(A) = d\Omega_{2n}$$

The e-m potential is the simplest example of a CS form

$$C_{2n+1} = \left\langle A \wedge (dA)^n + \alpha_1 A^3 \wedge (dA)^{n-1} + \cdots + \alpha_n A^{2n+1} \right\rangle$$

$$\Big\downarrow \textit{for } n = 0$$

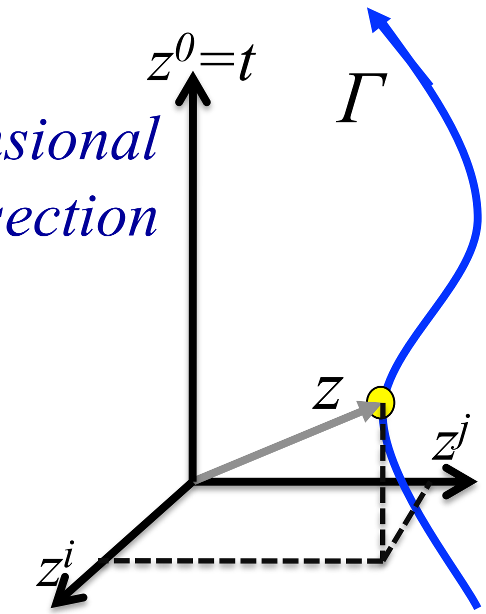
$$\boxed{C_{0+1}(A) = \langle A \rangle = A}$$

CS forms provide a generalization of the coupling (between point charges and the e-m field) to higher dimensional objects –branes- and nonabelian gauge fields.

Physicists have been playing with CS forms for 200 years!

Classical mechanics

Consider the trajectory of a point in a 2s+1 dimensional space. The projection on the 2s dimensional spatial section can be identified with the position on a phase space.



$$I = \int_\Gamma A_\mu(z)dz^\mu \ , \quad z^\mu = (z^0 = t, z^i)$$

$$= \int_\Gamma (A_0 dt + A_i dz^i), \ i = 1,\dots,2s$$

Identifying $z^i \equiv (q^1,\dots,q^s, p_1,\dots,p_s)$; $A_i = (p_1,\dots,p_s, 0,\dots,0)$, *and* $A_0 = -H(z)$, *one finds the action of a mechanical system of finite number of degrees of freedom,*

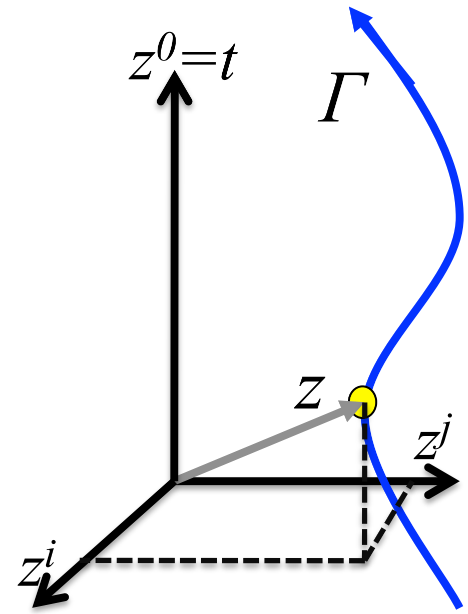
$$I[q,p] = \int_\Gamma [p_i dq^i - H(p,q)dt]$$

Dynamical equations

The equations read

$$\boxed{F_{ij}\dot{z}^j = E_i \;,}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i \,,\; E_i = -\partial_i A_0$. *Which can be checked to be Hamilton's equations,*



$$\boxed{\dot{q}^i = \frac{\partial H}{\partial p_i} \,,\;\; \dot{p}_i = \frac{\partial H}{\partial q^i} \,.}$$

Also,

Gauge invariance of CS action	=	*Invariance of mechanical system under canonical transformations*

A good part of classical physics is described by CS forms.

4. Chern-Simons couplings

Generalization to higher dimensions

Natural Ansatz:

Replace 1-forms *by* p-forms

$$\underset{\text{1-form}}{A_\mu} \longrightarrow \underset{p\text{-form}}{A_{\mu\nu\lambda\cdots}}$$

$$I[A,z] = \int_M j_{\mu_1\mu_2\ldots\mu_p} A^{\mu_1\mu_2\ldots\mu_p}$$

then… $$I[A,z] = \int_{\Gamma^p} A$$

(D-p)-form that projects on the p-dimensional history

This works fine for an abelian field: $A \longrightarrow A' = A + d\Omega$

but not for nonabelian one: $A \longrightarrow A' = g^{-1}Ag + g^{-1}dg$

There is no obvious analog of

$$A_\mu \rightarrow A'_\mu = g^{-1}(A_\mu + \partial_\mu)g, \quad g \in G$$

for a p-form:

$$A_{\mu_1\mu_2\ldots\mu_p} \rightarrow A'_{\mu_1\mu_2\ldots\mu_p} = g^{-1}(A_{\mu_1\mu_2\ldots\mu_p} + ?_{\mu_1\mu_2\ldots\mu_p})g$$

- *This p-form does not define a natural connection (covariant derivative)*
- *Nonabelian curvature:* $F = dA + A \wedge A$*?*
- *There is no clear relation between current conservation and (nonabelian) gauge transformation.*
- *There is no consistent Hamiltonian evolution*

Chern-Simons forms

A consistent (gauge invariant) coupling for any group G and for higher dimensional Γs is provided by a Chern-Simons form $C_{2n\text{-}1}(A)$:

$$dC_{2n-1}(A) = P_{2n}(F)$$

Then, under a gauge transformation, $C_{2n-1}(A)$ changes by a closed form: $0 = \delta dC = d\delta C$

$$\boxed{\Rightarrow \delta C = d\Omega}$$

Locally exact ✔

The Chern-Simons coupling

The minimal coupling between a particle and electromagnetism is generalized by the formula

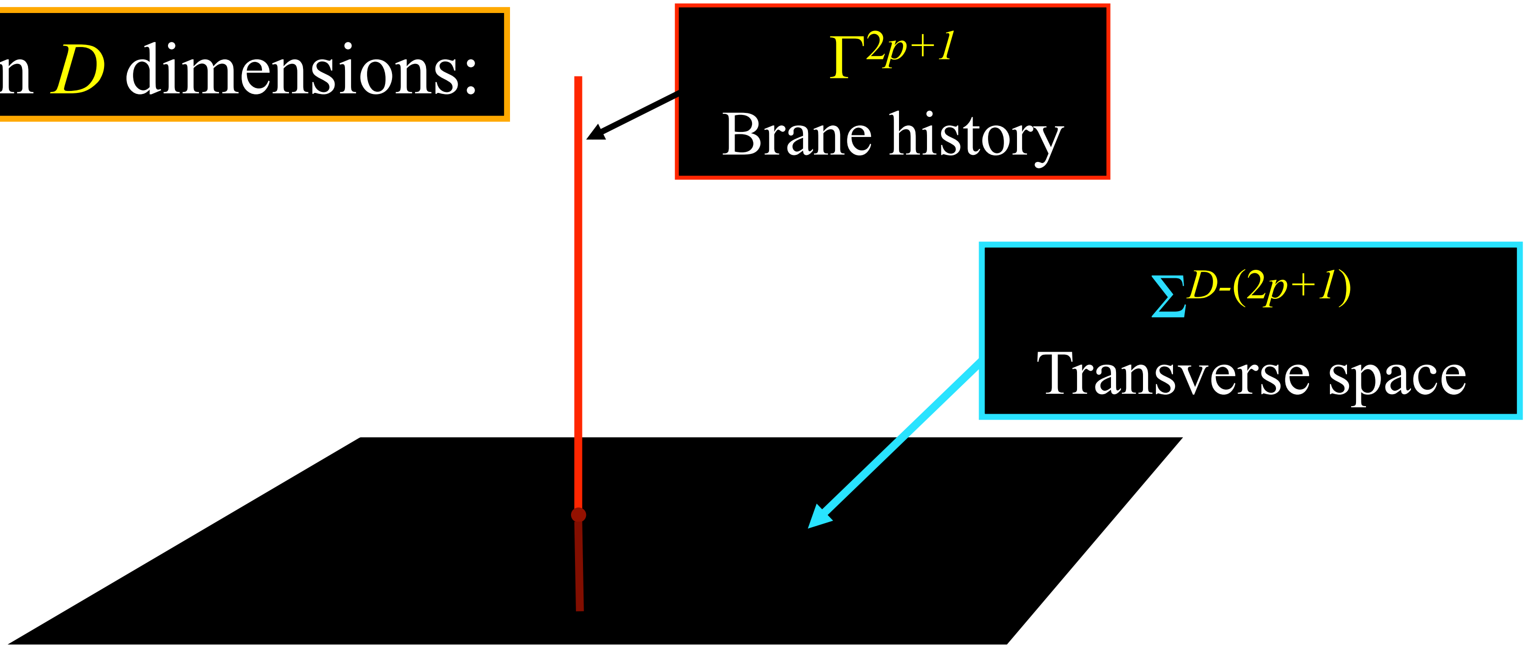
$$I[A] = \int_M \left\langle *j_{2p+1}\tilde{C}_{2p+1}(A) \right\rangle$$

(*where we have defined* $C_{2p+1}(A) = \left\langle \tilde{C}_{2p+1}(A) \right\rangle$).

◆ *This describes the coupling between an extended object, a 2p-dimensional (mem)brane and a nonabelian connection.*

◆ *As in electrodynamics, this coupling is invariant under gauge transformations provided* $*j_{2p+1}$ *is conserved,* $\mathrm{D}*j_{2p+1} = 0$.

$2p$-brane in D dimensions:



Source:

$$
{}^{*}j_{2p+1} = q\,\delta(\Sigma)\,d\Omega_{\Sigma}\,[S^{a_1 a_2 \cdots a_m} J_{a_1} \cdots J_{a_m}]
$$

Source with support at the center of Σ

Generators

This is a D-$(2p+1)$-form that couples to a $2p+1$-form

The Chern-Simons coupling

Lemma:

$$I[A] = \int_M \left\langle *j_{2p+1}\tilde{C}_{2p+1}(A)\right\rangle,$$

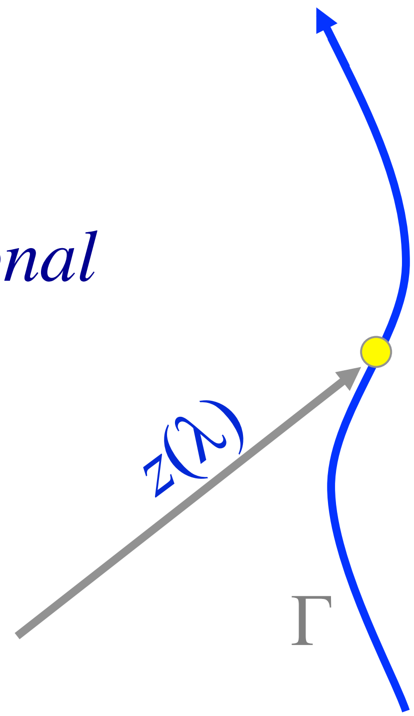
is invariant under gauge transformations, provided j_{2p+1} is conserved, $d*j_{2p+1} + [A, *j_{2p+1}] = D*j_{2p+1} = 0$.

Proof: This can be explicitly verified. //

N.B.: If the current is produced by particles or fields that are dynamically governed by an action principle, invariant under the same gauge symmetry, then Noether theorem guarantees that such conserved current can always be explicitly written out. //

5. Examples of branes

Branes are defects



- *An m-brane is a localized $(m+1)$-dimensional manifold, embedded in spacetime.*
- *They disrupt the homogeneity of spacetime (impurities)*
- *They are obstructions that change the topology of spacetime(topological defects)*
- *Examples: point particle (m=0),*
  domain wall (m=2),
  2p-brane (m=2p)

Branes are naked singularities

- *If not wrapped by a horizon, branes are naked singularities (NS).*

- *Near naked singularities the energy grows infinitely, the laws of physics break down*

  "Green slime, lost socks and TV sets could emerge from them" *(J. Earman)*

- *They better not exist (Cosmic Censorship)*

But sometimes they do exist and nothing terrible happens…

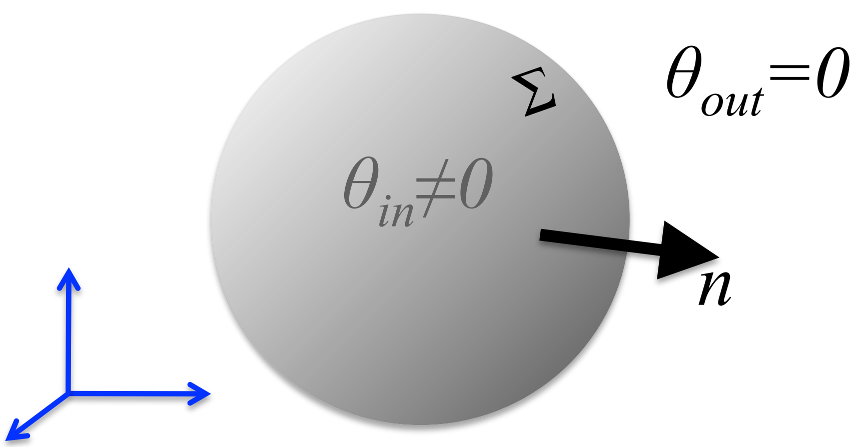
0- and 2- brane sources in EM (4*D*)

- 0-*brane* [*point charge*]:

$$*j = e\delta(\Gamma)d\Omega_{D-1} \Leftrightarrow j^{\mu} = e\dot{z}^{\mu}\delta^{(3)}(\vec{x}-\vec{z})d\tau \quad ✔$$

- 2-*brane* [*interface between two regions in 3-space*]:

$$*j = -\theta\delta(\Sigma)dn \Leftrightarrow j^{\mu\nu\lambda} = -\theta\varepsilon^{\mu\nu\lambda\perp}\delta(x^{\perp})d^{3}x(\Sigma)$$



This brane is the 2*D surface* Σ *of a solid volume in* 3*D*, *whose history is a* 2+1 *worldsheet* $\Sigma\times\mathrm{R}$

Calling the brane charge –θ,

$$\int *j \wedge C_3(A) = -\theta \int_{\Sigma\times\mathrm{R}} C_3(A)$$

$$= -\theta \int_{V\times\mathrm{R}} \tfrac{1}{4}\,\varepsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} d^4 x$$

The full Maxwell + brane *action is*

$$\boxed{I[A] = \tfrac{1}{2}\int_{M^4} F \wedge *F - \tfrac{\theta}{2}\int_{V\times\mathrm{R}} F \wedge F}$$

θ-vacuum

which describes a topological insulator:

$$\nabla \cdot \vec{E} = \theta\delta(\Sigma)\ \hat{n}\cdot\vec{B}$$

$$\nabla \times \vec{B} - \partial_t \vec{E} = \theta\delta(\Sigma)\ \hat{n}\times\vec{E}$$

Brane sources in gravity

- 0-*brane in GR* = *conical defect* ~ *point particle*

In 2+1 *dimensions,*

$$*j^a = \tfrac{1}{2} m\delta(\Gamma) d\Omega_2 \varepsilon^{abc} J_{bc},$$

J_{bc} = *Isometry generator that produced the conical defect.*

~ *black hole with* $M<0$, $M<|J|$, $M<|Q|$ ✔

Ok in 2+1 *dimensions; serious problem for* $D>3$

- *Ok in* $D>3$ *for codimension* 2 *branes* ✔
- *Ok for higher codimensions and commuting generators.* ✔

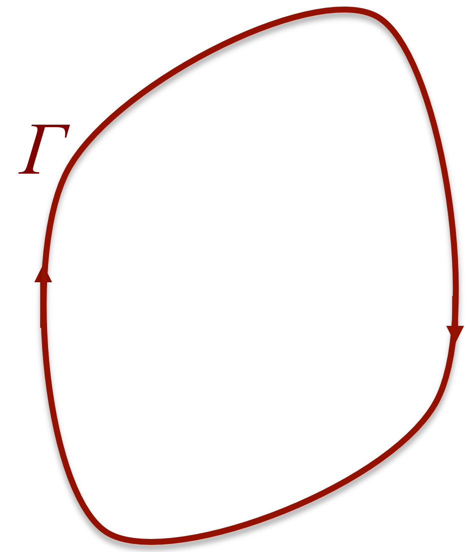
6. Quantization

Quantum mechanics

Consider

$$I[A,z] = e\oint_{\Gamma} A$$

where Γ *is a loop.*



Quantum mechanics is the requirement that this integral take integer values (maximum constructive interference):

$$\boxed{e\oint_{\Gamma} A = nh}$$

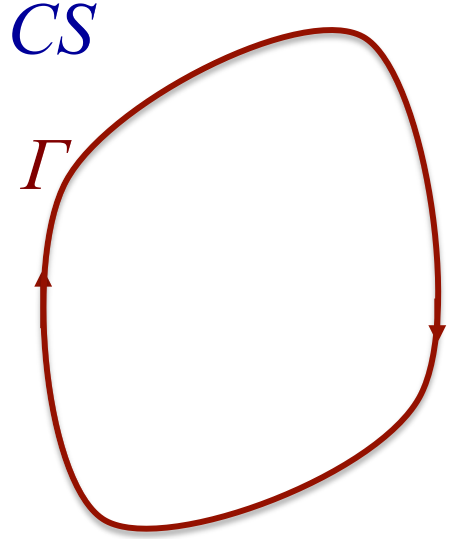
Aharonov-Bohm effect

Dirac's monopole quantization

Bohr-Sommerfeld quantization

Bohr-Sommerfeld quantization

Consider a mechanical system described by a CS lagrangian in a 2s+1 dimensional space.



$$\oint_{\Gamma} A = \int_{\Gamma} A_{\mu}(z)dz^{\mu} \;, \quad z^{\mu} = (z^0 = t\,,\, z^i\,)$$
$$= \int_{\Gamma} (A_0 dt + A_i dz^i\,),\;\; i = 1,\ldots,2s$$

If the orbit is periodic, one finds the B-S quantization condition

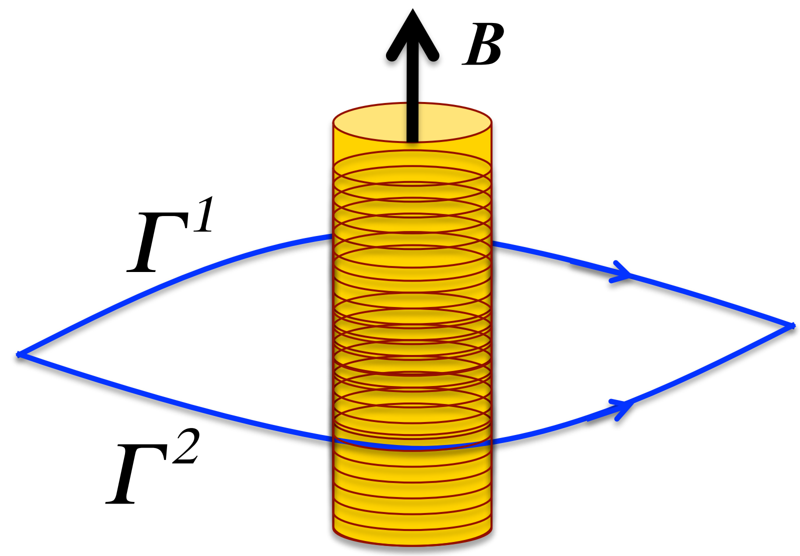
$$\oint_{\Gamma} A = \boxed{\oint_{\Gamma} [\,p_i dq^i - H(p,q)dt\,] = 2n\pi\hbar = nh}$$

Aharonov-Bohm effect

By Stoke's theorem,

$$e\int_{\Gamma^1} A - e\int_{\Gamma^2} A = e\int_M F$$

where $\Gamma^1 \bigcup (-\Gamma^2) = \partial M.$



Hence, for two paths enclosing a quantum of flux,

$$\boxed{e\int_M F = 2n\pi\hbar}$$

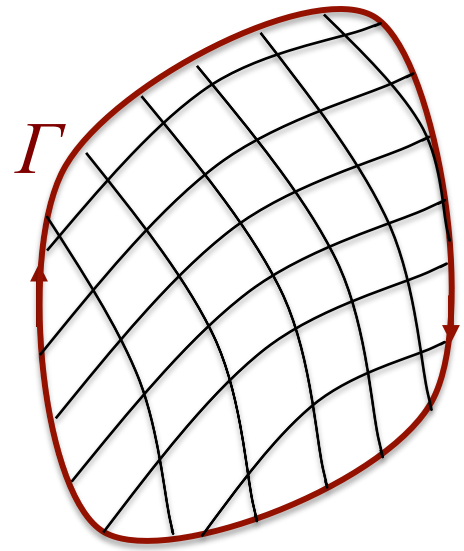
there is constructive interference (maximum).

Monopole quantization

By Stoke's theorem,

$$e\oint_{\Gamma} A = e\int_{M} F$$

where $F = dA$, *and* $\Gamma = \partial M$.



For a Dirac monopole, $F = \frac{g}{r^2} d^2 x$, *and therefore*

$$\boxed{eg = 2n\pi\hbar}$$

7. Summary

Summary

- *CS forms generalize the coupling between a point charge and the electromagnetic field, providing a natural gauge-invariant way to couple gauge fields (abelian or not) to extended sources:* $2p$*-branes.*

- *The simplest CS action can also describe an arbitrary classical mechanical system of finite degrees of freedom.*

- *Path integral quantization yields the Bohr-Sommerfeld quantum postulate, Dirac's monopole quantization and the Aharonov-Bohm effect.*

- *Mathematically, CS forms are related to topological invariants, the Chern characteristic classes.*

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We do not see how this helps to settle the Poincaré conjecture.