Gravity with Auxiliary Fields

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General Relativity: Lovelock Theorem

Lovelock Theorem

- *The most general 2-covariant divergence-free tensor, which is constructed solely from the metric $g_{\mu\nu}$ and its derivatives up to second differential order, is the Einstein tensor plus a cosmological constant term.*

- *This result suggests a natural route to Einstein's equations*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{1}$$

Gravity with Auxiliary Fields

We assume the theory admits a manifestly covariant Lagrangian formulation and that the equivalence principle holds:

$$L = L_g[\mathbf{g}, \phi] + L_M[\mathbf{g}, \psi]\,. \tag{2}$$

Field equations:

$$E_{ab}[\mathbf{g}, \phi] = T_{ab}\,, \tag{3}$$
$$\boldsymbol{\Phi}[\mathbf{g}, \phi] = 0\,. \tag{4}$$

Crucial assumption: it is possible to obtain an algebraic equation for ϕ

$$\mathcal{F}[\phi, \mathbf{g}, \mathbf{T}] = 0\,, \tag{5}$$

where ϕ *only appears at zeroth differential order.*

$\Rightarrow \phi$ **is an auxiliary field!**

Gravity with Auxiliary Fields

Trivial Case

The field equations imply an algebraic relation of the form

$$\mathbf{\Phi}[\phi, \mathbf{g}] = \mathcal{F}[\phi, \mathbf{g}] = 0 \,. \tag{6}$$

We can then eliminate the extra field from the eqs. and obtain

$$E_{ab}[\mathbf{g}] = T_{ab} \,. \tag{7}$$

Lovelock theorem guarantees that this tensor is the Einstein one plus a cosmological constant term.

$\Rightarrow$ **The gravitational theory is equivalent to GR!**

Gravity with Auxiliary Fields

General Case

If an algebraic equation for ϕ exists, it must be obtained by some combination of the field equations and of their derivatives.

The implicit solution for ϕ can be schematically written as

$$\phi = \phi[\mathbf{g}, \mathbf{T}] \,. \tag{8}$$

Then we can finally write the field equations for the metric as

$$E_{ab}[\mathbf{g}, \mathbf{T}] = T_{ab} \quad \Rightarrow \quad G_{ab} + \Lambda g_{ab} = T_{ab} + S_{ab}[\mathbf{g}, \mathbf{T}]. \tag{9}$$

Constructing S_{ab}

Properties of S_{ab}

- *Depends only on* **g**, **T** *and their derivatives*
- *Vanishes in vacuum*
- *Divergence-free*

$$\begin{aligned} S_{ab} &= \alpha_1\, g_{ab}\, T + \alpha_2\, g_{ab}\, T^2 + \alpha_3\, T\, T_{ab} + \alpha_4\, g_{ab}\, T_{cd}\, T^{cd} \\ &+ \alpha_5\, T^c{}_a\, T_{cb} + \beta_1\, \nabla_a \nabla_b\, T + \beta_2\, g_{ab}\, \Box\, T \\ &+ \beta_3\, \Box\, T_{ab} + 2\beta_4\, \nabla^c \nabla_{(a}\, T_{b)c} + \dots \end{aligned} \tag{10}$$

P. Pani, T. P. Sotiriou and D. Vernieri, arXiv:1306.1835v1 [gr-qc] (2013)

Relations Between the Coefficients

Using the derivative commutators and the eqs. at lowest order such as

$$(\Box\nabla_b - \nabla_b\Box)\,T = R_{ab}\nabla^a T$$

$$\sim T_{ab}\nabla^a T - \left(\alpha_1 + \frac{1}{2}\right) T\nabla_b T + \Lambda\nabla_b T + \ldots, \qquad (11)$$

it can be shown that there is a unique choice of the coefficients α_i *and* β_j *which makes* $\nabla^a S_{ab} = 0$ *identically, that is:*

$$\begin{aligned} \alpha_1 &= -\beta_1\Lambda, & 4\alpha_2 &= (1+2\alpha_1)(\beta_1-\beta_4)\,, \\ \alpha_3 &= \beta_4(1+2\alpha_1)-\beta_1, & 2\alpha_4 &= \beta_4\,, \\ \alpha_5 &= -2\beta_4\,, \quad \beta_2 = -\beta_1, & \beta_3 &= -\beta_4. \end{aligned} \qquad (12)$$

General Parametrization of the Field Equations

The field equations finally read

$$\begin{aligned} G_{ab} &= T_{ab} - \Lambda g_{ab} \\ &- \beta_1 \Lambda\, g_{ab}\, T + \frac{1}{4}\left(1 - 2\beta_1\Lambda\right)\left(\beta_1 - \beta_4\right) g_{ab}\, T^2 \\ &+ \left[\beta_4 \left(1 - 2\beta_1 \Lambda\right) - \beta_1\right] T\, T_{ab} + \frac{1}{2}\beta_4\, g_{ab}\, T_{cd}\, T^{cd} \\ &- 2\beta_4\, T^c{}_a\, T_{cb} + \beta_1 \nabla_a \nabla_b T - \beta_1\, g_{ab} \Box T \\ &- \beta_4 \Box T_{ab} + 2\beta_4 \nabla^c \nabla_{(a} T_{b)c} + \ldots, \end{aligned} \tag{13}$$

where all the coefficients have been expressed in terms of Λ, β_1 *and* β_4.

Particular Cases

- *It is easy to show that EiBI gravity in the small coupling limit is a particular case of what derived, with*

$$\beta_1 = 0\,, \qquad \beta_4 = -\kappa/2\,. \tag{14}$$

- *On the other hand, the case of Brans-Dicke theory with $\omega_0 = -3/2$ simply corresponds to $\beta_4 = 0$ with Λ and β_1 being related to the model parameters.*

- *It is interesting to note that these two particular cases are representative of two "orthogonal" classes of corrections.*

Newtonian Limit

In order to derive the Newtonian limit of the theory, for simplicity we set $\Lambda = 0$, *since it is a higher post-Newtonian order contribution.*

In the limit of small velocities and small matter fields we have:

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} \,, \tag{15}$$

$$T_{ab} = \epsilon \rho \delta^0_a \delta^0_b \,. \tag{16}$$

We can use the gauge freedom to impose

$$\epsilon \partial^a \left(h_{ab} - \frac{1}{2} \eta_{ab} h \right) = \zeta \partial_b T \,. \tag{17}$$

Note that the gauges for two different matter configurations would differ from each other inside matter but different gauges are equivalent in vacuum.

Newtonian Limit

Field equations to first order in ϵ

$$-\frac{\nabla^2 h_{00}}{2} = \frac{\rho}{2} + \frac{1}{2}\beta_- \nabla^2 \rho \,, \tag{18}$$

$$-\frac{\nabla^2 h_{ij}}{2} = \frac{\delta_{ij}}{2}\left[\rho - \beta_+ \nabla^2 \rho\right] - \left[\beta_1 - \zeta\right] \partial_i \partial_j \rho \,, \tag{19}$$

$$\nabla^2 h_{0i} = 0 \,, \tag{20}$$

where $\beta_\pm = \beta_1 \pm \beta_4$.

In general $h_{00} \neq h_{ii}$*, and there is an off-diagonal component* $h_{ij} \neq 0$*. However it can be set to zero by choosing* $\zeta = \beta_1$ *in the gauge-fixing choice (in the standard Einstein-Fock gauge* $\zeta = 0$*).*

Newtonian Limit

Solutions in the gauge $\zeta = \beta_1$

$$h_{00}(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} - \beta_- \rho(\vec{x}) \,, \tag{21}$$

$$h_{ij}(\vec{x}) = \delta_{ij} \int d^3x' \frac{\rho(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} + \beta_+ \rho(\vec{x})\delta_{ij} \,, \tag{22}$$

$$h_{0i}(\vec{x}) = 0 \,. \tag{23}$$

The only possibility to avoid second-order derivatives of ρ in the Newtonian equations (and to avoid the contributions proportional to ρ in the solutions) is setting $\beta_1 = \beta_4 = 0$, i.e. standard Newtonian gravity.

Viability Constraints

Let us compute the acceleration $\vec{a} = \nabla h_{00}$ *experienced within a thin layer in the interior and close to the surface of a spherical object with Newtonian mass* M *and radius* R_s. *It reads*

$$\vec{a} = \vec{a}_N - \beta_- \nabla \rho \,. \tag{24}$$

We model the density profile in the layer of width $L \ll R$, *as*

$$\rho(r) = \rho_0 \left(\frac{R - r}{L} \right)^n , \qquad R - L < r < R \tag{25}$$

where n *parametrizes the slope of the profile.*

Viability Constraints

The fact that the density can abruptly change in a region of width L produces strong extra forces in this class of theories.
Using that profile, in the region $R - L < r < R$ we find:

$$\frac{a}{a_N} = 1 + \frac{3n}{4\pi R_s L}\beta_- \left[(R_s - r)/L\right]^{n-1} . \tag{26}$$

In order not to affect the standard Newtonian force to measurable levels in tabletop experiments, the last term on the r.h.s. must be much smaller than unity.
Evaluating the force at $r \sim R - L$ we obtain the constraint

$$(\beta_1 - \beta_4) \ll \frac{4\pi R_s L}{3n} . \tag{27}$$

Viability Constraints

Similar abrupt changes in the density profile naturally occur at the interface of any macroscopic object. Assuming $R_s \sim$ m and $L \sim \mu$m, we get the following constraint for the characteristic lenght scale λ_β:

$$\lambda_\beta \ll n^{-1/2}\,\mathrm{mm}\,. \tag{28}$$

Consider that the **Hubble radius squared is roughly $\Lambda^{-1} \sim 10^{52}\mathrm{m}^2$**!

If these theories were to pass sensible macroscopic constraints, they have to accomodate at least two typical lenght scales, λ_β and the effective cosmological constant scale, which differ for many orders of magnitude.

Conclusions

- *Auxiliary fields in the gravitational action.*
- *Two possibilities: GR or higher order derivatives of the matter fields!*
- *General parametrization of theories with auxiliary fields: Two parameters control all of the possible theories!*
- *Newtonian limit and general results.*
- *Viability constraints: if passed, λ_β^2 and Λ^{-1} differ for many orders of magnitude.*
- *These theories are unlikely to play any role at large scales!*

First Example: Brans-Dicke Theory

The action for Brans-Dicke theory with $\omega_0 = -3/2$ *is*

$$S[g,\phi,\psi_m] = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(\phi R + \frac{3}{2\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) + S_M(g_{\mu\nu},\psi), \tag{29}$$

whose independent variation with respect to the metric and the scalar yields to the field eqs.

$$\begin{aligned} G_{\mu\nu} &= \frac{\kappa}{\phi}T_{\mu\nu} - \frac{3}{2\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi\right) \\ &+ \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) - \frac{V(\phi)}{2\phi}g_{\mu\nu}, \end{aligned} \tag{30}$$

$$\Box\phi = \frac{\phi}{3}(R - V'(\phi)) + \frac{1}{2\phi}\nabla^\mu\phi\nabla_\mu\phi. \tag{31}$$

First Example: Brans-Dicke Theory

- *We can use the trace of the first eq. to eliminate R in the second one. The outcome is*

$$2V(\phi) - \phi V'(\phi) = \kappa\, T. \tag{32}$$

- *Once $V(\phi)$ is assigned, the scalar field can be algebraically solved as $\phi = \phi(T)$, then it is a non-dynamical field.*

- *Substituting it back into the field eqs. one gets*

Higher order derivatives of the matter fields!

Second Example: EiBI Gravity

EiBI gravity is described by the following action

$$S[g,\Gamma,\psi_m] = \frac{1}{4\pi G\kappa}\int d^4x\left(\sqrt{|\det\left(g_{ab}+\kappa\mathcal{R}_{(ab)}\right)|} -(1+\kappa\Lambda)\sqrt{g}\right) + S_M\left[g_{ab},\psi_M\right] \,. \quad (33)$$

Independent variation of the action with respect to the metric and the connection yields after some manipulations to

$$\Gamma^c_{ab} = \frac{1}{2}q^{cd}\left(\partial_a q_{bd}+\partial_b q_{ad}-\partial_d q_{ab}\right)\,, \quad (34)$$

$$q^{ab} = \frac{(1+\kappa\Lambda)g^{ab}-\kappa T^{ab}}{\sqrt{g}\sqrt{\det\left((1+\kappa\Lambda)g^{ab}-\kappa T^{ab}\right)}}\,, \quad (35)$$

where $q_{ab}\equiv g_{ab}+\kappa\mathcal{R}_{(ab)}$.

Second Example: EiBI Gravity

The connection Γ^c_{ab} is not a dynamical field!
The field eqs. can be expanded at first order in κ by noting that

$$q^{ab} = g^{ab} - \kappa\tau^{ab} + \mathcal{O}(\kappa^2)\,, \tag{36}$$

where $\tau_{ab} \equiv T_{ab} - \frac{1}{2}g_{ab}T + \Lambda g_{ab}$. One finally gets a single equation for the metric g_{ab} only:

$$\begin{aligned} R_{ab} &= \Lambda g_{ab} + T_{ab} - \frac{1}{2}Tg_{ab} + \kappa\left[S_{ab} - \frac{1}{4}Sg_{ab}\right] \\ &+ \frac{\kappa}{2}\left[\nabla_a\nabla_b\tau - 2\nabla^c\nabla_{(a}\tau_{cb)} + \Box\tau_{ab}\right] + \mathcal{O}(\kappa^2)\,, \end{aligned} \tag{37}$$

where $S_{ab} = T^c{}_a T_{cb} - \frac{1}{2}TT_{ab}$.

M. Bañados and P. G. Ferreira, Phys. Rev. Lett. **105**, *011101 (2010)*
P. Pani and T. P. Sotiriou, Phys. Rev. Lett. **109**, *251102 (2012)*

Constructing S_{ab}

Why not terms with Ricci or Riemann contracted with T_{ab}, like

$$R_{ab}T\ ,\ \ RT_{ab}\ ,\ \ R_{ac}T^{c}{}_{b}\ ,\ \ \ldots\ ,\ R_{abcd}T^{bc}\ ? \tag{38}$$

- *R_{ab} and R can be calculated from the field eqs. and substitued above to give terms already present in the field eqs.*
- *For the term with R_{abcd} let us use the tensorial relation*

$$\nabla^{a}\nabla_{d}T_{ab} = R_{ed}T^{e}{}_{b} + R_{becd}T^{ec}, \tag{39}$$

which can be symmetrized giving

$$R_{becd}T^{ec} = \nabla^{a}\nabla_{(d}T_{b)a} - R_{e(d}T_{b)}{}^{e}. \tag{40}$$

These terms are already present into the field eqs.!

Asking for Divergence-Free Field Equations

Then, taking the divergence of both sides of the field eqs. we get:

$$\begin{aligned} 0 &= \alpha_1 \nabla_b T + 2\,\alpha_2\, T\,\nabla_b T + \alpha_3\, T_{ab}\,\nabla^a T + 2\,\alpha_4\, T^{cd}\,\nabla_b T_{cd} \\ &+ \alpha_5\, T^c{}_a\,\nabla^a T_{cb} + \beta_1 \Box \nabla_b T + \beta_2 \nabla_b \Box T \\ &+ \beta_3 \nabla^a \Box T_{ab} + 2\beta_4 \nabla^a \nabla^c \nabla_{(a} T_{b)c}. \end{aligned} \tag{41}$$

The various terms are not independent from each others, being related through the derivative commutators!

A Few Tensorial Algebra

With a few algebra one can obtain the following tensorial relations

$$\left(\Box\,\nabla_b - \nabla_b\,\Box\right) T = R_{ab}\nabla^a\, T, \tag{42}$$

$$\left(\nabla^a\,\nabla^c\,\nabla_a - \nabla^c\,\Box\right) T_{cb} = R_{abcd}\nabla^d\, T^{ca}, \tag{43}$$

$$\left(\nabla_c\,\nabla_d - \nabla_d\,\nabla_c\right) T_{ab} = R_{aecd}\, T^e{}_b + R_{becd}\, T_a{}^e\,, \tag{44}$$

and one can use the Bianchi identity, suitably contracted as

$$\nabla^a R_{abcd} = 2\nabla_{[c} R_{d]b}. \tag{45}$$