Higher-Spin Gauge Theories

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Plan

Lecture I

I Properties of HS theory

II Higher-spin algebra

Lecture II

III Nonlinear HS Theory

IV Space-time metamorphoses

V Conclusion

Symmetries

HS gauge theory: theory of maximal symmetries

Usual lower-spin symmetries

- **Relativistic theories: Poincaré symmetry:**

$\delta x^a = \varepsilon^a + \varepsilon^a{}_b x^b$ $\quad \varepsilon^a$: **translations;** ε^{ab} : **Lorentz rotations**

Lie algebra: $\delta x^a = [T, x^a]\,, \qquad T = \varepsilon^n P_a + \varepsilon^{ab} M_{ab}$

$$P_a = \frac{\partial}{\partial x^a}\,, \qquad M_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a}$$

$$[M_{ab}, P_c] \;=\; P_a \eta_{bc} - P_b \eta_{ac}$$

$$[M_{ab},\, M_{cd}] = M_{ad}\eta_{bc} - M_{bd}\eta_{ac} - M_{ac}\eta_{bd} + M_{bc}\eta_{ad}$$

$$[P_a\,, P_b] = 0$$

$(A)dS$ deformation

$$[P_a \, , P_b] = \Lambda \mathbf{M_{ab}}$$

$\Lambda < 0$: AdS, $o(d-1,2)$

$\Lambda > 0$: dS, $o(d,1)$

$\Lambda = 0$: **Minkowski space,** $iso(d-1,1)$

- **SUSY**

$$P_a, M_{ab} \longrightarrow P_a, M_{ab}, \mathbf{Q}_\alpha \,, \qquad \alpha = 1,2,3,4$$

$$\{Q_\alpha \,, Q_\beta\} = \gamma^a_{\alpha\beta} P_a$$

$$[M_{ab} \,, Q_\alpha] = \sigma_{ab\alpha}{}^\beta Q_\beta \,, \qquad \sigma_{ab} = \frac{1}{4}[\gamma_a \,, \gamma_b]$$

- **Inner symmetries: generators T_i are space-time invariant**

$$[T_i \,, (P_a, M_{ab})] = 0$$

Standard Model: $T_i \sim SU(3) \times SU(2) \times U(1)$

- **Conformal (super)symmetries**

Local Symmetries

Useful viewpoint: any global symmetry is the remnant of a local symmetry with parameters like $\varepsilon^a(x), \varepsilon^{ab}(x), \varepsilon^\alpha(x), \varepsilon^i(x)$ being arbitrary functions of space-time coordinates

Local symmetries are symmetries of the full theory

Global symmetries are symmetries of its particular solution

Example:

Infinitesimal diffeomorphisms $\delta x^a = \varepsilon^a(x)$ are symmetries of GR

Global symmetry with $\varepsilon^a(x) = \varepsilon^a + \varepsilon^a{}_b x^b$ are symmetries of the Minkowski solution $g_{ab} = \eta_{ab}$ of Einstein equations

Gauge fields

Let

$$S = \int_{M^d} L(\varphi(x), \partial_a \varphi(x), \dots)$$

be invariant under a global symmetry g with parameters ε^n $(n = a, \alpha, i, \dots)$

For $\varepsilon^n(x)$

$$\delta S = - \int_{M^d} J^a_n(\varphi) \partial_a \varepsilon^n(x)$$

$J^a_n(\varphi)$ are conserved currents since $\partial_a J^a_n(\varphi) \cong 0$ by virtue of field equations

To achieve local symmetry introduce gauge fields A^n_a

$$\delta A^n_a = \partial_a \varepsilon^n + \dots$$

$$S \longrightarrow S + \Delta S + \dots\,, \qquad \Delta S = \int_{M^d} J^a_n(\varphi) A^n_a(x)$$

ΔS: Noether current interaction.

Subtlety

If $\varphi(x)$ were gauge fields with gauge parameters ε', $J^a_n(\varphi)$ may not be invariant under the ε' symmetry

Noether current interaction for several gauge fields may be obstructed

Inner symmetry:

Yang-Mills fields - spin 1

$$A_a(x) = A_a^i(x)T_i\,, \qquad \varepsilon(x) = \varepsilon^i(x)T_i$$

$$\delta A_a(x) = D_a\varepsilon(x)\,, \qquad D_a\varepsilon(x) = \partial_a\varepsilon(x) + [A_a(x)\,,\varepsilon(x)]$$

$$[D_a\,,D_b] = R_{ab}\,, \qquad R_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]\,, \qquad \delta R_{ab} = g[R_{ab},\varepsilon]$$

Poincaré symmetry:

Cartan gravity - spin 2 $A^n_\nu = (e_\nu{}^a, \omega_\nu{}^{ab})$

$e_\nu{}^a$ relates indices ν with a identified in Minkowski space at $e_\nu{}^a = \delta^a_\nu$

Gauge transformation

$$\delta e_\nu{}^a(x) = \partial_\nu \varepsilon^a(x) + \omega_\nu{}^a{}_b(x)\varepsilon^b(x) - \varepsilon^a{}_b(x) e_\nu{}^b(x) + \Delta e_\nu{}^a$$

$$\delta \omega_\nu{}^{ab}(x) = \partial_\nu \varepsilon^{ab}(x) + \omega_\nu{}^a{}_c(x)\varepsilon^{cb}(x) - \omega_\nu{}^b{}_c(x)\varepsilon^{ca}(x) + \Delta\omega_\nu{}^{ab}$$

$\Delta e_\nu{}^a$, $\Delta\omega_\nu{}^{ab}$ corrections to YM transformation proportional to curvatures

$$R_{\nu\mu}{}^a = \partial_\nu e_\mu{}^a + \omega_\nu{}^a{}_b e_\mu{}^b - (\nu \leftrightarrow \mu)\,, \qquad R_{\nu\mu}{}^{ab} = \partial_\nu \omega_\mu{}^{ab} + \omega_\nu{}^a{}_c \omega_\mu{}^{cb} - (\nu \leftrightarrow \mu)$$

$R_{\nu\mu}{}^a = 0 \ \rightarrow \omega = \omega(e, \partial e)\,,$ $\quad R_{\nu\mu}{}^{\rho\sigma}$: Riemann tensor

Metric $g_{\nu\mu} = e_\nu{}^a e_\mu{}^b \eta_{ab}$

SUSY

SUGRA: spin 3/2 gauge field gravitino

$$\delta\psi_{\nu\alpha} = D_\nu \varepsilon_\alpha + \ldots$$

Spontaneous symmetry breaking

Equations of motion are G-invariant

while a solution that describes our world is not

Higgs: $H^i(x) = H_0^i + h^i(x)$

Unbroken part $\tilde{G} \subset G$ is a leftover symmetry of H_0^i

$\tilde{G} = SU(3) \times U(1)$ in SM

If H_0^i has non-zero dimension $[H_0^i] = cm^{-1} \sim GeV$

spontaneous symmetry breaking is a low-energy effect

symmetry restoration at $E > H_0^i$

In the unbroken regime, gauge fields associated with usual lower-spin symmetries describe massless particles

$s = 1$: $A_\nu{}^i$

$s = 3/2$: $\psi_{\nu\,\alpha}$

$s = 2$: $e_\nu{}^a \omega_\nu{}^{ab}$

General Features of HS Theories

Key question: is it possible to go to larger HS symmetries?

What are HS symmetries and HS counterparts of lower-spin theories including GR?

What are physical motivations for their study and possible outputs?

Fronsdal fields

All $m = 0$ HS fields are gauge fields

$\varphi_{\nu_1 \ldots \nu_s}$ **is a rank s symmetric tensor obeying** $\varphi^{\rho}{}_{\rho}{}^{\mu}{}_{\mu \nu_5 \ldots \nu_s} = 0$

Gauge transformation:

$$\delta \varphi_{\nu_1 \ldots \nu_s} = \partial_{(\nu_1} \varepsilon_{\nu_2 \ldots \nu_s)} \, , \qquad \varepsilon^{\mu}{}_{\mu \nu_3 \ldots \nu_{s-1}} = 0$$

Field equations: $G_{\nu_1 \ldots \nu_s}(x) = 0 \qquad G_{\nu_1 \ldots \nu_s}(x)$: **Einstein-like tensor**

$$G_{\nu_1 \ldots \nu_s}(x) = \Box \varphi_{\nu_1 \ldots \nu_s}(x) - s \partial_{(\nu_1} \partial^{\mu} \varphi_{\nu_2 \ldots \nu_s \mu)}(x) + \frac{s(s-1)}{2} \partial_{(\nu_1} \partial_{\nu_2} \varphi^{\mu}{}_{\nu_3 \ldots \nu_s \mu)}(x)$$

Action

$$S = \int_{M^d} \left(\frac{1}{2} \varphi^{\nu_1 \ldots \nu_s} G_{\nu_1 \ldots \nu_s}(\varphi) - \frac{1}{8} s(s-1) \varphi_{\mu}{}^{\mu \, \nu_3 \ldots \nu_s} G^{\rho}{}_{\rho \, \nu_3 \ldots \nu_s}(\varphi) \right)$$

No-go and the role of $(A)dS$

In 60th it was argued (Weinberg, Coleman-Mandula) that

HS symmetries cannot be realized in a nontrivial local field theory in Minkowski space

In 70th it was shown by Aragone and Deser that HS gauge symmetries are incompatible with GR if expanding around Minkowski space

Green light: AdS background with $\Lambda \neq 0$ Fradkin, MV, 1987

In agreement with no-go statements the limit $\Lambda \to 0$ is singular

HS Symmetries versus Riemann geometry

HS symmetries do not commute to space-time symmetries

$$[T^a, T^{HS}] = T^{HS}, \qquad [T^{ab}, T^{HS}] = T^{HS}$$

HS transformations map gravitational fields (metric) to HS fields

Consequence:

Riemann geometry is not appropriate for HS theory:

concept of local event may become illusive!

Differential forms: coordinate independence without metric

Differential forms are totally antisymmetric tensors

p-form: $\omega(x) = \theta^{\nu_1} \dots \theta^{\nu_p} \omega_{\nu_1 \dots \nu_p}(x)$

$$\theta^\nu \theta^\mu = -\theta^\mu \theta^\nu \,, \qquad (\theta^\nu = dx^\nu)$$

Invariant differentiation is provided by de Rham differential

$$d = \theta^\nu \frac{\partial}{\partial x^\nu} \,, \qquad d^2 = 0$$

Due to total antisymmetrization symmetric Christoffel symbols drop out

Connections $A = \theta^\nu A^i_\nu T_i$ are one-forms

Curvatures $R = D^2 \,,\ D = d + A$ are two-forms

Elaboration of this language in HS theory leads to new understanding of fundamental concepts of space-time including its dimension

HS Gauge Theory and Quantum Gravity

HS symmetry is in a certain sense maximal relativistic symmetry. Hence, it cannot result from spontaneous breakdown of a larger symmetry: HS symmetries are manifest at ultrahigh energies above any scale including Planck scale

- HS gauge theory should capture effects of Quantum Gravity: restrictive HS symmetry versus unavailable experimental tests
- Lower-spin theories as low-energy limits of HS theory: lower-spin symmetries: subalgebras of HS symmetry

HS theory and String theory

- String Theory as spontaneously broken HS theory?! $(s > 2, m > 0)$

  Recent conjecture (Chang, Minwalla, Sharma and Yin (2012)):

  String Theory = Quantum HS theory?!

HS AdS/CFT correspondence

AdS_4 HS theory is dual to $3d$ vectorial conformal models Klebanov- Polyakov (200

Giombi and Yin (2009)

AdS_3/CFT_2 correspondence Gaberdiel and Gopakumar (2010)

Analysis of HS holography helps to uncover the origin of AdS/CFT

Global HS Symmetry

HS symmetry in AdS_{d+1}:

Maximal symmetry of a d-dimensional free conformal field(s)=singletons

usually, scalar and/or spinor

Consider KG massless equation in Minkowski space

$$\Box C(x) = 0\,, \qquad \Box = \eta^{ab}\frac{\partial^2}{\partial x^a \partial x^b}$$

What are symmetries of KG equation? Shaynkman, MV 2001 $3d$; Eastwood 2002 $\forall d$

i **Poincaré**

ii **Scale transformation (dilatation)**

$$\delta C(x) = \varepsilon D C(x)\,, \qquad D = x^a \frac{\partial}{\partial x^a} + \frac{d}{2} - 1$$

iii **Special conformal transformations**

$$\delta C(x) = \varepsilon_a K^a C(x)\,, \qquad K^a = (x^2 \eta^{ab} - 2x^a x^b)\frac{\partial}{\partial x^b} + (2-d)x^a$$

Problem 1.1. Check invariance

Problem 1.2. Check: P_a, M_{ab}, K^a, D form a Lie algebra (conformal algebra)

Problem 1.3. Check that conformal algebra is $o(d, 2)$

Auxiliary problem

Consider equations

$$D\mathcal{C}_A(x) = 0\,, \qquad D^2 = 0 \tag{1}$$

$$D = d + \omega(x)\,, \qquad \omega_A{}^B(x) = \omega^\Omega(x) T_{\Omega\,A}{}^B$$

$\omega(x)$: **flat connection on the space V of $\mathcal{C}_A$**

(1) is invariant under the gauge transformation

$$\delta\mathcal{C}(x) = -\varepsilon(x)\mathcal{C}(x)\,, \qquad \varepsilon_A{}^B(x) = \varepsilon^\Omega(x) T_\Omega(x)$$

$$\delta\omega(x) = D\varepsilon(x) := d\varepsilon(x) + \omega(x)\varepsilon(x) - \varepsilon(x)\omega(x)$$

Problem 1.4. Check

Global symmetry parameters

For a particular $\omega(x) = \omega_0(x)$, **to keep the equations invariant demands**

$$\delta_0 \omega(x) \longrightarrow D_0 \varepsilon^{\Omega}_{gl}(x) = 0$$

Since $D_0^2 = 0$, $\varepsilon^{\Omega}_{gl}(x)$ **is reconstructed (locally) in terms of** $\varepsilon^{\Omega}(x_0)$ $\forall x_0$

$\varepsilon^{\Omega}(x_0)$: **global symmetry parameters of** $D_0 \mathcal{C}(x) = 0$

Solution: $\omega_0(x) = g^{-1}(x) dg(x)$, $\mathcal{C}(x) = g^{-1}(x)\mathcal{C}$. **For** $g(x_0) = 1$, $\mathcal{C} = \mathcal{C}(x_0)$

Massless scalar unfolded

Minkowski space: $\omega(x) = e^a(x)P_a + \omega^{ab}(x)M_{ab}$

Cartesian coordinates: $\omega^{ab} = 0$, $e^a(x) = \theta^a$

Introduce an infinite set of 0-forms

$$C_{a_1\dots a_n}(x) = C_{(a_1\dots a_n)}(x)\,, \quad \eta^{bc}C_{bca_3\dots a_n}(x) = 0$$

Unfolded KG equation

$$dC_{a_1\dots a_n}(x) = \theta^b C_{a_1\dots a_n b}(x)$$

This system is consistent since $\theta^b \wedge \theta^c = -\theta^c \wedge \theta^b$

First two equations imply

$$\partial_a C(x) = C_a(x)\,, \qquad \partial_a C_b(x) = C_{ab}(x) \longrightarrow C_{ab}(x) = \partial_a\partial_b C(x)$$

Tracelessness of $C_{nm}(x)$:

$$\Box C(x) = 0.$$

All other equations:

$$C_{a_1\dots a_n}(x) = \partial_{a_1}\dots\partial_{a_n}C(x)$$

$C_{a_1\dots a_n}(x)$: set of all on-mass-shell nontrivial derivatives of $C(x)$

Conformal HS algebra

Conformal HS algebra in d dimensions: algebra of linear transformations of the infinite-dimensional space V of various traceless symmetric tensors $C, C_a, C_{ab} \ldots$, i.e., $h = gl(V)$

h was carefully defined by Eastwood in 2002 by different methods

Algebraic construction simplifies in $d = 3$ using spinor formalism most relevant in the context of AdS_4/CFT_3 HS holography Shaynkman, MV (2001)

$3d$ multispinors

$3d$ **Lorentz algebra:** $o(2,1) \sim sp(2,R) \sim sl_2(R)$. $3d$ **spinors are real**

$$\chi^\dagger_\alpha = \chi_\alpha \,, \qquad \alpha = 1,2$$

$sp(2,R)$ **invariant tensor** $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ **relates lower and upper indices**

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta \,, \qquad \chi_\alpha = \chi^\beta \epsilon_{\beta\alpha}$$

Antisymmetrization of $3d$ spinor indices is equivalent to contraction

$$A_{\alpha,\beta} - A_{\beta,\alpha} = \epsilon_{\alpha\beta} A_{\gamma,}{}^{\gamma}$$

IRREPS of Lorentz algebra: totally symmetric multispinors $A_{\alpha_1\ldots\alpha_n}$

Consequence:

$$A_{a_1\ldots a_m} \sim A_{\alpha_1\ldots\alpha_{2m}} \,, \qquad A^b{}_{ba_3\ldots a_m} = 0$$

Problem 1.5. Prove by checking the number of independent components

Explicit map via 2×2 **real symmetric matrices**

$$A_{\alpha\beta} = \sigma^n_{\alpha\beta} A_n \,, \qquad \sigma^n_{\alpha\beta} = \sigma^n_{\beta\alpha}$$

Spinorial form of $3d$ massless equations

Space V of all $3d$ traceless symmetric tensors is the space of (even) functions of commuting spinor variable y^α

$$C(y|x) = \sum_{n=0}^{\infty} C^{\alpha_1 \dots \alpha_{2n}}(x) y_{\alpha_1} \dots y_{\alpha_{2n}}$$

Unfolded massless equations take the form

$$\theta^{\alpha\beta} \left(\frac{\partial}{\partial x^{\alpha\beta}} + \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \right) C(y|x) = 0 \qquad (2)$$

Problem 1.6. Check

Problem 1.7. Check that for odd $C(-y|x) = -C(y|x)$ (2) describes $3d$ massless spinor field $C_\alpha(x) = \frac{\partial}{\partial y^\alpha} C(y|x)\Big|_{y=0}$

$3d$ HS symmetry

$3d$ conformal HS algebra is the algebra of various differential operators $\epsilon(y, \frac{\partial}{\partial y})$ obeying $\epsilon(-y, -\frac{\partial}{\partial y}) = \epsilon(y, \frac{\partial}{\partial y})$

$$\delta C(y|x) = \epsilon(y, \frac{\partial}{\partial y}|x) C(y|x)$$

$$\epsilon(y, \frac{\partial}{\partial y}|x) = \exp\left[-x^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right]\epsilon_{gl}(y, \frac{\partial}{\partial y}) \exp\left[x^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right]$$

Problem 1.8. Check

For any polynomial $\epsilon_{gl}(y, \frac{\partial}{\partial y})$, $\epsilon(y, \frac{\partial}{\partial y}|x)$ is polynomial as well: polynomial. $\epsilon_{gl}(y, \frac{\partial}{\partial y})$ describe local HS transformations

Conformal subalgebra

$3d$ **Conformal algebra** $sp(4) \sim o(3,2)$

$$P_{\alpha\beta} = \frac{\partial^2}{\partial y^\alpha \partial y^\beta}\,, \qquad K^{\alpha\beta} = y^\alpha y^\beta\,, \qquad M_{\alpha\beta} = y_\alpha \frac{\partial}{\partial y^\beta} + y_\beta \frac{\partial}{\partial y^\alpha}\,, \qquad D = y^\alpha \frac{\partial}{\partial y^\alpha} + 1$$

Problem 1.9. Check that P, K, M, D form closed algebra

Problem 1.10. * Derive conformal transformations of $C(x)$

Weyl algebra

Weyl algebra A_n: associative algebra of polynomials of oscillators $\hat{Y}_A$

$$[\hat{Y}_A\,,\hat{Y}_B]=C_{AB}\,,\qquad A,B,\ldots=1,\ldots 2n\,,\qquad C_{AB}=-C_{BA}$$

$3d$ CHS algebra $=$ AdS_4 HS algebra is (even part of) Weyl algebra A_2

$$\hat{Y}_A=\begin{pmatrix} y^\alpha \\ \frac{\partial}{\partial y^\beta}\end{pmatrix}$$

Symbols of operators

$$\hat{f}(\hat{Y}) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{A_1 \dots A_n} \hat{Y}_{A_1} \dots \hat{Y}_{A_n}\,, \qquad \text{symmetric } f^{A_1 \dots A_n}$$

Weyl symbol $f(Y)$ of the operator $\hat{f}(\hat{Y})$ is a function of commuting variables Y^A of the same expansion

$$f(Y) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{A_1 \dots A_n} Y_{A_1} \dots Y_{A_n}$$

Y_A **is the Weyl symbol of** $\hat{Y}_A$.

Star-product

Weyl star–product algebra is defined by the rule

$(f * g)(Y)$ **is a symbol of** $\hat{f}(\hat{Y})\hat{g}(\hat{Y})$. **In particular,**

$$[Y_A, Y_B]_* = 2iC_{AB}\,, \qquad [a\,,b]_* = a*b - b*a$$

Problem 1.11. Prove

$$\{Y_A\,, f(Y)\}_* = 2Y_A f(Y)\,, \qquad [Y_A\,, f(Y)]_* = 2i\frac{\partial}{\partial Y^A} f(Y)\,, \qquad Y^A = C^{AB} Y_B$$

Weyl-Moyal formula

$$(f_1 * f_2)(Y) = f_1(Y)\,\exp\,[i\overleftarrow{\partial^A}\overrightarrow{\partial^B} C_{AB}]\, f_2(Y)\ , \qquad \partial^A \equiv \frac{\partial}{\partial Y_A}$$

Problem 1.12. Prove using Campbell-Hausdorf formula for $\exp J^A \hat{Y}_A$

Important properties

- **associativity:** $(f * g) * h = f * (g * h)$
- **regularity:** **star product of any two polynomials of Y is a polynomial**

Integral representation

$$(f_1 * f_2)(Y) = \frac{1}{\pi^{2M}} \int dS dT\ \exp(-iS_A T_B C^{AB}) f_1(Y+S)\, f_2(Y+T)$$

Properties of HS algebras

Global symmetry of symmetric vacuum of bosonic HS theory

Let T_s be a homogeneous polynomial of degree $2(s-1)$

$$[T_{s_1}, T_{s_2}] = T_{s_1+s_2-2} + T_{s_1+s_2-4} + \ldots + T_{|s_1-s_2|+2}\,.$$

Once spin $s > 2$ appears, the HS algebra contains an infinite tower of higher spins: $[T_s, T_s]$ gives rise to T_{2s-2} as well as T_2 of $o(3,2) \sim sp(4)$.

Usual symmetries: spin-$s \leq 2$ $u(1) \oplus o(3,2)$: maximal finite-dimensional subalgebra of $hu(1,0|4)$. $u(1)$ is associated with the unit element.

$4d$ HS systems

Three series of $4d$ **HS algebras:** $hu(n,m|4)$, $ho(n,m|4)$, $husp(2n,2m|4)$

Spin-one YM sector:

$g = u(n) \oplus u(m), o(n) \oplus o(m)$ **or** $usp(2n) \oplus usp(2m)$

fermions: bifundamental.

Odd spins: adjoint representation of g.

Even spins: the opposite symmetry second rank representation of g,

Particle spectrum always contains a singlet for

colorless graviton and colorless scalar

$ho(1,0|4)$ **is minimal HS algebra: even spins** $s = 0, 2, 4, 6, \ldots$

Colorless scalar is the prediction of HS symmetry!

Some Reviews

MV, hep-th/0401177; 9910096; 9611024

X. Bekaert, S. Cnockaert, C. Iazeolla and MV, hep-th/0503128

D. Sorokin, arXiv:hep-th/0405069

A. Fotopoulos and M. Tsulaia, 0805.1346

X. Bekaert, N.Boulanger and P.Sundell, 1007.0435

M. R. Gaberdiel and R. Gopakumar, 1207.6697

S. Giombi and X. Yin, 1208.4036

Lecture II

HS theory in AdS_4 and space-time metamorphoses

III Nonlinear HS Theory

IV Space-time metamorphoses

Summary of Lecture I

General consequences of HS symmetry

$3d$ conformal HS algebra is even part of Weyl algebra A_2 of functions

$$f(Y) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{A_1 \dots A_n} Y_{A_1} \dots Y_{A_n}$$

of oscillators

$$[Y_A\,, Y_B]_* = 2iC_{AB}\,, \qquad A = 1, 2, 3, 4$$

AdS_4 HS algebra coincides with $3d$ conformal HS algebra

Spinorial langauge in four dimensions

Key fact $2 \times 2 = 4$

Minkowski coordinates as 2×2 **hermitian matrices**

$$X^n \Rightarrow X^{\alpha\dot\alpha} = \sum_{n=0}^{3} X^n \sigma_n^{\alpha\dot\alpha}\,, \qquad \sigma_n^{\alpha\dot\alpha} = (I^{\alpha\dot\alpha}, \vec\sigma_k^{\alpha\dot\alpha})$$

$I^{\alpha\dot\alpha}$: **unit matrix**

$\vec\sigma_k^{\alpha\dot\alpha}$, $k = 1,2,3$: **Pauli matrices**

$\alpha, \beta, \ldots = 1,2,\ \dot\alpha, \dot\beta, \ldots = 1,2$ **two-component spinor indices**

$$\det |X^{\alpha\dot\alpha}| = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2$$

Lorentz symmetry: $sl(2,\mathbb{C}) \sim o(3,1)$.

Dictionary between tensors and multispinors by:

$$\sigma^a_{\alpha\dot\alpha}\,, \qquad \sigma^{ab}_{\alpha\beta} = \sigma^{[a}_{\alpha\dot\alpha}\sigma^{b]\dot\beta}_{\beta}\,, \qquad \bar\sigma^{ab}_{\dot\alpha\dot\beta} = \sigma^{[a}_{\alpha\dot\alpha}\sigma^{b]\alpha}{}_{\dot\beta}$$

Pair of dotted and undotted indices: vector

Pairs of symmetrized indices of the same type: antisymmetric tensors

Problem 1.13. Check

NonAbelian HS Algebra

$3d$ **Conformal HS symmetry** $=$ AdS_4 **HS symmetry**

HS gauge fields: $\omega(Y|X)$

$Y_A = (y_\alpha, \bar{y}_{\dot\alpha})$, $\alpha, \dot\alpha = 1, 2$ **two-component spinor indices**

$$\omega(Y|X) = \sum_{n,m=0}^{\infty} \frac{1}{2n!m!} \omega_{\alpha_1 \ldots \alpha_n \, , \dot\alpha_1 \ldots \dot\alpha_m}(X) y^{\alpha_1} \ldots y^{\alpha_n} \bar{y}^{\dot\alpha_1} \ldots \bar{y}^{\dot\alpha_m}$$

HS curvature and gauge transformation

$$R(Y|X) = d\omega(Y|X) + \omega(Y|X) * \wedge \omega(Y|X)$$

$$\delta\omega(Y|X) = D\epsilon(Y|X) = d\epsilon(Y|X) + [\omega(Y|X)\,, \epsilon(Y|X)]_*$$

$$[y_\alpha\,, y_\beta]_* = 2i\varepsilon_{\alpha\beta}\,, \qquad [\bar{y}_{\dot\alpha}\,, \bar{y}_{\dot\beta}]_* = 2i\varepsilon_{\dot\alpha\dot\beta}$$

Vacuum

Equation of AdS_4: $R_0 = 0$ **for** $\omega_0 \in sp(4) \sim o(3,2)$

$$\omega_0(Y|X) = \frac{1}{4i}(w^{\alpha\beta}(X)y_\alpha y_\beta + \bar{w}^{\dot\alpha\dot\beta}(X)\bar{y}_{\dot\alpha}\bar{y}_{\dot\beta} + 2\lambda h^{\alpha\dot\beta}(X)y_\alpha\bar{y}_{\dot\beta})$$

Problem 1.14. Check

Fluctuations

$$\omega = \omega_0 + \omega_1\,, \qquad R_1 = D_0\omega_1 = d\omega_1 + [\omega_0\,,\omega_1]_*$$

Central On-shell theorem

The full unfolded system for free massless fields of all spins is formulated in terms of one-form $\omega(Y|X)$ and zero-form $C(Y|X)$

$$\mathbf{R_1(y,\overline{y}\mid X) = \overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\overline{y}^{\dot\alpha}\partial\overline{y}^{\dot\beta}}C(0,\overline{y}\mid X) + H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}C(y,0\mid X)}\ ,$$

$$\mathbf{\tilde{D}_0 C(y,\overline{y}\mid X) = 0}\,,$$

where

$$H^{\alpha\beta} = h^{\alpha}{}_{\dot\alpha}\wedge h^{\beta\dot\alpha}\,,\qquad \overline{H}^{\dot\alpha\dot\beta} = h_{\alpha}{}^{\dot\alpha}\wedge h^{\alpha\dot\beta}\ ,$$

$$R_1(y,\bar{y}\mid X) = D^{ad}\omega(y,\bar{y}\mid X)$$

$$D_0^{ad}\omega = D^L - \lambda h^{\alpha\dot\beta}\left(y_\alpha\frac{\partial}{\partial\bar{y}^{\dot\beta}} + \frac{\partial}{\partial y^\alpha}\bar{y}_{\dot\beta}\right)\,,\qquad \tilde{D}_0 = D^L + \lambda h^{\alpha\dot\beta}\left(y_\alpha\bar{y}_{\dot\beta} + \frac{\partial^2}{\partial y^\alpha\partial\bar{y}^{\dot\beta}}\right)\,,$$

$$D^L A = d_X - \left(\omega^{\alpha\beta}y_\alpha\frac{\partial}{\partial y^\beta} + \bar{\omega}^{\dot\alpha\dot\beta}\bar{y}_{\dot\alpha}\frac{\partial}{\partial\bar{y}^{\dot\beta}}\right)$$

Pattern of massless equations

Gauge fields of different spins: homogeneous polynomials in Y

$$\omega^s(\nu y, \nu \bar{y}|X) = \nu^{2(s-1)}\omega(y, \bar{y}|X)\,, \qquad C^s(\nu y, \nu^{-1}\bar{y}|X) = \nu^{\pm 2s}C(y, \bar{y}|X)$$

Infinite set of spins $s = 0, 1/2, 1, 3/2, 2, 5/2\ldots$

$$\omega^s_{\alpha_1\ldots\alpha_n\,,\dot{\beta}_1\ldots\dot{\beta}_m}: \quad n+m = 2(s-1)\,, \qquad C^s_{\alpha_1\ldots\alpha_n\,,\dot{\beta}_1\ldots\dot{\beta}_m}: \quad |n-m| = 2s$$

$C(Y|X)$: gauge invariant HS curvatures and spin-zero matter fields

Dynamical fields

Frame-like fields $\omega^s_{\alpha_1\ldots\alpha_n\,,\dot{\beta}_1\ldots\dot{\beta}_n}$ contain Fronsdal fields

$C(0,0|x)$: scalar

Other components are expressed via higher derivatives of the dynamical fields which come in combination

$$\lambda^{-1}\frac{\partial}{\partial x}\,, \qquad \lambda^2 = -\Lambda$$

Higher derivatives source nonanaliticity in Λ.

Examples

$s = 0:$ $\qquad C^0_{\alpha_1\ldots\alpha_n\,,\dot\beta_1\ldots\dot\beta_n} \sim C_{a_1\ldots a_n}\,, \qquad C^b{}_{ba_3\ldots a_n} = 0$

Problem 1.15. Derive Maxwell equations in the spin-one sector

$s = 2$

Gauge fields: Lorentz connection $\omega_{\alpha\beta}, \bar\omega_{\dot\alpha\dot\beta}$ and vierbein $\omega_{\alpha,\dot\beta}$

Zero-forms $C_{\alpha_1\alpha_2\alpha_3\alpha_4}(X)$ and $\bar C_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(X)$: Weyl tensor in terms of two-component spinors.

Higher components $C^s_{\alpha_1\ldots\alpha_n\,,\dot\beta_1\ldots\dot\beta_m}$: $\quad |n-m| = 4$: all its derivatives

$R_{\alpha,\dot\beta} = 0$: **expresses Lorentz connection via vierbein**

$R_{\alpha\beta} = H^{\gamma\delta}C_{\alpha\beta\gamma\delta},\; R_{\dot\alpha\dot\beta} = \bar H^{\dot\gamma\dot\delta}\bar C_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$: **nonzero part of the Riemann tensor belongs to Weyl tensor**

Spin two in tensorial language

$$R_{\nu\mu}{}^{a} = 0\,, \qquad R_{\nu\mu}{}^{ab} = e_{\nu}{}^{c} e_{\mu}{}^{d} C_{cd,}{}^{ab}\,, \qquad C_{ab,}{}^{b}{}_{c} = 0$$

$C_{cd,}{}^{ab}$ coincides with Weyl tensor

$R_{\nu\mu} = R_{\nu\rho}{}^{\rho}{}_{\mu} = 0$: Einstein equations

HS counterparts impose Fronsdal equations $G_{\nu_1 \ldots \nu_s} = 0$ and express generalized Weyl tensors in terms of Fronsdal fields

From COST to nonlinear theory

$$R(y,\overline{y}\mid X)=\overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\overline{y}^{\dot\alpha}\partial\overline{y}^{\dot\beta}}\;C(0,\overline{y}\mid X)+H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\;C(y,0\mid X)+\dots$$

$$\tilde{D}C(y,\overline{y}\mid X)+\dots\,,\qquad\qquad \dots=O(C,\omega_1)$$

$$R(y,\overline{y}\mid X)=d\omega(y,\overline{y}\mid X)+\omega(y,\overline{y}\mid X)*\omega(y,\overline{y}\mid X)$$

$$\tilde{D}C(y,\overline{y}\mid X)=dC(y,\overline{y}\mid X)+\omega(y,\overline{y}\mid X)*C(y,\overline{y}\mid X)-C(y,\overline{y}\mid X)*\omega(y,-\overline{y}\mid X)$$

Such field equations are unfolded: exterior differential of any field is expressed via the fields themselves

Problem: find gauge invariant nonlinear corrections

General properties of HS interactions

HS interactions contain higher derivatives

Nonanaliticity in Λ

Background HS gauge fields contribute to higher-derivative terms in the evolution equations: evolution is determined by higher-spin fields along with the metric: no geodesic motion in presence of nonzero HS fields

Hence insufficiency of metric in presence of HS fields

HS fields source lower-spin fields via $\omega * \omega$ terms

lower-spin fields source HS fields via C^2 terms

gravity sources HS fields and vice-versa

Idea of Nonlinear Construction

Being possible in a few first orders, straightforward construction of nonlinear deformation quickly gets complicated

- Trick: to find a larger algebra g' such that a substitution

$$\star \qquad \omega \to W = \omega + \omega C + \omega C^2 + \ldots$$

into g' reconstructs nonlinear equations via a zero-curvature condition

$$dW + W \wedge W = 0$$

To find restrictions on W that reconstruct $\star$ in all orders

Doubling of spinors

$$\omega(Y|X) \longrightarrow W(Z;Y|X)\,, \qquad C(Y|X) \longrightarrow B(Z;Y|X)$$

to be accompanied by equations that determine dependence on the additional variables Z^A in terms of "initial data"

$$\omega(Y|X) = W(0;Y|X)\,, \qquad C(Y|X) = B(0;Y|X)$$

$S(Z,Y|X) = dZ^A S_A$ is connection along Z^A

HS star product

$$(f\star g)(Z,Y) = \int dSdT f(Z+S,Y+S)g(Z-T,Y+T)\exp -iS_A T^A$$

$$[Y_A,Y_B]_\star = -[Z_A,Z_B]_\star = 2iC_{AB}\,, \qquad Z-Y\ :\ Z+Y \text{ normal ordering}$$

Inner Klein operators:

$$\kappa = \exp iz_\alpha y^\alpha\,, \qquad \bar\kappa = \exp i\bar z_{\dot\alpha}\bar y^{\dot\alpha}\,, \qquad \kappa\star f(y,\bar y) = f(-y,\bar y)\star\kappa\,, \qquad \kappa\star\kappa = 1$$

Nonlinear HS Equations

$$\begin{cases} dW + W \star W = 0 \\ dB + W \star B - B \star \tilde{W} = 0 \\ dS + W \star S + S \star W = 0 \\ S \star B - B \star S = 0 \\ S \star S = i(dZ^A dZ_A + dz^\alpha dz_\alpha F(B) \star \kappa + d\bar{z}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}} \bar{F}(B) \star \bar{\kappa}) \end{cases}$$

Manifest gauge invariance

$$\delta \mathcal{W} = [\varepsilon, \mathcal{W}]_\star \,, \qquad \delta B = \varepsilon \star B - B \star \tilde{\varepsilon} \,, \qquad \varepsilon = \varepsilon(Z; Y; K|x)$$

Nontrivial equations are free of space-time differential d

The equations are formally consistent and regular: no divergences despite non-polynomial Klein operators: $\kappa = \exp i z_\alpha y^\alpha$ and $\bar{\kappa} = \exp i \bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}$

Perturbative analysis

Vacuum solution $B_0 = 0\,, \qquad S_0 = dZ^A Z_A\,, \qquad W_0 = \frac{1}{2}\omega_0^{\mu\nu}(x)Y_\mu Y_\nu$

$$dW_0 + W_0 \star W_0 = 0 \longrightarrow \omega_0^{\mu\nu}(x) \quad \text{describes} \quad AdS_d$$

First-order analysis reproduces Central On-Shell Theorem

A particular form of star product plays crucial role

Unfolded Dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \text{initial values:} \quad q^i(t_0)$$

degrees of freedom = # of dynamical variables

Unfolded dynamics: multidimensional generalization

$$\frac{\partial}{\partial t} \to d\,, \qquad q^i(t) \to W^\Omega(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p} W^\Omega_{n_1 \ldots n_p}(x)$$

$$dW^\Omega(x) = G^\Omega(W(x))\,, \qquad d = dx^n \partial_n$$

$G^\Omega(W)$: function of “supercoordinates” W^Ω

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega{}_{\Lambda_1 \ldots \Lambda_n} W^{\Lambda_1} \wedge \ldots \wedge W^{\Lambda_n}$$

Covariant first-order differential equations

$d > 1$: Compatibility conditions

$$G^\Lambda(W) \wedge \frac{\partial G^\Omega(W)}{\partial W^\Lambda} \equiv 0$$

Properties

- General applicability
- Manifest (HS) gauge invariance under the gauge transformation

$$\delta W^{\Omega} = d\varepsilon^{\Omega} + \varepsilon^{\Lambda} \frac{\partial G^{\Omega}(W)}{\partial W^{\Lambda}},$$

gauge parameter $\varepsilon^{\Omega}(x)$ is a $(p_{\Omega} - 1)$-form.

- Invariance under diffeomorphisms

Exterior algebra formalism

- Interactions: nonlinear deformation of $G^{\Omega}(W)$
- Degrees of freedom are in 0-forms $C^{i}(x_0)$ at any $x = x_0$ (as $q(t_0)$)

infinite-dimensional module dual to the space of single-particle states

realized as a space of functions of auxiliary variables (like $C(y, \bar{y})$ instead of phase space coordinates in the Hamiltonian approach

Unfolded dynamics provides a tool to control unitarity in presence of higher derivatives

Space-time metamorphoses

Independence of ambient space-time: **geometry is encoded by** $G^{\Omega}(W)$

Key observation: unfolded equation makes sense in any space-time

$$dW^{\Omega}(x) = G^{\Omega}(W(x))\,, \quad x \to X = (x, y)\,, \quad d_x \to= d_X = d_x + d_y\,, \quad d_y = dy^u \frac{\partial}{\partial y^u}$$

X**-dependence is reconstructed in terms of fields** $W^{\Omega}(X_0) = W^{\Omega}(x_0, y_0)$ **at any** X_0 **To take** $W^{\Omega}(x_0, y_0)$ **in space** M_X **with coordinates** X_0 **is the same as to take** $W^{\Omega}(x_0)$ **in the space** $M_x \in M_X$ **with coordinates** x

Unfolding as a covariant twistor transform

$$
\begin{array}{ccc}
 & C(Y|x) & \\
 \eta \swarrow & & \searrow \nu \\
M(x) & & T(Y)\,.
\end{array}
$$

$W^{\Omega}(Y|x)$ are functions on the "correspondence space" C.

Space-time M : coordinates x. Twistor space T : coordinates Y.

Unfolded equations: Penrose transform mapping functions on T to solutions of field equations in M.

Holographic duality: different space-times M for the same twistor space

$3d$ conformal fields and currents

$3d$ **massless equations**

$$\left(\frac{\partial}{\partial x^{\alpha\beta}} \pm i \frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right) C_j^\pm(y|x) = 0\,, \qquad \alpha,\beta = 1,2\,, \quad j = 1,\dots\mathcal{N}$$

Shaynkman, MV (2001)

Rank-two equations: conserved currents **Gelfond MV 2008**

$$\left\{ \frac{\partial}{\partial x^{\alpha\beta}} - \frac{\partial^2}{\partial y^{(\alpha}\partial u^{\beta)}} \right\} J(u,\, y|x) = 0$$

$J(u,\, y|x)$: **generalized stress tensor. Rank-two equation is obeyed by**

$$J(u,\, y\,|x) = \sum_{i=1}^{\mathcal{N}} C_i^-(u+y|x)\, C_i^+(y-u|x)$$

Bilocal fields in the twistor space.

Elementary currents

Primaries: $3d$ **currents of all spins**

$$J(u,0|x) = \sum_{2s=0}^{\infty} u^{\alpha_1} \dots u^{\alpha_{2s}} J_{\alpha_1 \dots \alpha_{2s}}(x)\,, \qquad \tilde{J}(0,y|x) = \sum_{2s=0}^{\infty} y^{\alpha_1} \dots y^{\alpha_{2s}} \tilde{J}_{\alpha_1 \dots \alpha_{2s}}(x)$$

$$J^{asym}(u,y|x) = u_\alpha y^\alpha J^{asym}(x)$$

$$\Delta J_{\alpha_1 \dots \alpha_{2s}}(x) = \Delta \tilde{J}_{\alpha_1 \dots \alpha_{2s}}(x) = s+1 \qquad \Delta J^{asym}(x) = 2$$

Differential equations: conservation condition

$$\frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial u_\alpha \partial u_\beta} J(u,0|x) = 0\,, \qquad \frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial y_\alpha \partial y_\beta} \tilde{J}(0,y|x) = 0$$

From $3d$ currents to $4d$ massless fields

Extension of $3d$ current equation to $4d$ massless equations is easy in unfolded dynamics: $x^{\alpha\beta} \to X^{\alpha\dot{\beta}}$

$$\left(\frac{\partial}{\partial X^{\alpha\dot{\alpha}}} + \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot{\beta}}}\right) C(y, \bar{y}) = 0$$

Unfolded equations for $4d$ massless fields of all spins at $\Lambda = 0$

AdS: $x^{\alpha\beta} = \frac{1}{2}(X^{\alpha\beta} + X^{\beta\alpha})$: **boundary coordinates**,

$z^{-1} = X^{\alpha\beta}\epsilon_{\alpha\beta}$: **radial coordinate**

At the non-linear level full HS theory in AdS_4 is equivalent to the theory of $3d$ currents of all spins interacting through conformal HS gauge fields

$sp(8)$ invariant setup

$4d$ massless field equations for all spins are $sp(8)$ symmetric (Fronsdal 1985)

To see $Sp(8)$ extend $4d$ massless equations to a ten-dimensional space

$\mathcal{M}_4$: with local coordinates $X^{AB} = X^{BA}$ $A = (\alpha, \dot{\alpha}) = 1, 2, 3, 4$

$$dX^{AB}\left(\frac{\partial}{\partial X^{AB}} + \frac{\partial^2}{\partial Y^A \partial Y^B}\right)C(Y|X) = 0\,, \qquad A, B = 1, \dots M$$

$M = 2$: $3d$ massless fields: $Sp(4)$ is $3d$ conformal group Shaynkman, MV (2001)

$M = 4$: $Sp(8)$ extends $4d$ conformal group $SU(2,2)$

From four to ten

Dynamical variables in $\mathcal{M}_4$:

$C(0|X)$ **describes all** $4d$ **integer spins**

$Y^A C_A(0|X)$ **describes all** $4d$ **half-integer spins**

Nontrivial field equations: **(2001)**

bosons :
$$\left(\frac{\partial^2}{\partial X^{AB}\partial X^{CD}} - \frac{\partial^2}{\partial X^{CB}\partial X^{AD}}\right) C(X) = 0$$

fermions :
$$\left(\frac{\partial}{\partial X^{AB}} C_C(X) - \frac{\partial}{\partial X^{CB}} C_A(X)\right) = 0$$

From ten to four

Usual space-time picture appears as a result of identification of a local event simultaneously with the metric tensor

Time in $\mathcal{M}_M$

$$X^{AB} = T^{AB} t$$

T^{AB} is positive-definite (2002)

Usual space in $\mathcal{M}_M$: Clifford algebra

$$X^{AB} = x^n \gamma_n^{AB}$$

$M = 2, 4, 8, 16$: $\quad d = 3, 4, 6, 10$ Bandos, Lukierski, Sorokin (1999); MV (2002); Bandos, Bekaert, de Azcarraga, Sorokin, Tsulaia (2005)

HS theory and quantum mechanics

Unfolded dynamics distinguishes between positive and negative frequencies

$$\left(\frac{\partial}{\partial X^{AB}} \pm i\frac{\partial^2}{\partial Y^A \partial Y^B}\right)C^{\pm}(Y|X) = 0\,,$$

$$C(X) = C^{+} + C^{-}\,, \qquad C^{\pm} = \int d^M\xi c^{\pm}(\xi) \exp \pm i\xi_A\xi_B X^{AB}$$

Time parameter $t = \frac{1}{M}X^{AB}T_{AB}$ **with any positive-definite** T_{AB}

Holographic duality between relativistic HS theory and nonrelativistic QM

Reduction of X^{AB} to t with $T_{AB} = \delta_{AB} t$

$$i\frac{\partial}{\partial t}C^{\pm} = \pm\frac{\partial^2}{\partial Y^A \partial Y^B}\delta^{AB}C^{\pm}(Y|X)$$

$C^{\pm}$ counterparts of ψ and $\bar{\psi}$

Twistor coordinates Y play a role of non-relativistic coordinates

AdS and dS harmonic potentials with correct and wrong sign 2012 related results Bekaert, Meunier and Moroz (2011)

HS theory explains both gravity and QM?!

nonlinear QM with gravtationally small coupling constant?!

Multiparticle algebra as a symmetry of a multiparticle theory

$l(U(h))$ (2012)

- contains h as a subalgebra
- admits quotients containing up to k^{th} tensor products of h:

  k Regge trajectories?!
- Acts on all multiparticle states of HS theory

Promising candidate for a HS symmetry algebra of HS theory with mixed symmetry fields like String Theory

String Theory as a theory of bound states of HS theory

Chang, Minwalla, Sharma and Yin (2012)

Conclusion

HS gauge theories contain gravity along with infinite towers of other fields with various spins including ordinary matter fields: singlet scalar!

HS gauge theories available in any $d \geq 3$ are analogues of pure SUGRA

HS theory contains non-minimal higher-derivative interactions

To achieve spontaneous breakdown of HS symmetries a string-like extension is needed

A multiparticle theory: quantum HS theory and String Theory

Remarkable interplay between classical and quantum physics

A very few exact solutions are known in $4d$ HS theory:

BH-like solution Didenko, MV 2009, Iazeola, Sundell 2010

A number of inequivalent BTZ-like solutions in $3d$ HS theory

Didenko, MV 2009, Gutperle, Kraus 2011,...