Beyond supergravity in AdS/CFT

Diana Vaman

Physics Department, University of Virginia

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Outline

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- Real-time AdS/CFT: A prescription
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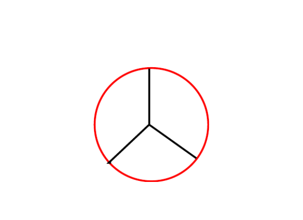
- Beyond supergravity in AdS/CFT
  Large but finite λ: higher derivative corrections to supergravity action
  - Effect on jet stopping distance: everything goes wrong at the same scale
  - Why? Tidal stretching of the gravitons into macroscopic strings

Real-time zero temperature AdS/CFT

AdS/CFT at $T = 0$: Strong-weak duality between a conformal field theory and string theory in a curved (Anti-de Sitter) background.
Trademark example: $\mathcal{N} = 4$ super Yang-Mills (sYM) and IIB string theory on $AdS_5 \times S^5$.
Identification of partition functions: $Z_{sYM} = Z_{IIBstring}$
In a simplified set-up, consider the supergravity (sugra) modes on $AdS_5 \times S^5$.
Witten:
$Z_{sYM}[J = \text{source for a BPS opt. } O] = Z_{sugra}[\phi_{sugra}(x^\mu, z) \longrightarrow J(x^\mu) \text{ at the AdS bdy}]$
$\Longrightarrow$ Correlators of BPS sYM operators can be computed at strong coupling by doing a weakly coupled gravity computation.



3-point function Witten diagram

Witten's prescription was for Euclidean AdS.
The correlators are then computed in imaginary time. What about real-time correlators?

Why do we need to worry about real time correlations?

Answer: The response of the system to some external perturbation is phrased in the language of retarded/causal correlators.

Examples:

In the hydrodynamic regime, the stress tensor gives the response of the plasma to a metric fluctuation via 2-point and higher order stress-stress retarded correlators. The real-time 2-point functions give (via Kubo formulae) the linear (the familiar shear viscosity coefficient) and a couple of second order coefficients. The real-time 3-point functions give the remaining second order hydrodynamic coefficients.

In the case when the system is perturbed by injecting some high energy excitation (a "jet"), the response is again given by real-time correlators, except that this time the leading order contribution comes from 3-point correlation fucntions.

Examples of real-time correlators:

- CFT 2-point functions

$$
\begin{aligned}
G_F(x) &= -i\langle 0|\mathcal{T}O(x)O(0)|0\rangle = -i\left(\frac{1}{-t^2+\vec{x}^2+i\epsilon}\right)^{\Delta} \\
G^+(x) &= -i\langle 0|O(x)O(0)|0\rangle = \theta(t)G_F(x)+\theta(-t)G_F^*(x) = -i\left(\frac{1}{-(t-i\epsilon)^2+\vec{x}^2}\right)^{\Delta} \\
G^-(x) &= -i\langle 0|O(0)O(x)|0\rangle = \theta(-t)G_F(x)+\theta(t)G_F^*(x) = -i\left(\frac{1}{-(t+i\epsilon)^2+\vec{x}^2}\right)^{\Delta} \\
G_R(x) &= G_F(x)-G^-(x) = \theta(t)\Big(G^+(x)-G^-(x)\Big),\ G_A(x) = \theta(-t)\Big(G^-(x)-G^+(x)\Big)
\end{aligned}
$$

- CFT 3-point functions

$$
\begin{aligned}
G_F(x_1,x_2,x_3) &= (-i)^2\langle 0|\mathcal{T}O(x_1)O(x_2)O(x_3)|0\rangle \\
&= (-i)^2\left(\frac{1}{-t_{12}^2+\vec{x}_{12}^2+i\epsilon}\,\frac{1}{-t_{23}^2+\vec{x}_{23}^2+i\epsilon}\,\frac{1}{-t_{31}^2+\vec{x}_{31}^2+i\epsilon}\right)^{\Delta/2}
\end{aligned}
$$

etc.

Comment: 2 and 3-point functions are completely determined (up to an overall coefficient) by the CFT algebra.

The problem:

AdS_5 can be described using different coordinates: global coordinates, which cover the whole space, or coordinates which make manifest the Poincare symmetry of the field theory (plus a radial coordinate)

$$ds^2_{AdS} = \frac{1}{z^2}(dx^\mu dx_\mu + dz^2)$$

and which cover only half of the space.

Which region of AdS must one integrate to get real-time correlators?

How are the different types of real-time correlators computed from sugra?

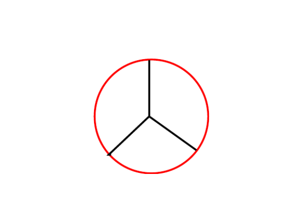
To find the answer, P.Arnold, E.Barnes, D.V. and C. Wu (2010) used reverse engineering:
-start from the known expressions of the real-time 3-point correlators:

$$\begin{aligned}
G_F(x_1,x_2,x_3) &= (-i)^2\langle 0|\mathcal{T}O(x_1)O(x_2)O(x_3)|0\rangle \\
&= (-i)^2\left(\frac{1}{-t_{12}^2+\vec{x}_{12}^2+i\epsilon}\,\frac{1}{-t_{23}^2+\vec{x}_{23}^2+i\epsilon}\,\frac{1}{-t_{31}^2+\vec{x}_{31}^2+i\epsilon}\right)^{\Delta/2} \\
G_{123}(x_1,x_2,x_3) &= (-i)^2\langle 0|O(x_1)O(x_2)O(x_3)|0\rangle \\
&= (-i)^2\left(\frac{1}{-t_{12}^2+\vec{x}_{12}^2+i\epsilon t_{12}}\,\frac{1}{-t_{23}^2+\vec{x}_{23}^2+i\epsilon t_{23}}\,\frac{1}{-t_{31}^2+\vec{x}_{31}^2-i\epsilon t_{31}}\right)^{\Delta/2} \\
G_R(x_1,x_2;x_3) &= \theta(t_{31})\theta(t_{12})\Big(G_{312}-G_{132}+G_{213}-G_{231}\Big) \\
&+ \theta(t_{32})\theta(t_{21})\Big(G_{321}-G_{231}+G_{123}-G_{132}\Big)
\end{aligned}$$

and manipulate until the structure of a cubic Witten diagram emerges.

Warm-up:

- begin with Euclidean signature 3-point CFT correlators;
- then Fourier-transform to momentum space;
- use Schwinger parameters to deal with the denominators, and perform the Gaussian integrals; use one more Schwinger parameter to deal with the intermediate result.
  Discover AdS/CFT!
  $G(k_1, k_2, k_3) \sim \delta(k_1 + k_2 + k_3) \int_0^\infty \frac{dz}{z^5} \prod_{i=1}^3 (\sqrt{k_i^2})^{\Delta-2} z^2 K_{\Delta-2}(\sqrt{k_i^2} z)$
  AdS_5 metric: $ds^2 = \frac{dz^2 + dx^\mu dx_\mu}{z^2}$
  Bulk-to-boundary propagators: $(\sqrt{k_i^2})^{\Delta-2} z^2 K_{\Delta-2}(\sqrt{k_i^2} z)$



3-point function Witten diagram

Retarded 3-point correlators

- What to expect: in field theory, the retarded/advanced 2-point function in momentum space are obtained by analytic continuation $\omega \to -i(\omega \pm i\epsilon)$ of the Euclidean 2-point function.
- Is there a similar analytic continuation which gives the retarded higher n-point functions computed from holography?

Consider the retarded 3-point with x_3 with the largest time:

$$\begin{aligned} G_R(x_1, x_2; x_3) &= \theta(t_{31})\theta(t_{12})\Big(G_{312} - G_{132} + G_{213} - G_{231}\Big) \\ &+ \theta(t_{32})\theta(t_{21})\Big(G_{321} - G_{231} + G_{123} - G_{132}\Big) \end{aligned}$$

In momentum space this yields

$$G_R(k_1, k_2; k_3) \sim \delta(k_1 + k_2 + k_3) \int_0^\infty \frac{dz}{z^5} \prod_{i=1,2} \Big((\sqrt{-(E_i - i\epsilon)^2 + \vec{p}_i^{\,2}})^{\Delta-2} z^2$$
$$\times K_{\Delta-2}(\sqrt{-(E_i - i\epsilon)^2 + \vec{p}_i^{\,2}} z)\Big)(\sqrt{-(E_3 + i\epsilon)^2 + \vec{p}_3^{\,2}})^{\Delta-2} z^2 K_{\Delta-2}(\sqrt{-(E_3 + i\epsilon)^2 + \vec{p}_3^{\,2}} z)$$

This is a Witten diagram with two advanced bulk-to-boundary propagators and one retarded propagator joined at a bulk vertex integrated over the Poincare patch of the AdS space.

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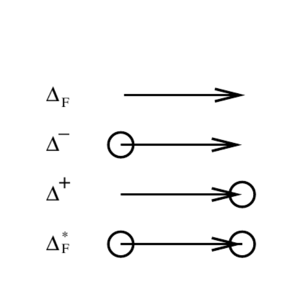
How are real-time correlators computed from sugra?

For Feynman (time-ordered) correlators use Feynman sugra propagators; for retarded/ (causal) correlators use causal sugra propagators. Tantamount to using Veltman's circling rules at $T = 0$ in supergravity (Arnold, Barnes, DV, Wu[1004.1179]).



$$\begin{aligned} \Delta_R &= \Delta_F - \Delta^- \\ \Delta_A &= \Delta_F - \Delta^+ \end{aligned}$$

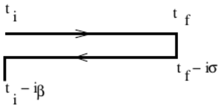
and in momentum space $(i\Delta_F) = Re(i\Delta_R) + iIm(i\Delta_R)\text{sign}(E)$

This is the jump-off point for real-time $T \neq 0$ computations from AdS/CFT.

Finite temperature real-time field theory

In real time formalism, fields have doublers.

Schwinger-Keldysh contour:



$$S = \int_C \mathcal{L}[\phi(\vec{x}, t)]$$

Or use a "single time" formalism, with two sets of fields by defining

$$\phi_1 \equiv \phi(\vec{x}, t), \qquad \phi_2 \equiv \phi(\vec{x}, t - i\sigma)$$

Finite temperature propagators

$$\begin{aligned} D_{11}(x) &= \langle e^{-\beta H} \mathcal{T} \phi_1(x)\phi_1(0)\rangle, \qquad D_{22} = \langle e^{-\beta H} \tilde{\mathcal{T}} \phi_2(x)\phi_2(0)\rangle, \\ D_{12} &= \langle e^{-\beta H} \phi_2(0)\phi_1(x)\rangle, \qquad D_{21} = \langle e^{-\beta H} \phi_2(x)\phi_1(0)\rangle \end{aligned}$$

In Heisenberg formalism $\phi(\vec{x}, t) = e^{i\hat{H}t}\phi(\vec{x})e^{-i\hat{H}t}$
$\Rightarrow$ for $\sigma = \beta/2$, $\qquad D_{12}(x) = D_{21}(-x), \quad \text{and} \quad D_{22}(x) = D_{11}(-x).$

The finite temperature propagator is a matrix, the Schwinger-Keldysh propagator

$$\begin{pmatrix} D_{11}(p) & D_{12}(p) \\ D_{21}(p) & D_{22}(p) \end{pmatrix} = \begin{pmatrix} \frac{i}{p^2+i\epsilon} + n(E)2\pi\delta(p^2) & (\theta(-E) + n(E))2\pi\delta(p^2)e^{\beta E/2} \\ (\theta(E) + n(E))2\pi\delta(p^2)e^{-\beta E/2} & \frac{-i}{p^2-i\epsilon} + n(E)2\pi\delta(p^2) \end{pmatrix}$$

where

$$n(E) = \frac{1}{e^{\beta|E|} - 1}$$

Finite temperature Feynman and Wightman propagators

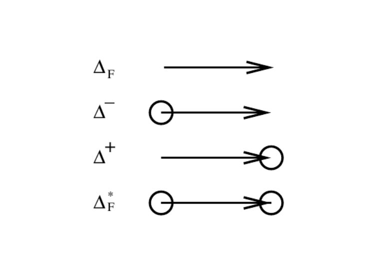
$$\begin{aligned} D_{12} &= \exp(\beta E/2)D^-, \qquad D_{21} = \exp(-\beta E/2)D^+, \\ D_{11} &= D_F, \qquad D_{22} = D_F^*. \end{aligned}$$

Retarded propagator $D_R = \theta(t)\langle e^{-\beta H}[\phi_1(x), \phi_1(0)]\rangle$ determines completely the other components of the SK propagator, with

$$G_F = Re\, G_R + iIm\, G_R \coth(\beta E/2), \qquad G_{R,F} = -iD_{R,F}$$

Zero temp limit: $D_F = \frac{-i}{-E^2+\vec{p}^2-i\epsilon}, D_R = \frac{-i}{-(E+i\epsilon)^2+\vec{p}^2},$ ✓

Kobes, Kobes& Semenoff (1985): Trade off the SK path integral contour for circling rules diagrams.



Any finite temperature "Feynman" diagram is given by the sum of all diagrams with vertices either circled or uncircled, with the exception of vertices connected to external lines which remain uncircled.

Largest time equation (identity): The sum of all diagrams with all vertices either circled or uncircled is 0.

A retarded n-point function, with one external vertex having the largest time is given by the sum of all diagrams, with all other vertices being either circled or uncircled, with the exception of the vertex with the largest time, which remains uncircled.

It is this causal Green's function computed in real-time formalism that coincides with the analytic continuation from imaginary time formalism.

Real-time finite temperature AdS/CFT

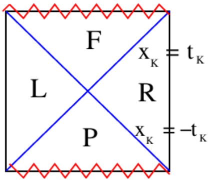
Finite temperature AdS/CFT: The finite temperature phase of the CFT on $R^{3,1}$ is holographically dual to AdS with a black hole in it: AdS-Schwarzschild

$$ds_{10}^2 = \frac{r^2}{R^2}(-f(r)dt^2 + d\vec{x}^2) + \frac{R^2}{r^2 f(r)}dr^2 + R^2 d\Omega_5^2$$

$$= R^2\left(\frac{-f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{z^2 f(z)}}{z^2} + d\Omega_5^2\right), \qquad z = \frac{R^2}{r}$$

$$f(r) = 1 - \frac{r_h^4}{r^4} = 1 - \frac{z^4}{z_h^2} \equiv 1 - u^2$$

The Hawking temperature is $T_H = \frac{r_h}{\pi R^2}$.



AdS-S Penrose diagram

How to integrate over the black hole bulk (given the presence of singularities, horizons)?

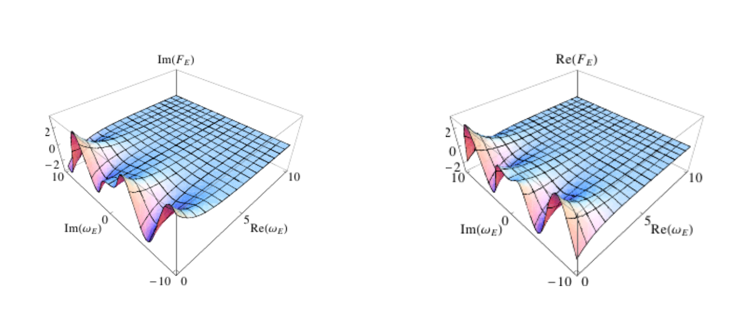
Tools of the trade: supergravity bulk-to-boundary propagators, and supergravity vertices.
Consider a massless scalar field $\phi(p^\mu, u) = F(\omega, \vec{p}, u)\phi_0(p^\mu)$ in the AdS-S background obeys $\Box F = 0$:

$$F'' - \frac{1+u^2}{u(1-u^2)}F' + \left(\frac{\omega^2}{u(1-u^2)^2} - \frac{\vec{p}^2}{u(1-u^2)}\right)F = 0$$

where $\omega, \vec{p}$ and u are dimensionless quantities

$$\omega = \frac{E}{2\pi T_H}, \qquad \vec{p} = \frac{\vec{P}}{2\pi T_H}, \qquad u = \frac{r_h^2}{L^2}z^2.$$

Retarded bulk-to-boundary propagator: F is an incoming wave at the horizon.
Re and Im parts of the Euclidean bulk-to-boundary propagator F_E, with $u = 0.5$ and $\vec{p}^2 = 1$:



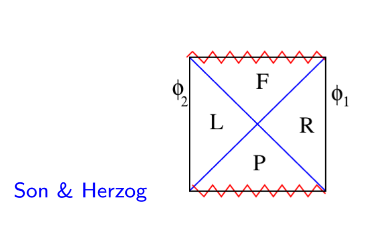
For a while what was known was how to compute 2-point functions:
Son & Starinets (2002) conjectured that the retarded 2-point CFT correlator at finite temperature is given by

$$\langle O(\omega,\vec{k})O(0)\rangle_\beta \propto \sqrt{g}g^{uu}\partial_u F(\omega,\vec{k},u)\Big|_{u=0}$$

based on the zero-temperature limit.

Son & Herzog (2002) gave a geometric interpretation of the finite temperature Schwinger-Keldysh matrix 2-point correlator by adding sources for the physical and ghost/doubler fields on the two boundaries of the AdS-S Penrose diagram.

Comments: Peculiar feature of 2-point functions which arise from $\int\sqrt{g}\partial\phi\cdot\partial\phi$: when evaluating the quadratic action on-shell, a 2-point function reduces to a boundary term. A genuine integration over the black hole bulk is not needed. Son and Herzog's prescription was not precise in how the integration over the black hole bulk needs to be carried out.



Construct the propagator matrix for the supergravity fields (e.g. a scalar):
-the field approaches ϕ_1 and ϕ_2 on the R and L boundaries.
-use the Unruh mode decomposition to keep incoming, positive frequency modes and outgoing negative frequency modes at the horizon.

$$\phi(p,u_R)=\left(\frac{e^{2\omega\pi}}{e^{2\omega\pi}-1}F(p;u_R)-\frac{1}{e^{2\omega\pi}-1}F(-p,u_R)\right)\phi_1(p)-2i\frac{e^{\omega\pi}}{e^{2\omega\pi}-1}ImF(p,u_R)\phi_2(p)$$

$$\phi(p,u_L)=2i\frac{e^{\omega\pi}}{e^{2\omega\pi}-1}ImF(p,u_L)\phi_1(p)+\left(\frac{e^{2\omega\pi}}{e^{2\omega\pi}-1}F(-p,u_L)-\frac{1}{e^{2\omega\pi}-1}F(p,u_L)\right)\phi_2(p)$$

$$\phi(p,u)\big|_a=\phi_b(p)\mathcal{G}_{ba}(p,u).$$

- Start with the retarded propagator: $\mathcal{G}_R \equiv F(\omega, \vec{k}, u)$.
- Follow Gibbons & Perry: The “Kruskal” vacuum Feynman propagator is the one which exhibits periodicity in imaginary “Schwarzschild” time. Hallmark characteristic of thermal propagators.

  Feynman propagator: $\mathcal{G}_F = Re\mathcal{G}_R + i\coth(\omega\pi) Im\mathcal{G}_R$ is the Feynman bulk-to-boundary Green’s function.
  $\mathcal{G}_R = \mathcal{G}_F - \mathcal{G}^-$ etc. $\Rightarrow$ get the other (Wightman) Green’s functions.
  Compare with the bulk-to-boundary propagator matrix $\mathcal{G}_{ab}$.

- D_{11}= Feynman; $D_{12} = D^- \exp(\omega\pi)$; $D_{21} = D^+ \exp(-\omega\pi)$ where $i\mathcal{G} = D$.
  Same relations as for a finite temperature field theory. Here $\sigma = \beta/2$.

How to integrate over the black hole bulk? ABVW 2010: To compute correlators, use a R-minus-L prescription. (Also used by Frolov and Martinez in a different context.) Even though the scalar field and its action were real to begin with, the boundary conditions imposed at the horizon break reality. The on-shell action is complex and yields complex 2-point functions

$$G_{ab}(p_1, p_2) = -(-1)^{a+b} \frac{\delta^2 \mathcal{S}_0}{\delta\phi_a(p_1)\delta\phi_b(p_2)}$$

and

$$G_R(p) = -2\bar{N}\sqrt{-g}g^{uu}F(p,u)\partial_u F(p,u)|_{u=0}$$

Define 3-point functions in AdS-S to be given by R-minus-L quadrant integration

$$G_{abc} \propto \int \sqrt{-g}\left(\mathcal{G}_{a1}\mathcal{G}_{b1}\mathcal{G}_{c1} - \mathcal{G}_{a2}\mathcal{G}_{b2}\mathcal{G}_{c2}\right)$$

$\mathcal{G}_{ab}$ is a propagator from the boundary $a = 1, 2$ (1=R,2=L) to the bulk $b = 1, 2$ (1=R,2=L).
Does it make sense?

- Real time finite temperature identities
  G_{abc} should obey the same KMS identities as the CFT correlator it computes

  $$G_{a_1 a_2 \ldots a_n} = (-)^{n-1} G^*_{\bar{a}_1 \bar{a}_2 \ldots \bar{a}_n}$$

  where $\bar{a} = 1$ if $a = 2$ and $\bar{a} = 2$ if $a = 1$, and we have assumed that $\sigma = \beta/2$.
  Here this follows from $\mathcal{G}_{ab} = \mathcal{G}^*_{\bar{a}\bar{b}}$.

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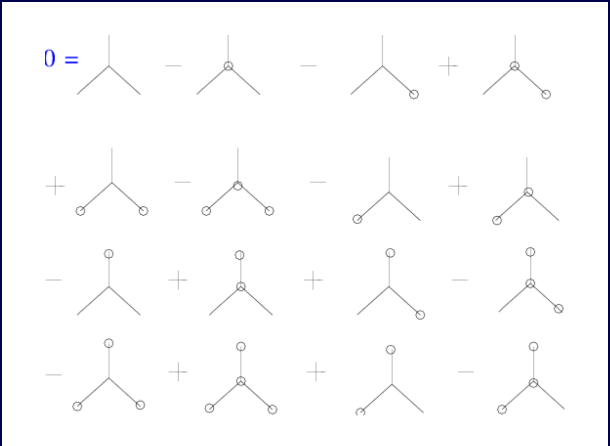
- Largest time



$$-\sinh(\omega_r\pi)ReG_{112}(q,p) = \sinh(\omega_q\pi)ReG_{211}(q,p) + \sinh(\omega_p\pi)ReG_{121}(q,p)$$

$$ImG_{111}(q,p) - \cosh(\omega_p\pi)ImG_{121}(q,p) - \cosh(\omega_q\pi)ImG_{211}(q,p) - \cosh(\omega_r\pi)ImG_{112}(q,p) = 0$$

q, p are incoming and r is outgoing.

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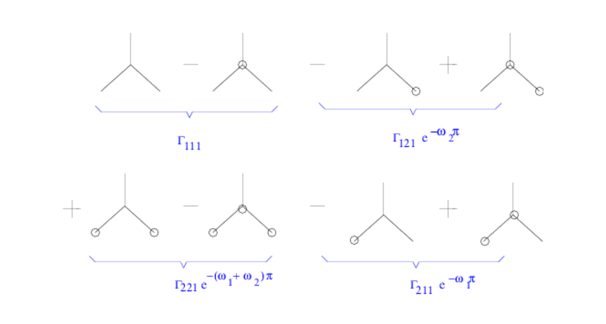
- Retarded 3-point definition

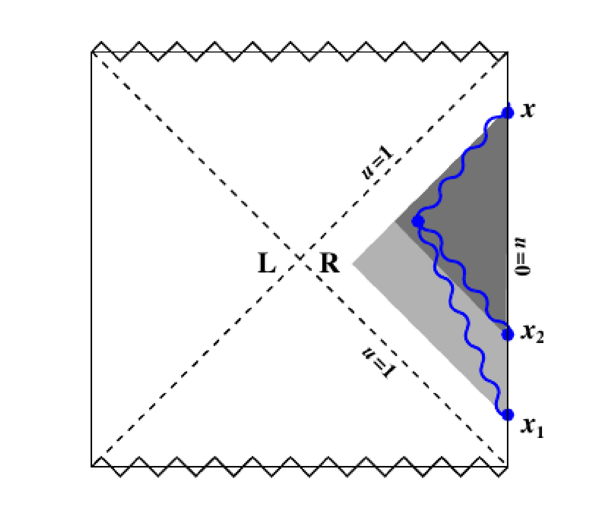


The retarded 3-point correlator is given by the sum of all diagrams above. After substituting the various G_{abc}, the final expression is very simple

$$G_R(q,p;r) \propto \int_0^1 du \sqrt{g}\, \mathcal{G}_A(q)\mathcal{G}_A(p)\mathcal{G}_R(r)$$

This is consistent with causality, analyticity and with the zero temperature limit.

Causality:



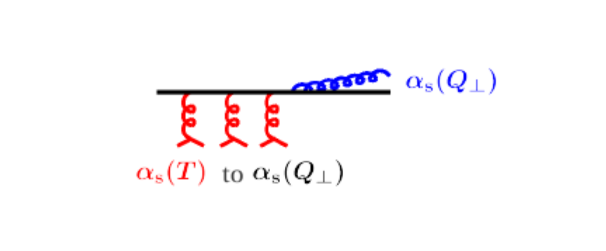
Simpler interpretation and main message:

- The R-minus-L prescription is merely enforcing on the gravity side Veltman circling rules at finite temperature.
- Dispose of the Penrose diagram all together.
- The bulk-to-boundary propagators which are causal are singled out by imposing incoming/outgoing wave condition at the horizon. The thermal Feynman propagators are constructed from the retarded ones by the same relation as for finite temperature field theory $\mathcal{G}_F = Re\mathcal{G}_R + i\coth(\omega\pi)Im\mathcal{G}_R$.
- The bulk vertex integration on the gravity side is done only up to the horizon. The Poincare coordinates are singled out since they are the preferred coordinates in the dual field theory.

Applications to strongly coupled plasmas

Jet quenching

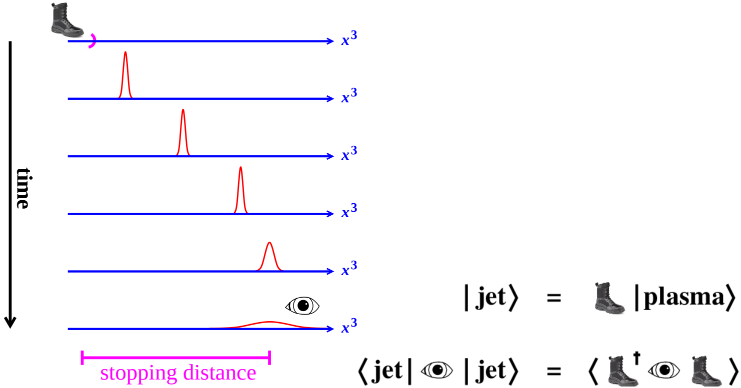
The problem: How far does a localized high-energy excitation travel through the quark-gluon plasma before stopping and thermalizing?



All weak-coupling: $l_{stop} \propto E^{1/2}$ (up to logs)
All strong-coupling $\mathcal{N} = 4$ super Yang-Mills: $l_{stop} \propto E^{1/3}$

(Maximal distance traveled $\sim (E/\sqrt{\lambda})^{1/3}$ for excitations dual to semi-classical strings(Chesler Jensen Karch Yaffe (2008), Gubser Gulotta Pufu Rocha (2008). No $\sqrt{\lambda}$-dependence for excitations dual to sugra modes (Hatta Iancu Mueller (2008), Arnold DV (2010).)

Arnold DV 1008.4023 re-opened the problem by posing the question on the field theory side: namely begin by specifying the nature of the excitation created on the gauge theory side, and by measuring the response (in terms of conserved charge densities) in the field theory as well.



Work at strong coupling and use AdS/CFT duality in the last step to compute the resulting correlator.

The Boot Operator:

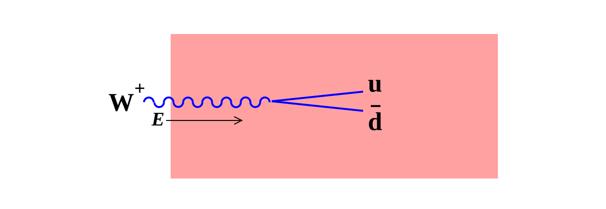
For simplicity consider the case when the external perturbation is a transverse polarized R-current (it's easier to track conserved R-charge as opposed to conserved energy/momentum, and also easier to discuss a perturbation created by the decay of a slightly off shell gauge boson than a slightly off shell graviton):

$$
\begin{aligned}
&\mathcal{L} \to \mathcal{L} + j^a_\mu A^{a\mu}_{\rm cl},\\
&A^\mu_{\rm cl}(x) = \bar\varepsilon^\mu \mathcal{N}_A \left[\frac{T^+}{2}\, e^{i\bar k\cdot x} + \text{h.c.}\right] e^{-\frac{1}{2}(x_0/L)^2} e^{-\frac{1}{2}(x_3/L)^2},\\
&\bar k^\mu = (E,0,0,E), \qquad \bar\varepsilon^\mu = (0,1,0,0)\\
&E \gg T, EL \gg 1.
\end{aligned}
$$

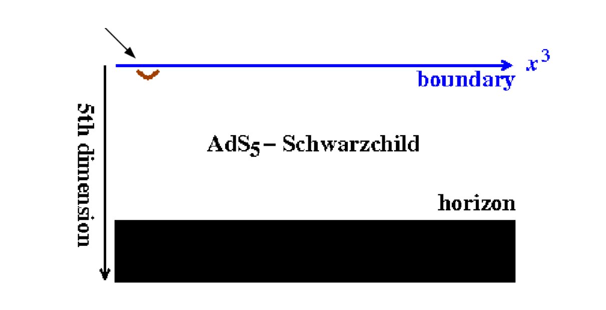
Jets

- our jets; also studied by Hatta Iancu Mueller (2008)

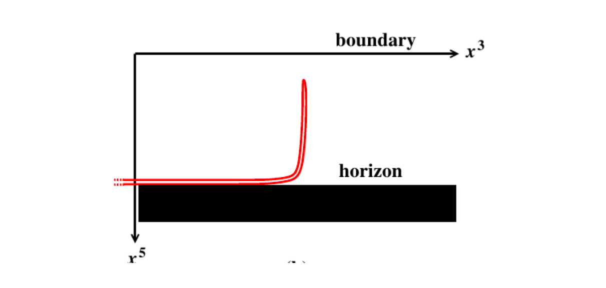
Analogy: A very high energy W^+ boson decaying inside a standard-model quark-gluon plasma and producing high-energy partons with net 3rd component of isospin, $\tau^3/2$:



- other jets
  - light-quark jets (Chesler Jensen Karch Yaffe(2008))



  - gluon jets (Gubser Gulotta Pufu Rocha (2008))



  - gluon beam jets via synchrotron radiation (Chesler Ho Rajagopal (2011))

The problem:

The Boot The source $A^{a\mu}_{\rm cl}$ creates an excitation that carries energy, momentum, and R charge.

The Eye Operator We track the R charge density, specifically the large-time behavior ($t \gg$ both T^{-1} and L) of

$$\left\langle j^{(3)0}(x) \right\rangle_{A_{\rm cl}}$$

if the system starts in thermal equilibrium at $t = -\infty$.

Measure the response in a "smeared" fashion, without resolving short distance scales (shorter than $1/E$ or $1/T$). This "smearing" eliminates the large momentum component of the respose function.
This reduces to a retarded 3-point function!
Consider a small perturbation $H = H_0 + \delta H$, where δH is short lived. The evolution of some observable $\mathcal{O}$ under H

$$\langle \mathcal{O}(t) \rangle_H = Z_0^{-1} Tr\left(e^{-\beta H_0} U^\dagger(t, -\infty) \mathcal{O} U(t, -\infty) \right)$$

where $U(t, t_0) = \mathcal{T} \exp(-i \int_{t_0}^t dt' H(t'))$ is the evolution operator.
Work in the interaction picture, and expand in powers of δH.

$$\langle \mathcal{O}(t) \rangle_H - \langle \mathcal{O}(t) \rangle_{H_0} = \int dt_1\, G_R(t_1; t) + \frac{1}{2!} \int dt_1\, dt_2\, G_R(t_1, t_2; t) + \ldots$$

where t is the "largest time" and

$$\begin{aligned} iG_R(t_1; t) &= \theta(t - t_1) \langle [\mathcal{O}(t), \delta H(t_1)] \rangle_{H_0} \\ i^2 G_R(t_1, t_2; t) &= \theta(t - t_2)\theta(t_2 - t_1) \langle [[\mathcal{O}(t), \delta H(t_2)], \delta H(t_1)] \rangle_{H_0} \\ &+ \theta(t - t_1)\theta(t_1 - t_2) \langle [[\mathcal{O}(t), \delta H(t_1)], \delta H(t_2)] \rangle_{H_0} \end{aligned}$$

Due to "smearing" the contribution coming from the 2-point function vanishes: the source carries large momenta (energetic jet) but the response function does not. The smearing in question is motivated by an interest in the hydrodynamic regime after the jet has stopped.
So, the leading order contribution to the response function comes from

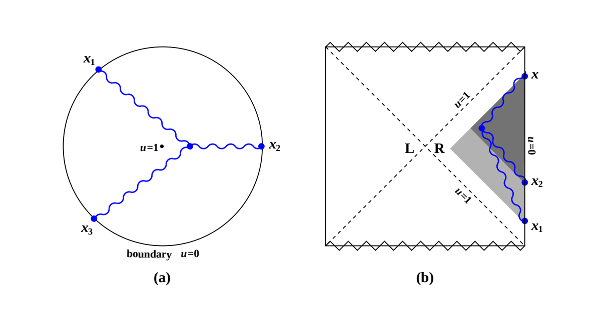
$$\left\langle j^{(3)\mu}(x) \right\rangle_{A_{\mathrm{cl}}} = \frac{1}{2} \int d^4x_1\, d^4x_2\, G_{\mathrm{R}}^{(ab3)\alpha\beta\mu}(x_1, x_2; x)\, A^a_{\alpha,\mathrm{cl}}(x_1)\, A^b_{\beta,\mathrm{cl}}(x_2)$$

where

$$\begin{aligned} G_{\mathrm{R}}^{(ab3)\alpha\beta\mu}(x_1, x_2; x) \quad &= \quad \theta(t - t_1)\theta(t_1 - t_2)\langle[[j^{(3)}(x), j^a(x_1)], j^b(x_2)]\rangle \\ &+ \quad \theta(t - t_2)\theta(t_2 - t_1)\langle[[j^{(3)}(x), j^b(x_2)], j^a(x_1)]\rangle \end{aligned}$$

The physical problem of tracking the jet evolution reduces to a technical problem: how to actually compute a retarded 3-point correlator.

Witten diagram for (a) 3-point boundary correlator in imaginary-time AdS_5-Schwarzschild and (b) retarded 3-point boundary correlator $G_{\mathrm{R}}(x_1, x_2; x)$ in real-time AdS_5-Schwarzschild.



Technical comments (helpful approximations):

-the jet has large energy (WKB approximation useful; analytic expressions are now available)

-the R-charge density is measured at scales which are large comparative to $1/E$ or even $1/T$ (one measures a "smeared response"); the Fourier-transform 3-point correlator factorizes (almost).

Charge deposition function

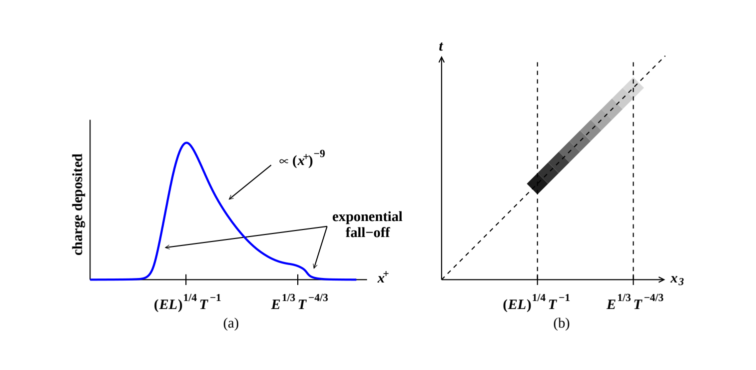
Final result:

$$(\partial_t - \frac{1}{2\pi T}\nabla^2)\left\langle j^{(3)0}(x)\right\rangle_{A_{\rm cl}} \simeq \bar{\mathcal{Q}}^{(3)}\,\Theta(x),$$

where the charge deposition function is

$$\Theta(x) \simeq 2\,\delta_L(x^-)\,\theta(x^+)\begin{cases}\dfrac{(4c^4EL)^2}{(2\pi T)^8(x^+)^9}\,\Psi\left(-\dfrac{c^4EL}{(2\pi Tx^+)^4}\right), & x^+ \ll E^{1/3}/(2\pi T)^{4/3};\\ \dfrac{(2\pi T)^4 2(c_2L)^2}{E}\,\Psi(0)\,\exp\left(-\dfrac{c_1(2\pi T)^{4/3}x^+}{E^{1/3}}\right), & x^+ \gg E^{1/3}/(2\pi T)^{4/3}.\end{cases}$$

with $\Psi(y) = e^{-2y^2}$, $c \equiv \dfrac{\Gamma^2(\frac{1}{4})}{(2\pi)^{1/2}}$, $c_1 \simeq 0.927$, $c_2 \simeq 3.2$.



Jet quenching simplified

P.Arnold, DV (2011): Consider a source

$$\text{source}(x) \sim e^{i\bar{k}\cdot x}\Lambda_L(x)$$

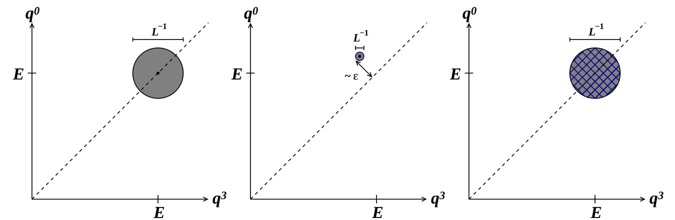
which creates a localized perturbation at the boundary, which then propagates in the 5th dimension, eventually falling into the horizon.
Previously, $\bar{k}$ was light-like:

$$\text{(a)} \qquad \bar{k}^\mu = (E, 0, 0, E)$$

Instead choose now $\bar{k}$ off the light-cone

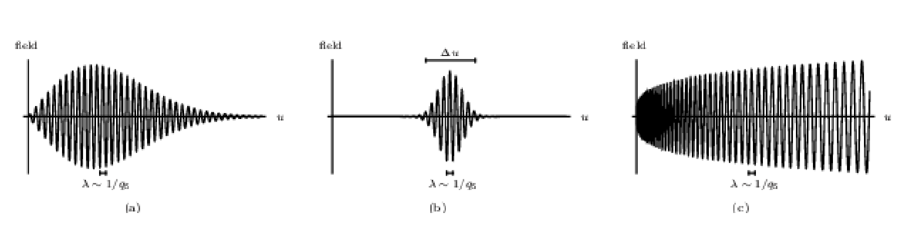
$$\text{(b)} \qquad \bar{k}^\mu = (E+\epsilon, 0, 0, E-\epsilon)$$

Qualitative picture of momenta used to generate jets:

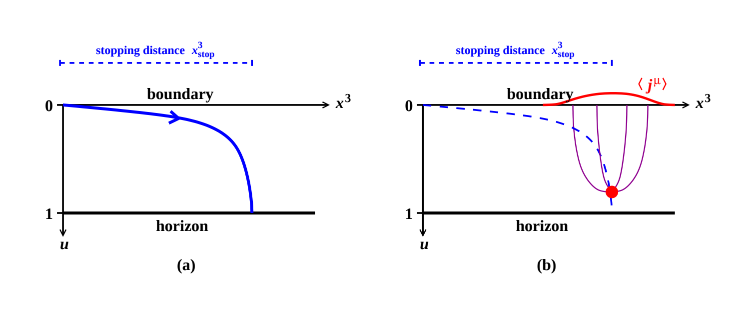


(a) (b) (c)

(a,b) A snapshot in time of waves in the fifth dimension u for times after the boundary source has turned off but relatively early (before the wave gets very close to the horizon).
(a) shows the type of wave generated by a localized source that superposes a range of q^2 values.
(b) shows the wave packet generated by a source with approximately well-defined q^2
(c) shows a single 4-momentum component, corresponding to a single, definite value of 4-momentum q_μ.



(a) A classical particle in the AdS_5-Schwarzschild space-time, moving in the $\mathbf{x}^3$ direction as it falls from the boundary to the black brane in the fifth dimension u. (b) The presence of the particle perturbs the boundary theory in a manner that spreads out diffusively as the particle approaches the horizon for $\mathbf{x}^0 \to \infty$.



The x^3 distance traveled is estimated from the geodesic equation:

$$\frac{dx^\mu}{dx^5} = \sqrt{g_{55}}\,\frac{g^{\mu\nu}q_\nu}{\sqrt{-q^\alpha q_\alpha}}$$

$$q^2 \equiv q_\mu q_\nu \eta^{\mu\nu}.$$

Consider that the original source is a superposition of wave packets with $\bar{k}$ off the light-cone by an amount ϵ, and the spread in momenta $1/L \ll \epsilon$. Each small wave packet may be approximated by a point-particle.
The particle falls into the horizon after having traveled a distance

$$x^{\mathbf{3}}_{\text{stop}} \simeq \frac{c}{\sqrt{2}} \left(\frac{|\mathbf{q}|^2}{-q^2} \right)^{1/4} \simeq \frac{c}{2} \left(\frac{E}{\epsilon} \right)^{1/4}$$

Each wave packet will travel a different distance, depending on its energy and ϵ. The total charge deposited by the initial source will be the weighted average of all these individual wave packets, with an weight equal to the probability that the source produces a jet of a given q^2, $\mathcal{P}$.

$$\text{Prob}(x^{\mathbf{3}}) \simeq \int d(q^2)\, \mathcal{P}(q^2)\, \delta \left(x^{\mathbf{3}} - x_{\text{stop}^{\mathbf{3}}}(q^2) \right)$$

For the original source, the typical value for $-q^2$ is $-q^2 \sim E/L$.
$x^{\mathbf{3}} \sim (EL)^{1/4}$ is then the typical distance traveled by the jet.

The distance traveled by the jet is determined by its 4d virtuality $-q^2$.

The distribution of stopping distances

Spectral density: $\rho(q) = 2Im(G_\perp^A(q))$ where $G_\perp^A$ is the gauge boson 2-point advanced correlator in momentum space.
For large momenta, we can approximate the 2-point function by the vacuum result

$$\begin{aligned}
\rho(q) &= Im\Big[-\frac{1}{g_{\rm SG}^2}\lim_{\bar u\to 0}\partial_{\bar u}\mathcal{G}_\perp^A\Big] \\
&= Im\Big[-\frac{q^2}{g_{\rm SG}^2}\Big(\ln(\bar u q^2)+2\gamma_E\Big)\Big] \\
&= \frac{\pi}{g_{\rm SG}^2}(-q^2)\theta(-q^2)\mathrm{sign}(q^0)
\end{aligned}$$

Back-of-envelope calculation:

$$x^{\mathbf{3}}_{\rm stop} \sim \frac{E^{1/4}}{q_+^{1/4}}, \qquad \mathcal{P}dq_+ \sim q^2 dq_+ \sim q_+ dq_+ \sim (x^{\mathbf{3}}_{\rm stop})^{-9} dx^{\mathbf{3}}_{\rm stop}$$

so

$$\mathrm{Prob}(x^{\mathbf{3}}) \simeq (x^{\mathbf{3}})^{-9}$$

The maximal $E^{1/3}$ scale from the typical $(EL)^{1/4}$ scale:
The classical particle picture must break down for a stopping distance of the order L.
Back-of-envelope calculation:

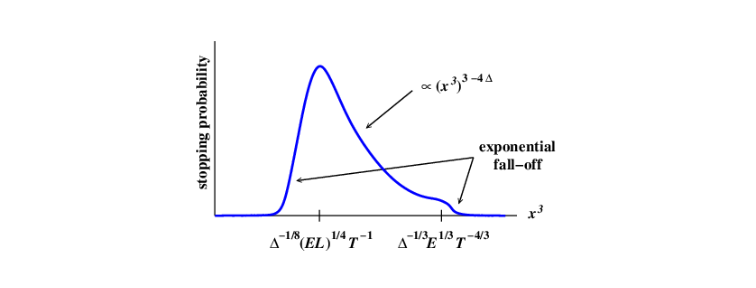
$$x_{\text{stop}}^{\mathbf{3}} \sim L \sim (EL)^{1/4} \qquad \Rightarrow \qquad L \sim E^{1/3} \sim x_{\text{stop max}}^{\mathbf{3}}$$

Massive particles

P.Arnold, DV (2011) For supergravity modes on $AdS_5 \times S^5$ mass is related to the conformal dimension of the CFT BPS operator

$$(Rm)^2 = \Delta(\Delta - d), \qquad d = 4.$$

The probability distribution of jet stopping distances for scalar or transverse BPS sources with conformal dimension Δ. (R-charge current case corresponds to Δ=3.) Here we assume that Δ is held fixed when taking the limit of large energy E (as well as large coupling $g^2 N_c$ and large N_c).



The typical scale, with the same $(EL)^{1/4}$ dependence, is again where most of the charge is being deposited; however, the heavier a KK mode, the sooner it stops; this is similar to weakly coupled field theory where the more partons are available to carry the total momentum, the shorter the stopping distance.

Adding a finite chemical potential

DV, C.Wu:

- Start with $\mathcal{N}=4$ $d=5$ $SU(2)\times U(1)$ gauged supergravity: metric g_{mn}, dilaton ϕ, $SU(2)\times U(1)$ gauge fields A^I_m and a_m, and two antisymmetric tensor fields B^α_{mn} which are charged under the action of the $U(1)$ field. It is a consistent truncation.
- Finite chemical potential (μ) in $\mathcal{N}=4$ sYM $\leftrightarrow$ supergravity in AdS-Reissner-Nordstrom background.

$$d\bar{s}^2 = \frac{4\pi^2 \mathrm{R}^2 T_H^2}{(2-\zeta)^2 u}\left(-f(u)dt^2 + d\vec{x}^2\right) + \frac{\mathrm{R}^2}{4u^2 f(u)}du^2, \qquad u = \frac{r_+^2}{r^2}$$

$$f(u) = (1-u)(1+u-\zeta u^2), \qquad \bar{Z}_0 = \frac{\sqrt{3\zeta}\,r_+}{2\mathrm{R}^2}(u-1)$$

$$T_H = \frac{(2-\zeta)r_+}{2\pi \mathrm{R}^2}, \qquad \mu = \frac{\sqrt{3\zeta}}{2\mathrm{R}^2}r_+ = \frac{\sqrt{3\zeta}}{2(2-\zeta)}2\pi T_H, \qquad 0 \leq \zeta \leq 2.$$

- Typical stopping distance ($\mu \ll E, \quad 1/L \ll E, \quad T \ll E$):

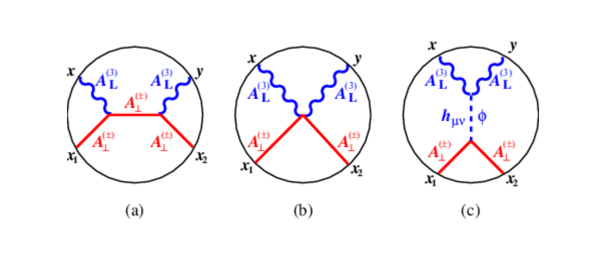
$$x^3_{\text{stop}} \simeq \frac{2-\zeta}{2}\,\frac{1}{2\pi T_H}\,\frac{8\Gamma^2(\frac{5}{4})}{\sqrt{\pi}(1+\zeta)^{1/4}}\left(\frac{\vec{q}^2}{-\mathbf{q}^2}\right)^{1/4}$$

Finite chemical potential leads to a jet quenching enhancement.

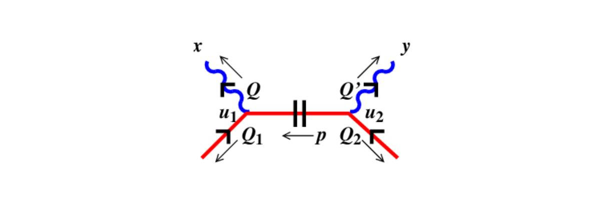
Charge-charge correlations

P.Arnold, DV (2011): More evidence that the charge deposition function is the result of averaging over uncorrelated point-particle stopping distances. Compute charge-charge correlator in the background of the source. This is a real-time 4-point function.

Typical Witten diagrams:



The only relevant real-time diagram is



where || denotes a symmetrized rr propagator.

Bulk-to-bulk propagators

The bulk-to-bulk symmetrized propagator is constructed knowing the bulk-to-bulk causal propagators

$$G_{\rm rr}(u_1, u_2, \omega, \vec{k}) = \coth(\tfrac{\beta\omega}{2})\left[G_R(u_1; u_2, \omega, \vec{k}) - G_A(u_1; u_2, \omega, \vec{k})\right]$$

where

$$G_R(u_1; u_2, \omega, \vec{k}) \propto \mathcal{G}_R(u_>, \omega, \vec{k})[\mathcal{G}_R(u_<, \omega, \vec{k}) - \mathcal{G}_A(u_<, \omega, \vec{k})]$$

and $\mathcal{G}$ denotes bulk-to-boundary propagators.
An explicit calcuation shows that for $x \neq y$,

$$\langle\{\Theta(x), \Theta(y)\}\rangle_{\rm jet} = 0$$

Outcome: no correlation for separated space-time points.

Hydrodynamic regime: second order hydro coefficients

Baier, Romatschke, Son, Starinets, Stephanov (2007):
In the hydro regime, the stress tensor of a CFT can be writen as an expansion in small gradients

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu}, \qquad T^{\mu\nu}_{\rm eq} = (\epsilon + P)U^\mu U^\nu + P g^{\mu\nu}$$

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_\Pi\left({}^{\langle}U\cdot\nabla\sigma^{\mu\nu\rangle} + \tfrac{1}{3}\nabla\cdot U\sigma^{\mu\nu}\right) + \kappa\left(R^{\langle\mu\nu\rangle} - 2U_\rho U_\sigma R^{\rho\langle\mu\nu\rangle\sigma}\right)$$
$$+\lambda_1\sigma^{\langle\mu}{}_\rho\sigma^{\nu\rangle\rho} + \lambda_2\sigma^{\langle\mu}{}_\rho\Omega^{\nu\rangle\rho} + \lambda_3\Omega^{\langle\mu}{}_\rho\Omega^{\nu\rangle\rho} + \ldots$$

where σ and Ω are the fluid's shear and vorticity tensors:

$$\sigma^{\mu\nu} = 2\nabla^{\langle\mu}U^{\nu\rangle} \equiv \tfrac{1}{2}\Delta^{\mu\rho}\Delta^{\nu\sigma}(2\nabla_\rho U_\sigma + 2\nabla_\sigma U_\rho) - \tfrac{1}{3}\Delta^{\mu\nu}\Delta^{\rho\sigma}2\nabla_\rho u_\sigma$$
$$\Omega^{\mu\nu} = \tfrac{1}{2}\Delta^{\mu\rho}\Delta^{\nu\sigma}(\nabla_\rho U_\sigma - \nabla_\sigma U_\rho)$$

where $\Delta^{\mu\nu}$ are transverse (to the fluid's velocity) projectors:

$$\Delta^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu$$

How to compute the hydro coefficients: until recently η, τ_Π, κ were obtained via Kubo formulae from 2-point stress correlators. What about the others? Answer: use 3-point retarded stress correlators!

- Moore and Sohrabi (2010): compute the fluid's response to a small, slowly varying gravitational perturbation, and derive Kubo-type formulae for 2nd order hydro coefficients.

$$\langle T^{\mu\nu}(z)\rangle_h = \langle T^{\mu\nu}\rangle_{h=0} - \frac{1}{2}\int d^4x\, G_R^{\mu\nu|\rho\sigma}(z;x)h_{\rho\sigma}(x)$$
$$+ \frac{1}{8}\int d^4x \int d^4y\, G_R^{\mu\nu|\rho\sigma|\tau\zeta}(z;x,y)h_{\rho\sigma}(x)h_{\tau\zeta}(y) + \dots$$

- Solve the conservation law $\nabla_\mu T^{\mu\nu} = 0$, and $T^\mu_\mu = 0$ iteratively, for the fluid's velocity U^μ and energy density ϵ, order-by-order in the metric fluctuations; compare with the previous expansion in terms of correlators $\Longrightarrow$ get Kubo-type formulae!
- P.Arnold, D.V., C.Wu (2011) Formulae for 2nd order hydro coefficients ($q \equiv (\omega, 0, 0, k)$, etc.):

$$\lim_{\substack{\omega_1\to 0\\ \omega_2\to 0}} \partial_{\omega_1}\partial_{\omega_2} \lim_{\substack{k_1\to 0\\ k_2\to 0}} G^{xy|xz|yz}(q;q_1,q_2) = -\lambda_1 + \eta\tau_\Pi$$

$$\lim_{\substack{\omega_2\to 0\\ k_1\to 0}} \partial_{k_2}\partial_{\omega_1} \lim_{\substack{\omega_2\to 0\\ k_1\to 0}} G^{xy|yz|tx}(q;q_1,q_2) = -\frac{1}{4}\lambda_2 + \frac{1}{2}\eta\tau_\Pi$$

$$\lim_{\substack{k_1\to 0\\ k_2\to 0}} \partial_{k_1}\partial_{k_2} \lim_{\substack{\omega_1\to 0\\ \omega_2\to 0}} G^{xy|0x|0y}(q;q_1,q_2) = -\frac{1}{4}\lambda_3$$

- Retarded supergravity bulk-to-boundary propagators:

$$
\begin{aligned}
\delta g_y^x &= C_5(1-u)^{-i\omega/2}\Big(1 - i\frac{\omega}{2}\ln(1+u) + \omega^2(-\frac{1}{2}\mathrm{Li}(2,\frac{1-u}{2})\\
&+ \frac{1}{8}\ln^2(1+u) + (1-\frac{\ln 2}{2})\ln(1+u)) - k^2\ln(1+u) + \ldots\Big)\\
C_5 &= \Big(1 + \frac{\omega^2(\pi^2 - 6\ln^2 2)}{24} + \ldots\Big)h_y^x
\end{aligned}
$$

- Supergravity quadratic action:

$$
\begin{aligned}
\delta^{(2)}\mathcal{S} &= \frac{N_c^2}{2^6\pi^2}\Big[\frac{1}{8}\int_{u=0}\frac{1}{u}\partial_5\Big(-\delta g_\mu^\mu\delta g_\nu^\nu + \delta g_\nu^\mu\delta g_\mu^\nu\Big)\\
&+ \frac{1}{4}\int_{u=0}\Big(\frac{3}{4}(h_0^0)^2 - \frac{1}{2}h_0^0 h_i^i + h_i^0 h_0^i + \frac{1}{4}h_i^i h_j^j - \frac{1}{2}h_j^i h_i^j\Big)\Big], \qquad i,j,k = 1,2,3
\end{aligned}
$$

Recover Baier, Romatschke, Son, Starinets, Stephanov:

$$
\begin{aligned}
G^{xy|xy}_{AdS} &= -\frac{\delta^2 \mathcal{S}}{\delta^2 h_{xy}} \\
&= \frac{N_c^2}{2^7\pi^2} - i\frac{N_c^2\omega}{2^6\pi^2} + \frac{(\omega^2(1-\ln 2)-k^2)N_c^2}{2^6\pi^2} + \ldots
\end{aligned}
$$

The result derived in the hydrodynamic limit from solving the conservation law of the stress tensor is

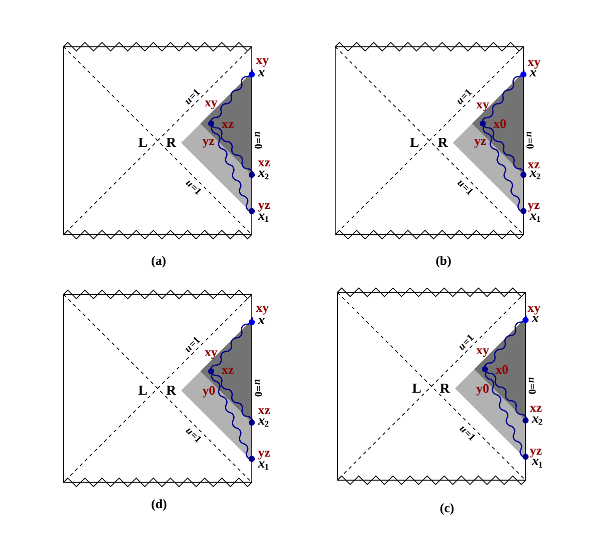
$$
G^{xy|xy}_{\text{hydro}} = \tfrac{1}{3}\bar{\epsilon} - i\eta\omega + \eta\tau_\Pi\omega^2 - \tfrac{1}{2}\kappa(\omega^2+k^2) + \ldots
$$

where the background energy density is $\bar{\epsilon} = \frac{3}{8}N_c^2\pi^2 T^4$.
Determine

$$
\eta = \frac{\pi N_c^2 T^3}{8}, \kappa = \frac{N_c^2 T^2}{8}, \eta\tau_\Pi = \frac{N_c^2(2-\ln 2)T^2}{16}.
$$

What about λ_1, λ_2 and λ_3?
Need 3-point stress tensor correlators!
This is an independent check of the values determined by Bhattacharyya, Hubeny, Minwalla, Rangamani (2007).

Real-time Witten diagrams for the retarded 3-point correlator $G^{xy|yz|xz}(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2)$ with the boundary point $\mathbf{x}$ having the largest time; $\mathbf{x}_1$ and $\mathbf{x}_2$ can have any time order.



$$\lim_{\substack{k_1\to 0\\ k_2\to 0}} G^{xy|yz|xz}_{AdS} = \frac{N_c^2}{2^4\pi^2}\left[\frac{1}{2^3} - i\frac{\omega_1+\omega_2}{2^2} - \frac{(\omega_1\omega_2+\omega_1^2+\omega_2^2)(\ln 2 - 1)}{2^2} + \dots\right]$$

$$\lim_{\substack{k_1\to 0\\ k_2\to 0}} G^{xy|yz|xz}_{\text{hydro}} = \frac{N_c^2}{2^4\pi^2}\left[\tfrac{1}{3}\bar{\epsilon} - i\eta(\omega_1+\omega_2) + \eta\tau_\Pi(\omega_1^2+\omega_2^2+\omega_1\omega_2) - \tfrac{1}{2}\kappa(\omega_1^2+\omega_2^2) - \lambda_1\omega_1\omega_2 + \dots\right]$$

- λ_1: $\lambda_1 = \frac{N_c^2}{2^6\pi^2}, \Rightarrow \lambda_1 = \frac{N_c^2 T^2}{16}.$

$$\lim_{\substack{\omega_1\to 0\\ \omega_2\to 0}} G^{xy|ty|tx}_{AdS} = \frac{N_c^2}{2^6\pi^2}\left[-\frac{1}{2} + (k_1^2+k_2^2) + \dots\right]$$

$$\lim_{\substack{\omega_1\to 0\\ \omega_2\to 0}} G^{xy|ty|tx}_{\text{hydro}} = -\tfrac{1}{3}\bar{\epsilon} + \tfrac{1}{2}\kappa(k_2^2+k_1^2) - \tfrac{1}{4}\lambda_3 k_1 k_2 + \dots$$

- λ_3: $\lambda_3 = 0.$

$$\lim_{\substack{k_1\to 0\\ \omega_2\to 0}} G^{xy|yz|tx}_{AdS} = \frac{N_c^2}{2^6\pi^2}\omega_1 k_2 + \dots$$

$$\lim_{\substack{k_1\to 0\\ \omega_2\to 0}} G^{xy|yz|tx}_{\text{hydro}} = (-\tfrac{1}{4}\lambda_2 + \tfrac{1}{2}\eta\tau_\Pi)\omega_1 k_2 + \dots$$

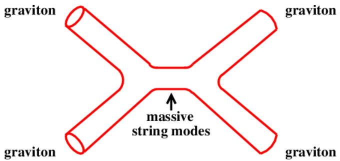
- λ_2: $\lambda_2 = -\frac{N_c^2}{2^5\pi^2}\ln 2 \Rightarrow \lambda_2 = -\frac{1}{8}N_c^2 T^2 \ln 2.$

Higher Derivative Corrections

• $\mathcal{N}=4$ conformal SYM all N, g_{YM} • $g_s = g_{YM}^2$	$\Leftrightarrow$	• Full Quantum Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ • $L^4 = 4\pi g_s N \alpha'^2$
• 't Hooft limit of $\mathcal{N}=4$ SYM $\lambda = g_{YM}^2 N$ fixed, $N \to \infty$ • $1/N$ expansion	$\Leftrightarrow$	• Classical Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ • g_s string loop expansion
• Large λ limit of $\mathcal{N}=4$ SYM (for $N \to \infty$) • $\lambda^{-1/2}$ expansion	$\Leftrightarrow$	• Classical Type IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$ • α' expansion

The first corrections to low-energy SUGRA arise from the low-energy limit of the string-string scattering amplitude, for which $N_c{=}\infty$ corresponds to the tree-level amplitude.

$$R \to R + \tfrac{1}{8}\,\zeta(3)\,\alpha'^3 \left[C^{hmnk} C_{pmnq} C_h{}^{rsp} C^q{}_{rsk} + \tfrac{1}{2} C^{hkmn} C_{pqmn} C_h{}^{rsp} C^q{}_{rsk} \right],$$



At tree level (appropriate for $N_c{=}\infty$), the energy dependence of the amplitude is captured by an overall factor (Green Schwarz (1982), Gross Witten (1986))

$$T(s,t,u)=\frac{\Gamma(-\alpha' s/8)\,\Gamma(-\alpha' t/8)\,\Gamma(-\alpha' u/8)}{\Gamma(1+\alpha' s/8)\,\Gamma(1+\alpha' t/8)\,\Gamma(1+\alpha' u/8)}\,,$$

where s, t, and u are the Mandelstam variables (in 10 dimensions). This is an "on-shell" result, which means $s+t+u=0$.
Green Vanhove (1999):

$$\begin{aligned} T &= \frac{64}{\alpha'^3 stu}\,\exp\left[\sum_{n=1}^{\infty}\frac{2\,\zeta(2n+1)}{2n+1}\left(\frac{\alpha'}{4}\right)^{2n+1}(s^{2n+1}+t^{2n+1}+u^{2n+1})\right] \\ &= \frac{3}{\sigma_3}+2\,\zeta(3)+\zeta(5)\,\sigma_2+\tfrac{2}{3}\,\zeta^2(3)\,\sigma_3+\tfrac{1}{2}\,\zeta(7)\,(\sigma_2)^2+\tfrac{2}{3}\,\zeta(3)\,\zeta(5)\,\sigma_2\sigma_3+\cdots\,, \end{aligned}$$

where

$$\sigma_k\equiv\left(\frac{\alpha'}{4}\right)^k(s^k+t^k+u^k).$$

In general, from the string scattering of higher n-point gravitons one reads off (Stieberger 2009):

$$\mathcal{L} \sim R + \left[\alpha'^3 R^4 + \alpha'^5 D^4 R^4 + \alpha'^6 D^6 R^4 + \cdots\right] + \left[\alpha'^5 D^2 R^5 + \alpha'^6 D^4 R^5 + \alpha'^7 D^6 R^5 + \cdots\right] + \cdots,$$

	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
$\alpha'^3\ \zeta(3)$	R^4				
$\alpha'^4\ \cancel{\zeta(4)}$	$\cancel{D^2R^4}$	$\cancel{R^5}$			
$\alpha'^5\ \zeta(5)$	D^4R^4	D^2R^5	R^6		
$\alpha'^5\ \cancel{\zeta(2)\zeta(3)}$	$\cancel{D^4R^4}$	$\cancel{D^2R^5}$	$\cancel{R^6}$		
$\alpha'^6\ \zeta(3)^2$	D^6R^4	D^4R^5	D^2R^6	R^7 ?	
$\alpha'^6\ \zeta(6)$	$\cancel{D^6R^4}$	$\cancel{D^4R^5}$	$\cancel{D^2R^6}$	R^7 ?	
$\alpha'^7\ \zeta(7)$	D^8R^4	D^6R^5	D^4R^6	D^2R^7 ?	R^8 ?
$\alpha'^7\ \zeta(3)\zeta(4)$	$\cancel{D^8R^4}$	$\cancel{D^6R^5}$	$\cancel{D^4R^6}$	D^2R^7 ?	R^8 ?
$\alpha'^7\ \zeta(2)\zeta(5)$	$\cancel{D^8R^4}$	$\cancel{D^6R^5}$	$\cancel{D^4R^6}$	D^2R^7 ?	R^8 ?
$\alpha'^8\ \zeta(3)\zeta(5)$	$D^{10}R^4$	D^8R^5	D^6R^6	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(8)$	$\cancel{D^{10}R^4}$	$\cancel{D^8R^5}$	$\cancel{D^6R^6}$	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(2)\zeta(3)^2$	$\cancel{D^{10}R^4}$	$\cancel{D^8R^5}$	$\cancel{D^6R^6}$	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(5,3)$	$\cancel{D^{10}R^4}$	$\cancel{D^8R^5}$	$\cancel{D^6R^6}$	D^4R^7 ?	D^2R^8 ?

TABLE I. *Tree–level higher order gravitational couplings and their corresponding zeta value coefficients probed by the N–graviton superstring amplitude. Vanishing terms are crossed out. Those terms, which have not yet been probed by the relevant N–graviton amplitude, are marked by a question mark.*

M. Paulos (2008): Include the 5-form field strengthcontribution. To leading order, the Weyl^4 term gets promoted to

$$C^4 \to \frac{1}{86016} \sum_i n_i M_i$$

n_i	M_i
-43008	$C_{abcd}C_{abef}C_{cegh}C_{dgfh}$
86016	$C_{abcd}C_{aecf}C_{bgeh}C_{dgfh}$
129024	$C_{abcd}C_{aefg}C_{bfhi}\mathcal{T}_{cdeghi}$
30240	$C_{abcd}C_{abce}\mathcal{T}_{dfghij}\mathcal{T}_{efhgij}$
7392	$C_{abcd}C_{abef}\mathcal{T}_{cdghij}\mathcal{T}_{efghij}$
-4032	$C_{abcd}C_{aecf}\mathcal{T}_{beghij}\mathcal{T}_{dfghij}$
-4032	$C_{abcd}C_{aecf}\mathcal{T}_{bghdij}\mathcal{T}_{eghfij}$
-118272	$C_{abcd}C_{aefg}\mathcal{T}_{bcehij}\mathcal{T}_{dfhgij}$
-26880	$C_{abcd}C_{aefg}\mathcal{T}_{bcehij}\mathcal{T}_{dhifgj}$
112896	$C_{abcd}C_{aefg}\mathcal{T}_{bcfhij}\mathcal{T}_{dehgij}$
-96768	$C_{abcd}C_{aefg}\mathcal{T}_{bcheij}\mathcal{T}_{dfhgij}$
1344	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cdeijk}\mathcal{T}_{fghijk}$
-12096	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cdfijk}\mathcal{T}_{eghijk}$
-48384	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cdfijk}\mathcal{T}_{egihjk}$
24192	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cefijk}\mathcal{T}_{dghijk}$
2386	$\mathcal{T}_{abcdef}\mathcal{T}_{abcdgh}\mathcal{T}_{egijkl}\mathcal{T}_{fijhkl}$
-3669	$\mathcal{T}_{abcdef}\mathcal{T}_{abcghi}\mathcal{T}_{dejgkl}\mathcal{T}_{fhkijl}$
-1296	$\mathcal{T}_{abcdef}\mathcal{T}_{abcghi}\mathcal{T}_{dgjekl}\mathcal{T}_{fhjikl}$
10368	$\mathcal{T}_{abcdef}\mathcal{T}_{abcghi}\mathcal{T}_{dgjekl}\mathcal{T}_{fhkijl}$
2688	$\mathcal{T}_{abcdef}\mathcal{T}_{aghdij}\mathcal{T}_{bgkeil}\mathcal{T}_{chkfjl}$

$$\mathcal{T}_{abcdef} = iD_a F_{bcdef} + \frac{1}{16}(F_{abcmn}F_{def}{}^{mn} - 3F_{abfmn}F_{dec}{}^{mn}).$$

Effect on hydrodynamic coefficients

$$\lambda \equiv g_{YM}^2 N_c \overset{\text{AdS/CFT}}{\Longleftrightarrow} \lambda = \frac{\mathrm{R}^4}{4\pi\alpha'^2}$$

Take $N_c \to \infty$: string loops are suppressed. The α' expansion turns into a $1/\sqrt{\lambda}$ expansion.

Buchel, Liu, Starinets (2004), Buchel (2008): The Weyl^4 term leads to a modification of the ratio of the shear viscosity hydrodynamic coefficient by entropy density

$$\frac{\eta}{s} = \frac{1}{4\pi}(1 + 15\zeta(3)\lambda^{-3/2}).$$

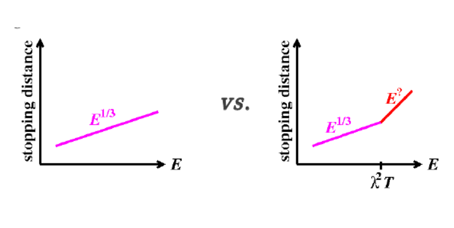
Myers, Paulos, Sinha (2008) included the 5-form field strength contribution to the study of hydrodynamic coefficients.
Their conclusion: no effect on top on the previously reported one.

Saremi, Sohrabi (2011) studied the effect of the Weyl^4 term on second order hydrodynamic coefficients.
In particular, to order $\lambda^{-3/2}$, λ_3 is no longer zero:

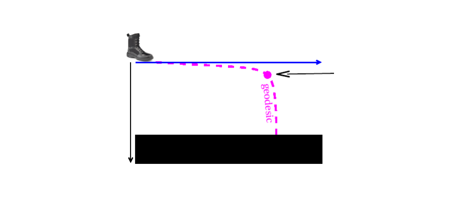
$$\lambda_3 = -\frac{25}{16}\zeta(3)\lambda^{-3/2} N^2 T^2.$$

Effect of higher derivative corrections on jet stopping

Suppose we are interested in the jet stopping distance at large, but finite (as opposed to infinite) coupling. This opens the possibility a new scale might enter which could invalidate the maximal stopping distance high energy dependence $E^{1/3}$.
Which of these scenarios is realized?



Recall that we have recast the jet stopping distance computation into the question of how far is dual gravity wavepacket (massless gauge boson, graviton etc) traveling before falling into the horizon:



So, we can ask what is the effect of the higher derivatives on the wavepacket's trajectory:
i) the background is modified (Klebanov Gubser Tseytlin (1998)); small corrections suppressed by powers of α'
ii) potentially large corrections to the geodesic equation: the quadratic action for the gravity fluctutation leads to an equation of motion which departs significantly from the geodesic equation.

As opposed to the hydrodynamic coefficient corrections, in the jet story there is a large scale, E, and so small corrections are potentially offset by powers of E.

In more detail

E.g. Take a scalar 5d Kaluza-Klein field originating into the 10d graviton

$$\Phi_{ab}(x,y) = e^{iS(x^{\mathbf{5}})} e^{iq_\mu x^\mu} Y_{ab}(y), \qquad \mu = 0,1,2,3, \qquad a = 6,7,8,9,10,$$
$$q_\mu = (E,0,0,q_{\mathbf{3}}) \qquad q^2 \equiv q_\mu q_\nu \eta^{\mu\nu} \ll E^2$$

with Y_{ab} some traceless S^5 spherical harmonic, and with $z \equiv x^{\mathbf{5}}$.
In the WKB limit, with $S(x^{\mathbf{5}})$ large, the KK mass of $\phi(x)$ (which is of order 1) can be negelected, the covariant derivatives can be replaced by

$$\nabla_I \to iQ_I \equiv i(q_\mu, q_5 = \frac{\partial S}{x^{\mathbf{5}}}), \qquad I = 0,1,2,3,5,$$

and the $\Box_{10d}\Phi = 0$ becomes

$$Q_I Q_J g^{IJ} = 0.$$

This leads immediately to the usual geodesic equation.

How is the geodesic derived:
-solve for q_5:

$$q_5 = \sqrt{-g_{55} q_\mu q_\nu g^{\mu\nu}};$$

-substitute into the wavepacket obtained by convolution with an appropriate localizing $\Lambda(x)$

$$\phi(x) = \int d^4 q \, e^{i q_\mu x^\mu + i \int q_5 dx^5} \tilde{\Lambda}(q);$$

-use saddle point to evaluate the integral:

$$0 = \frac{\partial}{\partial q_\mu} (q_\mu x^\mu + \int q_5 dx^5);$$

-and lastly derive

$$x^\mu = - \int dx^5 \frac{\partial q_5}{\partial q_\mu}.$$

The stopping distance formula is reproduced:

$$\begin{aligned} l_{stop} &= - \int_0^{z_h} dz \frac{dq_5}{dq_3} = \int_0^{z_h} dz \frac{\sqrt{g_{55}} g^{33} q_3}{-q_\mu q_\nu g^{\mu\nu}}. \\ &= \int_0^{z_h} dz \frac{q_3}{-q^2 + \frac{z^4}{z_h^4} q_3^2} \propto \left(\frac{E^2}{-q^2} \right)^{1/4} \end{aligned}$$

We can similarly turn the new dispersion relation (modified by Weyl^4 terms) into a particle trajectory equation.

Including the Weyl^4 term contribution, the new dispersion relation is

$$q_5 = \sqrt{g_{55}\left(- q_\mu q_\nu g^{\mu\nu} + \frac{\varepsilon z^{12} q_3^4}{z_h^8 L^2} \right)}$$

with $\varepsilon = 6\zeta(3)\lambda^{-3/2}$.

Define "the importance" of the Weyl^4 correction by comparing the size of the correction

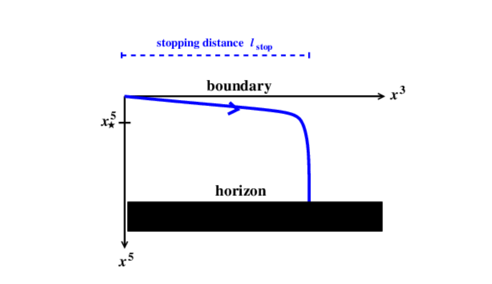
$$\frac{\varepsilon z^{12} q_3^4}{z_h^8 L^2}$$

and of the leading order term

$$-q_\mu q_\nu g^{\mu\nu} = -q^2 + \frac{z^4}{z_h^2} q_3^2$$

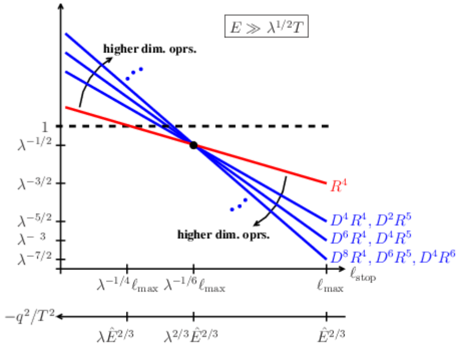
at a scale

$$z_\star \sim z_h \left(\frac{q^2}{E^2} \right)^{1/4}$$



$$\text{Importance}(C^4) \sim \frac{\frac{\epsilon z_\star^{10}}{z_h^8} E^4}{-q^2} \sim \frac{(-q^2)^{3/2}}{\lambda^{3/2} E T^2}$$

$$\text{Importance}(C^4) \sim \left(\frac{\lambda^{-1/4}\ell_{\max}}{\ell_{\text{stop}}}\right)^6 \sim \lambda^{-1/2}\left(\frac{\lambda^{-1/6}\ell_{\max}}{\ell_{\text{stop}}}\right)^6 .$$



Reminder: $l_{\max} \sim E^{1/3}$.

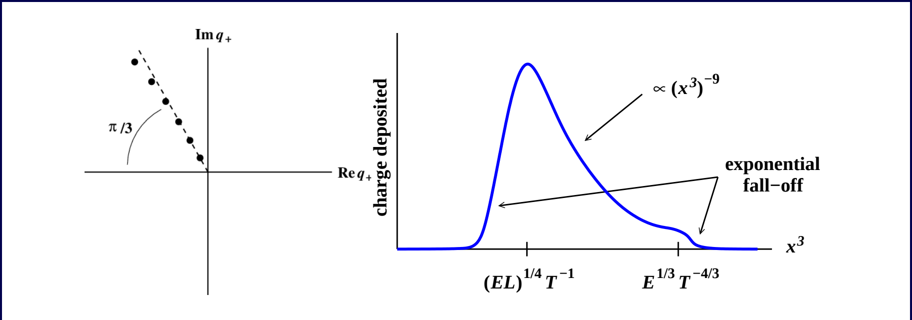
Exponential tails as stand in for stoping distance

The stopping distance depends on the kind of jets we make. However, for computing the maximal stoping distance we can use (with caveat) as a stand in the scale of the exponential fall off of the distribution of charge deposition

$$\mathrm{deposition}(x^3) \propto \exp(-x^3/l_{\mathrm{tail}})$$

where l_{tail} is determined by the quasinormal modes (poles of the retarded bulk-to-boundary propagator).
For $\lambda = \infty$, and for R-charge transverse polarized currents, the result is

$$l_{\mathrm{tail}} \simeq \frac{0.539E^{1/3}}{(2\pi T)^{4/3}}.$$



Note: $q_+ \equiv \frac{1}{2}(q^3 - q^0)$.

The Weyl^4 terms yield the following contribution to the quasinormal mode evaluation of $l_{\rm tail}$:

$$l_{\rm tail} = l_{\rm tail}^{\lambda=\infty}[1 + 82.174\lambda^{-3/2} + \dots]$$

where

$$l_{\rm tail}^{\lambda=\infty} = \frac{0.1704E^{1/3}}{(2\pi T)^{4/3}}$$

for the jet sourced by the traceless h_{ab} (these are fluctuation which transform under $\underline{84}$ rep of $SU(4) \simeq SO(6)$ and source a BPS operator of conformal dimension $\Delta = 6$ $Tr(\lambda\lambda\bar{\lambda}\bar{\lambda})$.
This result is well defined for jets which travel the farthest, and the first order correction leads to an increase of the stoping distance.

Conclusion:

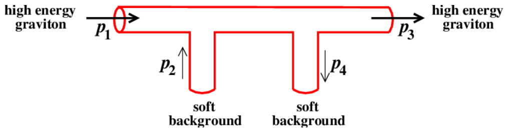
Those jets which travel a distance larger than $\lambda^{-1/6} l_{\rm max}$ have a well defined series in $1/\sqrt{\lambda}$.

For those jets which travel a distance shorter than $\lambda^{-1/6} l_{\rm max}$ the expansion in $1/\sqrt{\lambda}$ breaks down.

Interesting!!!

The distance $\lambda^{-1/6} l_{\rm max} \sim T^{-4/3} \left(\frac{E}{\sqrt{\lambda}} \right)^{1/3}$ is the maximal distance traveled by jets which are holograms of classical strings.(Chesler Jensen Karch Yaffe (2008), Gubser Gulotta Pufu Rocha (2008))

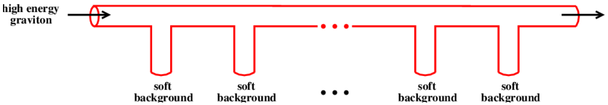
Why is the α' expansion breaking down?



If $\sqrt{s} \geq$"Mass of typical string excitations", such processes can excite on-shell string states. Then the small momentum expansion (i.e. the higher derivative expanion) breaks down.

Since $\sqrt{s} \propto \sqrt{E}$ and the typical string excitation has a mass of the order of $\frac{1}{\sqrt{\alpha'}} \propto \lambda^{1/4}$ this gives $E \geq T\lambda^{1/2}$, which is precisely the regime we discussed earlier.

Similarly,

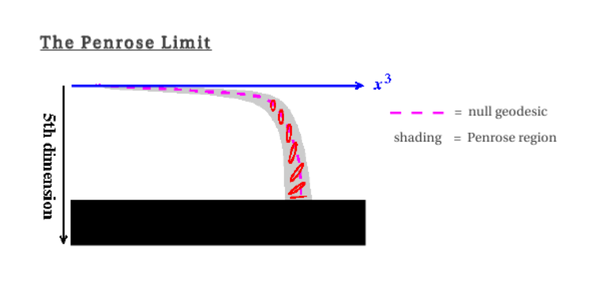


are unsupressed.

Tidal stretching of gravitons into classical strings

P.Arnold, P. Szepietowski, D.V., G. Wong (1212.3321)

Despite the fact that we do not know how to quantize the string in a generic curved background, we still have an analytic handle on the physics we want to study.
To leading order, the graviton follows a classical null geodesic trajectory. The string excitations about the ground state, the graviton, will probe only the spacetime in the immediate vicinity of the null geodesic.
But this space is the Penrose limit about the null geodesic and yields a plane wave metric. $\Longrightarrow$ We know how to quantize strings in plane wave backgrounds!



The Penrose limit

Define an affine parameter u along the null geodesic

$$du = \omega \, \frac{\sqrt{g_{55}}\, dx^5}{(-q_\alpha g^{\alpha\beta} q_\beta)^{1/2}} \, ,$$

and distances to the null geodesic

$$\Delta x^\mu \equiv x^\mu - \bar{x}^\mu(x^5).$$

Change coordinates from x^5 and Δx^0 to $u{=}u(x^5)$ and

$$v \equiv \frac{q_\mu \, \Delta x^\mu}{E} = -\Delta x^0 + \frac{q_3}{E} \, \Delta x^3$$

The AdS_5-Schwarzschild metric becomes

$$(ds)^2 = 2\, du\, dv + \frac{\mathrm{R}^2}{z^2} \Bigg[(dx^1)^2 + (dx^2)^2 + \frac{(E^2 - f(q_3^2))}{\omega^2} \, (d\Delta x^3)^2 + 2f \, \frac{q_3}{E} \, dv \, d\Delta x^3 - f\,(dv)^2 \Bigg],$$

where f is now implicitly a function of u.
Zooming onto the null geodesic (the Penrose limit):

$$u \to u, \quad v \to \gamma^{-2} v \quad x^i \to \gamma^{-1} x^i, \qquad \gamma \to \infty.$$

The plane wave metric in Brinkmann coordinates is

$$ds^2_{\mathrm{pp}} = 2\,du\,d\hat{v} + (d\hat{x}^{\mathbf{1}})^2 + (d\hat{x}^{\mathbf{2}})^2 + (d\Delta\hat{x}^{\mathbf{3}})^2 + \mathcal{G}(u,\hat{x}^{\mathbf{1}},\hat{x}^{\mathbf{2}},\Delta\hat{x}^{\mathbf{3}})\,(du)^2$$

with

$$\mathcal{G}(u,\hat{x}^{\mathbf{1}},\hat{x}^{\mathbf{2}},\Delta\hat{x}^{\mathbf{3}}) = \mathcal{G}_1(u)\left[(\hat{x}^{\mathbf{1}})^2 + (\hat{x}^{\mathbf{2}})^2\right] + \mathcal{G}_3(u)\,(\Delta\hat{x}^{\mathbf{3}})^2,$$

$$\mathcal{G}_1(u) = \mathcal{G}_2(u) \simeq -2\,\frac{z^6}{z_h^4\mathrm{R}^4}\,,$$

$$\mathcal{G}_3(u) \simeq 4\,\frac{z^6}{z_h^4\mathrm{R}^4}\,.$$

Write now the sigma model action for the bosonic string coordinates in the pp wave metric by choosing

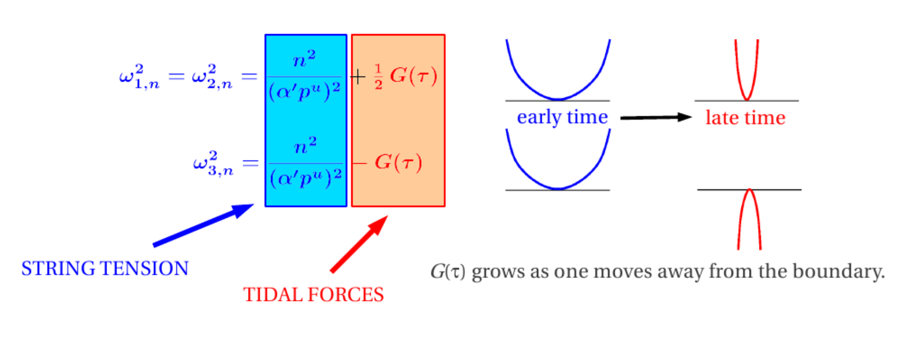
$$u \equiv \text{worldsheet time}$$

$$L = \frac{p^u}{2} \sum_i \sum_{n=-\infty}^{\infty} \left(\partial_\tau \Delta \hat{X}_n^{i\,*} \, \partial_\tau \Delta \hat{X}_n^i - \omega_{i,n}^2(\tau) \, \Delta \hat{X}_n^{i\,*} \, \Delta \hat{X}_n^i \right)$$

where $p^u = p_v \simeq E, \Delta \hat{X}^i = \sum_n \Delta \hat{X}_n^i e^{in\sigma}$ and

$$\omega_{i,n}^2(\tau) \equiv \frac{n^2}{(\alpha' p^u)^2} - \mathcal{G}_i(\tau).$$

The effect of the curved background: $\Delta \hat{X}^{1,2}$ are tidally compressed as one moves away from the boundary and $\Delta \hat{X}^3$ are stretched. (Similar to problems studied by Papadopoulos Russo Tseytlin (2002); also by Kim Lee (2000), Kim Page (2001).)



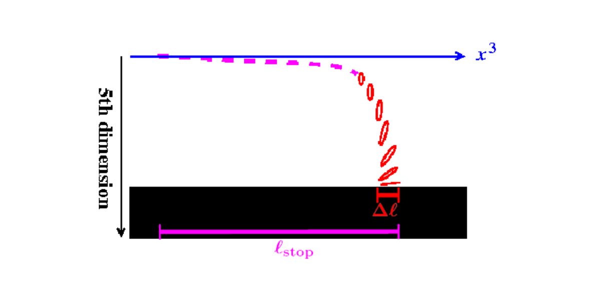
When tidal forces dominate over the string tension potential:

$$z_n \simeq \left(\frac{n z_h^2 \mathrm{R}^2}{2\alpha' E}\right)^{1/3} = \lambda^{1/6}\left(\frac{n z_h^2}{2E}\right)^{1/3} .$$

Since the stopping distance is essentially determined by evolution up to $z_\star$, only modes up to

$$n_\star \sim \left(\frac{l_{stop}}{\lambda^{-1/6} l_{max}}\right)^{-3}$$

become tidally unstable.
[Even so, these modes do not have time to stretch significantly until reaching $z = z_*$.]
Due to the unstable inverted harmonic oscillator potential at late times the dynamics becomes classical: we can calculate the prob distribution for each mode $\hat{X}_n^i$. $\Longrightarrow$ can calculate the late-time size of the classical string.



Averaging with the late-time probability distribution, the size of the string in the direction which is tidally stretched is

$$(\delta X^3)_{\rm rms} \simeq 0.8660\,\lambda^{-1/4} l_{stop}\,\ln^{1/2}\left(\frac{\lambda^{-1/6} l_{max}}{l_{stop}}\right),$$

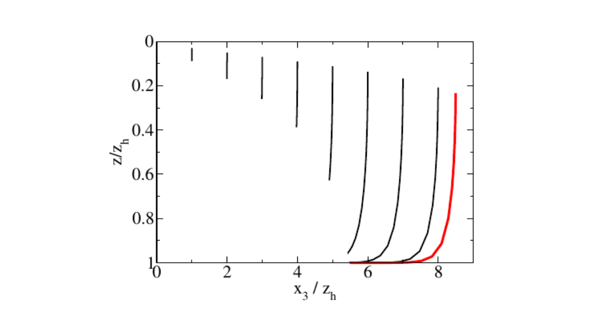
The prefactor $\lambda^{-1/4}$ is essentially coming from the proper size of the initial quanta, $\sqrt{\alpha'}$.

The conclusion of this story:

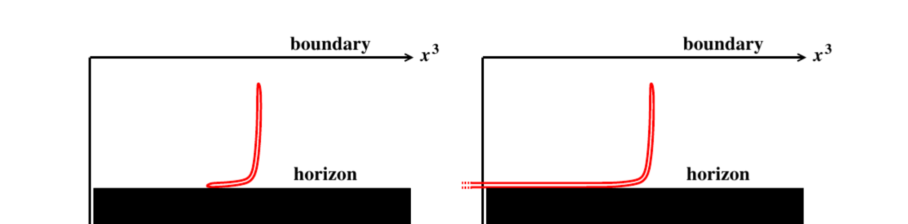
-the quantum string (graviton) is stretched into a classical closed string when $l_{stop} \ll \lambda^{-1/6} l_{max}$.

What happens if the Penrose limit breaks down (i.e. if the number of modes exicted $n_\star$ becomes exponentially big s.t. to compensate for the smalleness of the prefactor $\lambda^{-1/4}$)?

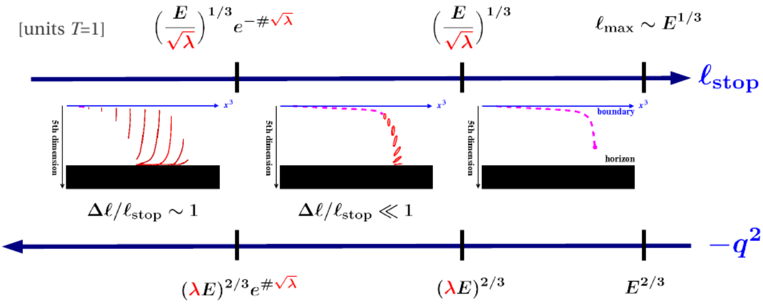
The string becomes tidally stretched all the way to the horizon:



This bears a striking similarity to the folded back strings dual to gluons considered by Gubser Gulotta Pufu and Rocha (2008):



Conclusion



A mystery persists: We expected to find some interpolating function $F(\lambda)$ such that

$$l_{stop} = E^{F(\lambda)}, \qquad \text{with } F(\lambda = 0) = 1/2 \text{ and } F(\lambda = \infty) = \frac{1}{3}$$

We expected that expansion around strong coupling could be of the form

$$F(\lambda) = \frac{1}{3} + \sum_{n=3,4\ldots} c_n \lambda^{-n/2}$$

which means that

$$l_{stop} = e^{ln(E)\times F(\lambda)} = E^{1/3}(1 + \# \ln(E)\lambda^{-3/2} + \ldots)$$

However, in our analysis we found no trace of a $\ln(E)$ term!