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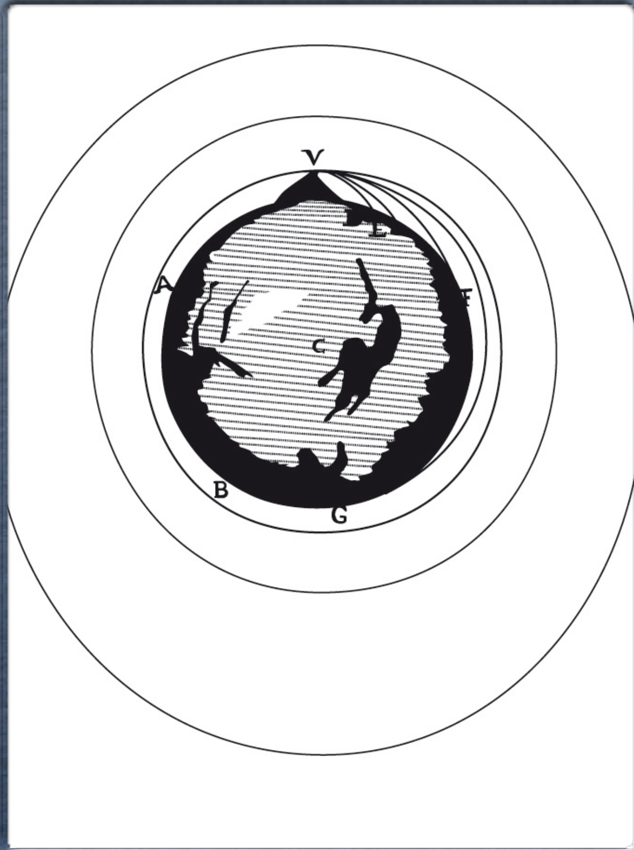
Higher Dimensional Conformally Invariant theories

(work in progress with Ricardo Troncoso)



- Modifying gravity
  - Extra dimensions (string-theory, braneworlds)
  - Massive Gravity
  - Horava-Lifshitz gravity
  - Scalar tensor theories (galileons-Horndenski)
- Modifying gravity means changing the dynamical behavior of the theory
- A simple but still remarkable modification of gravity: Lovelock theory
  - d>4 but still giving second order field equations which are divergent free
  - the Lagrangian is constructed out of the sum higher powers of the curvature 2-form, up to powers k in the curvature
  - In d=5 and d=6 it reduces to the Gauss-Bonnet (GB) combination, which involves quadratic curvature invariants
  - $\int d^d x \sqrt{-g} \left[R - 2\Lambda + \alpha \hat{G} \right]$ where $\hat{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$

- Here we will concentrate on theories with scalar fields which remain invariant under a conformal transformation (c.t.)
- Why scalar tensor theories???
  - the simplest modification, involving only one more d.o.f.
  - from string theory point of view the graviton is accompanied by the dilaton
  - Kaluza-Klein theory and braneworlds
  - Resent interest in selftunning senarios
- Why conformal invariance???
  - $\bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$ $\qquad d\bar{s}^2 = \Omega^2(x) ds^2$
  - GR is not invariant under conformal transformations
  - Conformal transformation is a localized scale transformation
  - GR due to G is not renormalisable. Maybe conformal invariance is important
  - Useful laboratory for black hole physics

OUTLINE

- Standard Conformal Invariance
- More General Conformal Invariance
- Field equations and a special factorization
- Non-trivial solutions
- Conclusions

Standard Conformal Invariance

- $\bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$ $\qquad$ $\bar{\phi} = \Omega^{\frac{2-d}{2}}(x)\phi$

  we have the following Lagrangian: $\mathcal{L} = \sqrt{-\bar{g}}\left(\bar{\phi}^2\bar{R} + \frac{4(d-1)}{(d-2)}\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\phi}\bar{\nabla}_\nu\bar{\phi}\right)$

  applying the transformation we see that we end up in the same form of the Lagrangian

  - the coefficient $\frac{1}{\xi} = \frac{4(d-1)}{(d-2)}$ is important
  - we could also add a proper self-interacting term without spoiling the invariance of the Lagrangian
  - its form is $\phi^{\frac{2d}{d-2}}$ where again the proper power plays a key role

- Can we construct more general actions???

More General Conformal Invariance

- We have already seen conformal transformations in d-dimensions

$$g_{\mu\nu} = \Omega^2(x)\hat{g}_{\mu\nu} \qquad \phi = \Omega^{\frac{2-d}{2}}(x)\hat{\phi}$$

- A special conformal frame...

$$\tilde{g}_{\mu\nu} = \phi^{\frac{4}{d-2}} g_{\mu\nu}$$

- First tests...

$$\mathcal{L}_0 = \sqrt{-\tilde{g}}\Lambda \quad \rightarrow \quad \sqrt{-g}\Lambda\phi^{\frac{2d}{d-2}}$$

conformally invariant term

$$\mathcal{L}_1 = \sqrt{-\tilde{g}}\tilde{R} \quad \rightarrow \quad \sqrt{-g}\left(\phi^2 R - \frac{4(d-1)}{(d-2)}\phi\Box\phi\right)$$

conformally invariant term

- This case of change of frame creates conformally invariant actions, since the metric is invariant by construction

$$\bar{g}_{\mu\nu} = \phi^{\frac{4}{d-2}} g_{\mu\nu} \quad \rightarrow \quad \hat{\phi}^{\frac{4}{d-2}}\hat{g}_{\mu\nu}$$

- Can we construct more general actions that are still conformally invariant???

- Let us start with the Riemann tensor...

$$\tilde{R}^{\rho}{}_{\sigma\mu\nu} = R^{\rho}{}_{\sigma\mu\nu} - 2\left(\delta^{\rho}_{[\mu}\delta^{\alpha}_{\nu]}\delta^{\beta}_{\sigma} - g_{\sigma[\mu}\delta^{\alpha}_{\nu]}g^{\rho\beta}\right)\Omega^{-1}\left(\nabla_{\alpha}\nabla_{\beta}\Omega\right)$$
$$+2\left(2\delta^{\rho}_{[\mu}\delta^{\alpha}_{\nu]}\delta^{\beta}_{\sigma} - 2g_{\sigma[\mu}\delta^{\alpha}_{\nu]}g^{\rho\beta} + g_{\sigma[\mu}\delta^{\rho}_{\nu]}g^{\alpha\beta}\right)\Omega^{-2}\left(\nabla_{\alpha}\Omega\right)\left(\nabla_{\beta}\Omega\right)$$

Applying the afore mentioned change of frame we get

$$\tilde{R}_{\alpha\beta}{}^{\zeta\delta} = \phi^{\frac{-4}{(d-2)}} R_{\alpha\beta}{}^{\zeta\delta} - 4\frac{2d}{(d-2)^2}\phi^{\frac{-2d}{(d-2)}}\delta^{[\delta}_{[\alpha}\nabla_{\beta]}\phi\nabla^{\zeta]}\phi$$
$$-\frac{4}{(d-2)^2}\phi^{\frac{-2d}{(d-2)}}\delta^{\zeta\delta}_{\alpha\beta}\nabla^{\kappa}\phi\nabla_{\kappa}\phi + 4\frac{2}{(d-2)}\phi^{\frac{2+d}{2-d}}\delta^{[\delta}_{[\alpha}\nabla_{\beta]}\nabla^{\zeta]}\phi$$

Now we can see that the above tensor, after a LOT of algebra, remains invariant under the transformation

$$g_{\mu\nu} = \Omega^2(x)\hat{g}_{\mu\nu} \qquad \phi = \Omega^{\frac{2-d}{2}}(x)\hat{\phi}$$

- Since c.t.'s are like local scale transformations we should be able to define a covariant derivative that transforms appropriately and produces the previous tensor, which is suitable to construct gauge invariant actions

$$[D_\mu, D_\nu] V^\alpha = \tilde{R}^\alpha{}_{\beta\mu\nu} V^\beta$$

- We now have the building blocks in order to write a general action. A suitable action that is conformally invariant has to be a linear combination of terms of the form

$$I_p = \int d^d x \sqrt{-g} \phi^{\frac{2d}{(d-2)}} X^{\alpha_1 \cdots \alpha_{2p}}_{\beta_1 \cdots \beta_{2p}} \left(\tilde{R}^{\beta_1 \beta_2}{}_{\alpha_1 \alpha_2} \cdots \tilde{R}^{\beta_{2p-1}\beta_{2p}}{}_{\alpha_{2p-1}\alpha_{2p}} \right)$$

where $X^{\alpha_1 \cdots \alpha_{2p}}_{\beta_1 \cdots \beta_{2p}}$ is an invariant tensor

- We demand second order field eqs $\longrightarrow$ $\ddot{\phi}$ must appear at most linearly
This condition restricts the invariant tensor to be fully antisymmetric so that the only allowed possibility is that it turns out to be proportional to the generalized Kronecker delta

$$X^{\alpha_1 \cdots \alpha_{2p}}_{\beta_1 \cdots \beta_{2p}} = \gamma_p \delta^{\alpha_1 \cdots \alpha_{2p}}_{\beta_1 \cdots \beta_{2p}} \qquad \alpha_p = \gamma_p \frac{d-2p}{2^{p+1}}$$

- Finally the most general action that is conformally invariant and leads to second order field equations for the scalar field is given by

$$I[\phi] = -\sum_{p=0}^{k} I_p \qquad \text{where} \qquad 1 \leq k \leq [\frac{d-1}{2}]$$

with [x] given by the integer part, stands for the higher power of the nonminimal couplings and

$$I_p = \frac{2\alpha_p}{d-2p}\int d^dx\sqrt{-g}\phi^{\frac{2d}{d-2}}\delta^{\alpha_1\cdots\alpha_{2p}}_{\beta_1\cdots\beta_{2p}}\left(\tilde{R}^{\beta_1\beta_2}_{\quad\alpha_1\alpha_2}\cdots\tilde{R}^{\beta_{2p-1}\beta_{2p}}_{\quad\alpha_{2p-1}\alpha_{2p}}\right)$$

- Notice that the action can be mapped to the Lovelock action for a suitably rescaled metric

$$I[\phi] = -I_L\left[\phi^{\frac{4}{d-2}}g_{\mu\nu}\right]$$

where

$$I_L\left[\tilde{g}_{\mu\nu}\right] := \sum_{p=0}^{k}\frac{2\alpha_p}{d-2p}\int d^dx\sqrt{-\tilde{g}}\delta^{\alpha_1\cdots\alpha_{2p}}_{\beta_1\cdots\beta_{2p}}\left(\tilde{R}^{\beta_1\beta_2}_{\quad\alpha_1\alpha_2}\cdots\tilde{R}^{\beta_{2p-1}\beta_{2p}}_{\quad\alpha_{2p-1}\alpha_{2p}}\right)$$

Field equations and a special factorization

- The field equation $\mathcal{E}_\phi = 0$ can be obtained by extremizing the action under variations of the scalar field

$$\delta I[\phi] = -\frac{\delta I_L[\tilde{g}^{\alpha\beta}]}{\delta \tilde{g}^{\mu\nu}} \frac{\delta \tilde{g}^{\mu\nu}}{\delta \phi} \delta\phi \quad \longrightarrow \quad \mathcal{E}_\phi := \tilde{\mathcal{E}}_{\mu\nu} \tilde{g}^{\mu\nu} = 0$$

where

$$\tilde{\mathcal{E}}_{\mu\nu} = \frac{1}{\sqrt{\tilde{g}}} \frac{\delta I_L[\tilde{g}^{\alpha\beta}]}{\delta \tilde{g}^{\mu\nu}} = \mathcal{E}_{\mu\nu}[\tilde{g}_{\alpha\beta}]$$

$$\tilde{\mathcal{E}}^{\alpha}{}_{\beta} = -\sum_{p=0}^{k} \frac{\alpha_p}{d-2p} \delta^{\alpha\alpha_1\cdots\alpha_{2p}}_{\beta\beta_1\cdots\beta_{2p}} \left(\tilde{R}^{\beta_1\beta_2}{}_{\alpha_1\alpha_2} \cdots \tilde{R}^{\beta_{2p-1}\beta_{2p}}{}_{\alpha_{2p-1}\alpha_{2p}} \right)$$

Alternatively the field equation can be written as $\mathcal{E}_\phi = -\sum_{p=0}^{k} \alpha_p \tilde{\mathcal{E}}_{2p} = 0$

where $\tilde{\mathcal{E}}_{2p} = \delta^{\alpha_1\cdots\alpha_{2p}}_{\beta_1\cdots\beta_{2p}} \tilde{R}^{\beta_1\beta_2}{}_{\alpha_1\alpha_2} \cdots \tilde{R}^{\beta_{2p-1}\beta_{2p}}{}_{\alpha_{2p-1}\alpha_{2p}}$

stands for the dimensional continuation of the Eyler density from 2p to d dimensions

- The E.M.T.
$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta I[\phi]}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{-g}}\frac{\delta I_L[\tilde{g}^{\sigma\gamma}]}{\delta \tilde{g}^{\alpha\beta}}\frac{\delta \tilde{g}^{\alpha\beta}}{\delta g^{\mu\nu}}$$

so that it reduces to
$$T_{\mu\nu} = \phi^2 \tilde{\mathcal{E}}_{\mu\nu}$$

- The stress-energy tensor transforms homogeneously with conformal weight -d, and also the trace of the E.M.T. vanishes on shell
$$T^{\mu}_{\ \nu} \to \Omega^{-d} T^{\mu}_{\ \nu}$$
$$T^{\mu}_{\ \mu} = \phi^{\frac{2d}{d-2}} \mathcal{E}_{\phi}$$

- The field equation and the E.M.T. can be factorized according to
$$\mathcal{E}_\phi = c_0 \delta^{\alpha_1\cdots\alpha_{2k}}_{\beta_1\cdots\beta_{2k}} \left(\tilde{R}^{\beta_1\beta_2}_{\ \ \alpha_1\alpha_2} + c_1 \delta^{\beta_1\beta_2}_{\alpha_1\alpha_2}\right)\cdots\left(\tilde{R}^{\beta_{2k-1}\beta_{2k}}_{\ \ \alpha_{2k-1}\alpha_{2k}} + c_p \delta^{\beta_{2k-1}\beta_{2k}}_{\alpha_{2k-1}\alpha_{2k}}\right) = 0$$
$$T^{\alpha}_{\beta} = \frac{c_0}{d-2k}\phi^{\frac{2d}{d-2}} \delta^{\alpha\alpha_1\cdots\alpha_{2k}}_{\beta\beta_1\cdots\beta_{2k}} \left(\tilde{R}^{\beta_1\beta_2}_{\ \ \alpha_1\alpha_2} + c_1 \delta^{\beta_1\beta_2}_{\alpha_1\alpha_2}\right)\cdots\left(\tilde{R}^{\beta_{2k-1}\beta_{2k}}_{\ \ \alpha_{2k-1}\alpha_{2k}} + c_p \delta^{\beta_{2k-1}\beta_{2k}}_{\alpha_{2k-1}\alpha_{2k}}\right)$$

where the coefficients c_i's are related to the a_i's through the relation
$$-\frac{4}{d-2}\sum_{p=0}^{k}\alpha_p x^p = c_0 \prod_{i=1}^{k}(x+c_i)$$

Examples

- The standard conformally coupled scalar field: k=1

$$I[\phi] = \int d^dx\sqrt{-g}\left(\frac{1}{2}\phi\Box\phi - \frac{1}{8}\frac{d-2}{d-1}\phi^2 R + \frac{(d-2)^2}{8d(d-1)}\lambda\phi^{\frac{2d}{d-2}}\right)$$

$$\Box\phi - \frac{1}{4}\frac{d-2}{d-1}R\phi + \frac{d-2}{4(d-1)}\lambda\phi^{\frac{d+2}{d-2}} = 0 \qquad \longrightarrow \qquad \tilde{R} = \lambda$$

- k=2 is the most general case in d=5 and d=6

$$\begin{aligned}
I_2 &= \gamma_2 \frac{1}{2}\int d^dx\sqrt{-g}\phi^{\frac{2d}{d-2}}\delta^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\beta_1\beta_2\beta_3\beta_4}\tilde{R}^{\beta_1\beta_2}_{\alpha_1\alpha_2}\tilde{R}^{\beta_3\beta_4}_{\alpha_3\alpha_4} \\
&= \gamma_2\int d^dx\sqrt{-g}(\phi^{\frac{2(d-4)}{(d-2)}}(R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}) \\
&\quad - 16\frac{(d-3)d}{(d-2)^2}\phi^{\frac{-4}{(d-2)}}R^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + 16\frac{(d-3)}{(d-2)}\phi^{\frac{d-6}{d-2}}R^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi - 8\frac{(d-3)}{(d-2)}\phi^{\frac{d-6}{d-2}}R\Box\phi \\
&\quad + 16\frac{(d-3)}{(d-2)^2}\phi^{-\frac{4}{d-2}}R\nabla_\alpha\phi\nabla^\alpha\phi - 16\frac{(d-3)(d-1)d}{(d-2)^3}\phi^{\frac{2d}{d-2}}(\nabla_\alpha\phi\nabla^\alpha\phi)^2 \\
&\quad - 32\frac{(d-3)}{(d-2)^2}\phi^{-\frac{d+2}{d-2}}\nabla_\alpha\phi\nabla^\alpha\phi\Box\phi + 32\frac{(d-3)d}{(d-2)^2}\phi^{-\frac{d+2}{d-2}}\nabla_\alpha\nabla_\beta\phi\nabla^\alpha\phi\nabla^\beta\phi \\
&\quad + 16\frac{(d-3)}{(d-2)}\phi^{-\frac{4}{d-2}}(\Box\phi)^2 - 16\frac{(d-3)}{(d-2)}\phi^{-\frac{4}{d-2}}\nabla_\alpha\nabla_\beta\phi\nabla^\alpha\nabla^\beta\phi)
\end{aligned}$$

Non-trivial solutions

- Configurations of constant rescaled curvature $\tilde{R}_{\alpha\beta}{}^{\zeta\delta} = -\tilde{c}\delta_{\alpha\beta}{}^{\zeta\delta}$ not only solve the field equation but also have a vanishing E.M.T., so they can be regarded as non-trivial vacua.

- On flat euclidean space $ds^2 = d\rho^2 + \rho^2 d\Omega^2_{d-1}$ the solution is given by

$$\phi = \left[\frac{\rho^2}{a} - \frac{\tilde{c}}{4}a\right]^{1-\frac{d}{2}}$$

  - regularity requires that $\tilde{c} < 0$ and in odd dimensions the integration constant α has to be positive.
  - value of the action: $I = -\sum_{p=0}^{k} I_p$ with $I_p^0 = \sqrt{\pi}\Omega_{d-1}\gamma_p(-\tilde{c})^{p-\frac{d}{2}}\frac{d!}{(d-2p)!}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1+d}{2}\right)}$

- For the case of a unit sphere S^d $ds^2 = d\theta^2 + \sin^2\theta d\Omega^2_{d-1}$ the solution is given by

$$\phi = \frac{C}{\left(\cos\theta + \epsilon\sqrt{1-\tilde{c}C^{2\frac{2}{d-2}}}\right)^{\frac{d-2}{2}}}$$

  - regularity requires that $\epsilon = 1$ and $\tilde{c}C^{\frac{4}{d-2}} < 0$
  - value of the action: $I = -\sum_{p=0}^{k} I_p$ with $I_p^1 = (-1)^d I_p^0$

Conclusions

- We have presented a generalization of the standard action for the conformally coupled scalar field with second order field eqs.

- C.I. strongly restricts the possible non-minimal couplings with higher powers of the curvature in the action

- Configurations of constant rescaled curvature, correspond to non-trivial vacua of the theory (vanishing of the scalar field equation and identically vanishing E.M.T.)

- In Euclidean constant curvature spaces, this class of solutions describe instantons, since they are regular everywhere and possess finite action

- A generalization of the YAMABE problem...

  similarities with Lovelock theory are not a coincidence

  the YAMABE problem could be extended to Lovelock theory in the following sense:

Given a compact Riemannian manifold (M,g) of dimension $n>3$, is there a metric conformal to g such that the linear combination of the dimensional continuation the lower-dimensional Euler densities is constant?

Thank you