New Massive Gravity

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- Part 1: Massive spin-1, Poincaré UIRs, Parity & helicity, Ghosts & Tachyons, 3D Fierz-Pauli, Linearized NMG, NMG, Unitarity & Renormalizability.

- Part 2: 'Cosmological' NMG, dS & AdS, Fierz-Pauli in AdS_3, Bulk vs Boundary unitarity, CS-like & Hamiltonian NMG, NMG to ZDG. Open questions.

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Proca equation

Consider action for 1-form A in D spacetime dimensions:

$$S = \int d^D x \left\{ -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} m^2 A_\mu A^\mu \right\} \qquad (F = dA).$$

Field equation is Proca equation $\partial^\mu F_{\mu\nu} = m^2 A_\nu$, which is equivalent to the two equations

$$\left(\Box - m^2\right) A = 0 \qquad \& \qquad \partial \cdot A = 0\,.$$

The latter equation is a subsidiary condition. It implies that there are only $(D-1)$ independent solutions of the form $A = ae^{ip\cdot x}$ for given 3-momentum p. For $D > 3$ these solutions span an *irreducible* representation of the rotation group $SO(D-1)$.

- For $D = 3$ we get two one-dimensional irreps of $SO(2)$.

3D Proca

We can write the 3D Proca eq. as

$$P(m)P(-m)A = 0\,, \qquad [P(m)]_\mu{}^\nu = \tfrac{1}{2}\left(\delta_\mu^\nu - \tfrac{1}{m}\varepsilon_\mu{}^{\tau\nu}\partial_\tau\right)\,.$$

Acting on solutions of the Proca equation, we have

$$P^2(\pm m) = P(\pm m)\,,$$

which shows that $P(\pm m)$ are (on-shell) projection operators. They project onto the irreducible Poincaré irreps.

➽ Parity takes $P(m) \to P(-m)$. The two polarization states propagated by Proca are a parity doublet.

Aside: 3D Poincaré UIRs

- 3D Poincaré group generated by 3-vectors $\mathcal{P}_\mu$ and $\mathcal{J}^\mu$. Massive UIRs classified by Casimirs

$$-\mathcal{P}^2 \equiv M^2 \,, \qquad \mathcal{P}\cdot\mathcal{J} \equiv Mh$$

M is mass and h is “relativistic helicity”. Define $|h|$ to be “spin”.

- $\mathcal{J}$ is pseudovector, so parity takes $h \to -h$.

- $2h \notin \mathbb{Z} \quad \Rightarrow \quad$ “Anyon” (by 3D spin/statistics theorem)

- Spin not defined if $M^2 = 0$, but still 3 UIRs: Boson & Fermion, and “infinite spin” (analog of 4D “continuous spin”)

Parity and helicity

➽ In any parity preserving 3D free field theory, every state of mass m and helicity h is paired with a state of the same mass but helicity $-h$. In other words, states of spin s come in parity doublets with helicities $(s,-s)$.

➽ If parity is violated, the degeneracy due to parity may be lifted. A doublet of states of helicities $\pm s$ may now have masses $m_\pm$ with $m_+ \neq m_-$.

➽ Proca preserves parity. The two spin-1 states it propagates are a parity doublet of helicities $(1,-1)$.

➽ We can generalize Proca equation to $P(m_+)P(-m_-)A = 0$. This propagates helicity 1 with mass m_+ and helicity -1 with mass m_-.

Generalized 3D Proca

The generalized 3D Proca equation $P(m_+)P(-m_-)A = 0$ is the field equation for the parity violating Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2\mu}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \frac{1}{2}m^2 A_\mu A^\mu\,,$$

where

$$m^2 = m_+ m_-\,, \qquad \mu = \frac{m_+ m_-}{m_- - m_+}\,.$$

The limit in which $m_+ = m_-$ is the limit in which $\mu = \infty$ and the parity-breaking "$\varepsilon A\partial A$" Chern-Simons term is absent.

➽ We can decouple the helicity -1 mode by taking $m_- \to \infty$ for fixed m_+ This yields the first-order equation $P(m_+)A = 0$.

Square root of 3D Proca

- The first-order equation $P(\pm m)A = 0$ is equivalent to

$$\tilde{F}^{\mu} = \pm m\, A^{\mu}\,, \qquad \left(\tilde{F}^{\mu} = \varepsilon^{\mu\nu\rho}\partial_\nu A_\rho\right)$$

This is a kind of "square root" of the Proca equation that propagates a single massive spin-1 mode of helicity ± 1. It is the field equation for the "self-dual" spin-1 Lagrangian

$$\mathcal{L}_{SD} = \pm\tfrac{1}{2} m\, \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \tfrac{1}{2} m^2 A_\mu A^\mu\,.$$

- N.B. The $\sqrt{\text{Proca}}$ equation still implies the subsidiary condition $\partial \cdot A = 0$.

TME

Any divergence free vector can be written locally as a curl, so

$$\partial \cdot A = 0 \quad \Rightarrow \quad A_\mu = \tilde{G}_\mu \,, \qquad \tilde{G}^\mu \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\rho}\partial_\nu B_\rho \,.$$

This is the general *local* solution.

➸ Back-substitute in the $\sqrt{\text{Proca}}$ equation to get the new equation $P(m)G = 0$. This is the field equation of **Topologically Massive Electrodynamics**, defined by

$$\mathcal{L}_{TME} = -\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} \pm \tfrac{1}{2}m\,\varepsilon^{\mu\nu\rho}B_\mu\partial_\nu B_\rho \,, \qquad \left(G_{\mu\nu} = 2\partial_{[\mu}B_{\nu]}\right)$$

We now have have a gauge theory description of massive spin 1, but only of one helicity (± 1).

Extended Proca

➽ Let's apply the same "trick" to 3D Proca. We solve the subsidiary condition as before, $A = \tilde{G}$, and back-substitute into Proca equation to get new equation $\left(\Box - m^2\right)\tilde{G} = 0$. This is a 3rd order equation that propagates (by construction) two spin-1 modes of mass m. It is also the field equation for the parity odd **Extended Proca** Lagrangian

$$\mathcal{L}_{EP} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho}\left[\tilde{G}_\mu\partial_\nu\tilde{G}_\rho - m^2\, B_\mu\partial_\nu B_\rho\right] .$$

➽ Higher derivative theories usually propagate "ghosts" (modes of negative energy). That is the case here. Because the Lagrangian is parity odd, parity flips both helicity **and** the sign of the energy: one of the two spin-1 states is a ghost.

Aside: Ghosts vs Tachyons

There are various ways in which a free field theory may be unphysical. It can have have either ghosts or tachyons. Both are defined as unitary irreps of the Poincaré group.

- Ghosts are Poincaré UIRs with $\mathcal{P}^2 \leq 0$ (i.e. real M) but $P^0 < 0$ (negative energy)

- Tachyons are Poincaré UIRs with $\mathcal{P}^2 > 0$ (i.e. imaginary M).

Supersymmetry excludes tachyons because there are no tachyonic UIRs of the super-Poincaré group, but it does not exclude ghosts. Ghosts can appear in supermultiplets. In this case, the Hilbert space necessarily includes states of negative norm (because otherwise susy implies that $P^0 \geq 0$).

➽ The spin-2 Fierz-Pauli Lagrangian for symmetric tensor φ is

$$\mathcal{L}_{FP} = -\tfrac{1}{2}\varphi_{\mu\nu}\,\mathcal{G}^{\mu\nu,\rho\sigma}\varphi_{\rho\sigma} - \tfrac{1}{4}m^2\left(\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi^2\right), \qquad (\varphi = \varphi^{\mu}{}_{\mu}),$$

where $\mathcal{G}$ is the 2nd-order operator, symmetric under interchange of its index pairs, such that $\mathcal{G}\varphi$ is the linearized Einstein tensor, so the linearized Bianchi identity is $\partial_\mu\mathcal{G}^{\mu\nu,\rho\sigma} \equiv 0$. The field equations are equivalent to the FP equations

$$\left(\Box - m^2\right)\varphi_{\mu\nu} = 0\,, \qquad \partial^\mu\varphi_{\mu\nu} = 0\,, \qquad \varphi = 0\,.$$

Now there is a differential subsidiary condition (as for spin 1) but also an algebraic subsidiary condition. These reduce number of modes propagated to $D(D+1)/2 - D - 1 = (D-2)(D+1)/2$. For $D > 3$ this is traceless symmetric tensor irrep. of $SO(D-1)$.

3D FP

➽ The number of modes propagated by spin-2 FP equations for $D = 3$ is 2: i.e. a parity doublet of spin-2 modes. The FP equations can be written as

$$[P(m)P(-m)]_\mu{}^\nu \varphi_{\nu\rho} = 0\,, \qquad \varphi^\mu{}_\mu = 0\,,$$

where $P(\pm m)$ are the spin-1 on-shell projection operators.

➽ There is a parity-violating "generalized FP" theory as for Proca. It propagates helicity ± 2 with mass $m_\pm$.

➽ The linearized 3D Einstein operator can be written as

$$\mathcal{G}_{\mu\nu}{}^{\rho\sigma} = -\tfrac{1}{2}\varepsilon_{(\mu}{}^{\eta\rho}\varepsilon_{\nu)}{}^{\tau\sigma}\partial_\tau\partial_\sigma$$

Linearized NMG

- As for Proca, we can solve the differential constraint:

$$\partial^\mu \varphi_{\mu\nu} = 0 \quad \Rightarrow \quad \varphi_{\mu\nu} = \mathcal{G}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \equiv G^{(lin)}_{\mu\nu}(h)$$

As a check, note that this equation can be interpreted as an equation for a metric perturbation h given a source with stress tensor φ. In 3D the metric is locally flat in the absence of a source, so h must be pure gauge when $\varphi = 0$.

- Substitute this solution into remaining FP equations to get

$$\left(\Box - m^2\right) G^{(lin)}_{\mu\nu}(h) = 0\,, \qquad R^{(lin)} = 0\,.$$

These are the equations of linearized "New Massive Gravity". By construction, they propagate a parity doublet of spin-2 modes.

NMG action

We can replace the two equations of linearized NMG by

$$G_{\mu\nu}^{(lin)} - \frac{1}{m^2}\left[\Box G_{\mu\nu}^{(lin)} - \frac{1}{4}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right) R^{(lin)}\right] = 0\,.$$

The trace of $[\dots]$ is zero, so the trace implies $R^{(lin)} = 0$. This equation is what one gets by variation of the quadratic approximation to the NMG Lagrangian

$$\mathcal{L}_{NMG} = \sqrt{|g|}\left[-R + \frac{1}{m^2}K\right]\,, \qquad K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\,.$$

The Einstein Hilbert term has the "wrong sign" when compared to the standard 3D GR action!

➽ Changing the relative sign makes the spin-2 modes tachyons. Changing the overall sign makes them ghosts.

Auxiliary field form of action

- We can rewrite the NMG action as

$$\mathcal{L}_{NMG} = \sqrt{|g|}\left\{-R + \frac{2}{m^2}\left[f^{\mu\nu}G_{\mu\nu} - \frac{1}{2}\left(f^{\mu\nu}f_{\mu\nu} - f^2\right)\right]\right\}$$

where $f_{\mu\nu}$ is a new symmetric tensor field, and $f = g^{\mu\nu}f_{\mu\nu}$. The $f_{\mu\nu}$ field equation is algebraic:

$$f_{\mu\nu} = S_{\mu\nu}\,, \qquad S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\,.$$

The S-tensor is the 3D Schouten tensor. Using this equation to eliminate $f_{\mu\nu}$ we recover the 4th order action.

- This shows that $K = G^{\mu\nu}S_{\mu\nu}$, which has special properties in all dimensions.

Unitarity

➳ We may linearize about the Minkowski vacuum by setting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \frac{2}{m^2} f_{\mu\nu}$ and expanding to quadratic order in h (the action is already quadratic in f). This gives

$$S^{(2)}_{NMG}[h,f] = -S^{(2)}_{EH}[h] + \frac{2}{m^4} S_{FP}[f;m]\,.$$

The Einstein-Hilbert term comes with the wrong sign but it propagates no modes. The FP term propagates two spin-2 modes of mass m. $\quad\Rightarrow\quad$ NMG is unitary.

➳ Any change in the relative coefficients of $R^{\mu\nu}R_{\mu\nu}$ and R^2 in the NMG action leads to a change in the relative coefficients of $f^{\mu\nu}f_{\mu\nu}$ and f^2 in $S_{FP}[f]$, and this leads to an additional scalar ghost. Unitarity requires "fine-tuning".

Scalar Massive Gravity

- There is one other curvature-squared extension of 3D GR that is unitary:

$$\mathcal{L}_{SMG} = \sqrt{|g|}\left[R + \frac{1}{8m^2}R^2\right]$$

The Einstein-Hilbert term now comes with the “right sign”. Linearization shows that there is a single scalar mode of mass m. This is a special case of 3D “$f(R)$ gravity”.

- General curvature squared extension of 3D GR is

$$\mathcal{L}(a,b) = \sqrt{|g|}\left[\sigma R + \frac{1}{m^2}\left(aK + \frac{1}{8}bR^2\right)\right]$$

Unitarity requires $\left\{\begin{array}{ccc} \textbf{either} & a = 0 \;\&\; \sigma = 1 & SMG \\ \textbf{or} & b = 0 \;\&\; \sigma = -1 & NMG \end{array}\right.$

Renormalizability?

Expanding $\mathcal{L}(a,b)$ about Minkowski space, and adding a suitable gauge-fixing term, we find

$$\mathcal{L}^{(2)}(a,b) + \mathcal{L}_{\text{fix}} = \tfrac{1}{2} h^{\mu\nu} \mathcal{O}_{\mu\nu\rho\sigma}(a,b) h^{\rho\sigma} \,,$$

where $\mathcal{O}$ is a linear combination of a spin-2 projection operator P_2 and a spin-0 projection operator P_0. Inverting in each of the two subspaces, we get the propagator

$$\frac{2m^2}{p^2(ap^2 - m^2\sigma)} P_2 + \frac{2m^2}{p^2(bp^2 + m^2\sigma)} P_0$$

If **both** terms go like $1/p^4$ the model is renormalizable. Otherwise it is non-renormalizable. So renormalizability requires $ab \neq 0$. But unitarity requires $ab = 0$ (SMG or NMG). The conclusion is: Unitarity implies non-renormalizability!

NMG for $D > 3$?

➽ In D dimensions $S_{\mu\nu} = \frac{1}{D-2}\left[R_{\mu\nu} - \frac{1}{2(D-1)}g_{\mu\nu}R\right]$, and we have the identity

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2 \equiv W^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma} - 4(D-3)G^{\mu\nu}S_{\mu\nu}$$

The LHS is the Gauss-Bonnet integrand, which is a total derivative for $D = 4$ and the Lovelock Lagrangian for $D > 4$.

➽ In any dimension, the addition of $G^{\mu\nu}S_{\mu\nu}$ leads to massive spin-2 for "wrong-sign" EH term, but unless $D = 3$ there is, in addition, a massless spin-2 ghost. With any other curvature-squared term, there is also a scalar ghost.

➽ There is a unitary $D > 3$ version of linearized NMG involving a "dual graviton" field, but no non-linear extension is known.

"Cosmological" NMG

- Add a cosmological term to NMG to get

$$\mathcal{L}_{CNMG} = \sqrt{|g|}\left[\sigma R + \frac{1}{m^2}K - 2\Lambda_0\right] \qquad (\sigma = \pm 1)$$

For $\Lambda_0 = 0$ we know that we must choose $\sigma = -1$, but now we allow for either sign (and either sign of m^2) and seek maximally symmetric solutions with $G_{\mu\nu} = -\Lambda g_{\mu\nu}$, i.e. with cosmological constant Λ. One finds that

$$\sigma\Lambda - \Lambda_0 + \frac{1}{m^2}\Lambda^2 = 0 \qquad \Rightarrow \qquad \Lambda = -2m^2\left[\sigma \pm \sqrt{1 + \Lambda_0/m^2}\right]$$

- Must have $\lambda_0 \equiv \Lambda_0/m^2 \geq -1$.

- Even for $\Lambda_0 = 0$ there is an (A)dS solution with $\Lambda = -4m^2\sigma$.

Linearization of CNMG

Given vacuum metric $\bar{g}$ with cosmological constant $\Lambda \neq 2m^2\sigma$, linearize $S_{NMG}[g, f]$ by setting

$$f_{\mu\nu} = k_{\mu\nu} + \frac{\Lambda}{2m^2} g_{\mu\nu}\,, \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} - \frac{4}{\Lambda - 2m^2\sigma} k_{\mu\nu}\,.$$

This gives quadratic Lagrangian

$$\mathcal{L}^{(2)}_{CNMG} = \frac{(2m^2\sigma - \Lambda)}{2m^2} \mathcal{L}^{(2)}_{EH}(h) + \frac{4}{m^2(\Lambda - 2m^2\sigma)} \mathcal{L}_{FP}(k; M)\,,$$

where $\mathcal{L}_{FP}(k; M)$ is the FP Lagrangian in the background metric g with $M^2 = \frac{1}{2}\left(\Lambda - 2m^2\sigma\right)$.

➽ Unitarity (no ghosts or tachyons) $\Leftrightarrow$ $m^2 > 0$ & $\Lambda > 2m^2\sigma$.

dS vacua

➻ In dS vacua, unitarity of FP theory imposes Higuchi bound: $M^2 \geq \Lambda$. For NMG this is equivalent to $\Lambda \leq -2m^2\sigma$, which has no $\Lambda > 0$ solution unless $\sigma = -1$, in which case the Higuchi bound is $\Lambda \leq 2m^2$. In this case we have

$$\sigma = -1\,, \qquad \Lambda = 2m^2\left[1 - \sqrt{1+\lambda_0}\right]\,, \qquad \lambda_0 \geq -1$$

➻ Higuchi bound is saturated when $\lambda_0 = -1$. In this vacua the gravitons are "partially massless". There is a gauge invariance of the FP action in partially massless vacua that eliminates one of the two propagating modes. However, this is an artefact of linearization – there are still two local degrees of freedom but only one appears in the linear theory.

AdS vacua

➽ The unitarity conditions $m^2 > 0$ & $\Lambda > 2m^2\sigma$ have no $\Lambda < 0$ solution unless $\sigma = -1$, i.e. "wrong sign" EH term, and then

$$\Lambda = 2m^2\left[1 - \sqrt{1+\lambda_0}\right], \qquad 0 \le \lambda_0 < 3,$$

where $\lambda_0 < 3$ is due to the NMG unitarity bound $\Lambda > -2m^2$

➽ At $\lambda_0 = 3$ we have $\Lambda = 2m^2\sigma$. In this case $\mathcal{L}^{(2)}_{NMG}$ is not diagonalizable. This case defines "critical" 3D gravity.

➽ N.B. The Breitenlohner-Freedman bound for scalars in AdS is $M^2 > \Lambda$, which would allow $-1 \ge \lambda_0 < 0$ if it also applied to spin-2 fields. However, Unitarity for spin-2 requires $M^2 \ge 0$.

Factorizing FP in AdS

Let $\bar{D}$ be the covariant derivative in the background AdS metric $\bar{g}$ with $\Lambda = -1/\ell^2$, and define

$$[\mathcal{D}(\eta)]_\mu{}^\nu = \ell^{-1}\delta_\mu^\nu + \frac{\eta}{\sqrt{|g|}}\varepsilon_\mu{}^{\tau\nu}\bar{D}_\tau\,.$$

The FP equations in AdS can be written, for traceless symmetric tensor φ, in the factorized form

$$[\mathcal{D}(\eta)\mathcal{D}(-\eta)]_\mu{}^\rho\varphi_{\rho\nu} = 0\,, \qquad \eta^2 = 2/\left(1 - 2m^2\ell^2\sigma\right)\,.$$

The unitarity bound $\Lambda > 2m^2\sigma$ can now be expressed as $|\eta| \leq 1$.

➽ Linearized Einstein equations saturate the bound. Transverse traceless metric perturbation H satisfies $[\mathcal{D}(1)\mathcal{D}(-1)]_\mu{}^\rho H_{\rho\nu} = 0$.

Factorizing NMG in AdS

In an appropriate gauge, the 4th order NMG equations take the factorized form

$$[\mathcal{D}(1)\mathcal{D}(-1)\mathcal{D}(\eta)\mathcal{D}(-\eta)]_\mu{}^\rho H_{\rho\nu} = 0\,, \qquad \eta^2 \leq 1\,.$$

For $\eta^2 \neq 1$, this is **equivalent to two second-order equations**: linearized Einstein and FP (as we saw from linearization of auxiliary field form of action). The FP equation propagates a parity doublet of spin-2 modes, now in AdS_3.

➳ In the unitarity limit $\eta^2 = 1$ we get the equation

$$\left[\mathcal{D}^2(1)\mathcal{D}^2(-1)\right] H_{\rho\nu} = 0\,.$$

This is the equation for linearized 3D critical gravity.

Bulk vs Boundary unitarity

➻ In asymptotically AdS space, the asymptotic symmetry group is the 2D conformal group, with algebra $\mathrm{Vir}_L \oplus \mathrm{Vir}_R$. The central charges of the two Virasoro algebras are equal ($c_L = c_R = c$) if parity is preserved. For NMG one finds that

$$c = \frac{3\ell}{2G_3}\left(\sigma - \frac{\Lambda}{2m^2}\right), \qquad \left(\ell = 1/\sqrt{-\Lambda}\right)$$

where G_3 is the 3D Newton constant.

➻ Modern interpretation: CFT is holographic dual of 3D quantum gravity with NMG as semi-classical limit, valid for $c >> 1$.

➻ Unitarity of CFT requires $c > 0$, but the bulk unitarity condition $\Lambda > 2m^2\sigma$ implies $c < 0$. We have a clash between bulk and boundary unitarity!

Einstein-Cartan and Chern-Simons

In its "Einstein-Cartan" formulation, 3D GR involves only the 3-vector 1-forms e^a (dreibein) and ω^a (spin connection) from which we construct the torsion and curvature 2-forms

$$T^a = de^a + \epsilon^{abc}\omega_b e_c\,, \qquad R^a = d\omega^a + \tfrac{1}{2}\epsilon^{abc}\omega_b\omega_c\,.$$

The EC action is the integral of the following 3-form Lagrangian

$$L_{EC} = -\sigma e_a \wedge R^a + \tfrac{1}{6}\Lambda_0 \epsilon^{abc} e_a \wedge e_b \wedge e_c$$

This is also a Chern-Simons 3-form for the dS ($\Lambda_0 > 0$), AdS ($\Lambda_0 < 0$) or Poincaré ($\Lambda_0 = 0$) group. The field equations are $T^a = 0$ and $R^a = 0$. Assuming invertibility of $e_\mu{}^a$, these are equivalent to the 3D Einstein equations.

CS-like formulation of NMG

An alternative EC Lagrangian 3-form is $L'_{EC} = L_{EC} + h_a \wedge T^a$, where h^a is a Lagrange multiplier field for the constraint $T^a = 0$. The ω^a field equation now determines h^a, and the e^a equation is then again equivalent to the Einstein equations. Now consider

$$L_{NMG} = L'_{EC} + \frac{1}{m^2}\left[f_a \wedge R^a - \frac{1}{2}\epsilon^{abc} e_a \wedge f_b \wedge f_c\right] .$$

The 1-forms f^a are auxiliary; field equation $\Rightarrow$ $f_{\mu\nu} = S_{\mu\nu}$, and the $[\ldots]$ term becomes $G^{\mu\nu}S_{\mu\nu}$. So L_{NMG} is a Lagrangian 3-form for NMG. It takes the Chern-Simons-like form

$$L_{NMG} = \frac{1}{2} g_{rs} A^r \cdot dA^s + \frac{1}{6} f_{rst} A^r \cdot A^s \times A^t ,$$

where A^r are four flavours $(r = 1, \ldots 4)$ of 3-vector 1-forms, g_{rs} is a metric on "flavour space", and f_{rst} are coupling constants.

Hamiltonian NMG

A time/space split $A_\mu \to (A_0, A_i)$ leads to Lagrangian

$$\mathcal{L} = -\frac{1}{2}\varepsilon^{ij} g_{rs} A_i^r \cdot \dot{A}_j^s + A_0^r \cdot \Phi_r \,,$$

where Φ_r are four 3-vector primary constraints. Equations of motion imply secondary constraints $\varepsilon^{ij} f_{ij} = 0$ and $\varepsilon^{ij} h_{ij} = 0$. No tertiary constraints, so we have $4\times 3+2 = 14$ (local) constraints on a phase space of (local) dimension $4 \times 3 \times 2 = 24$.

There are 6 gauge invariances (Diff$_3$ and Local Lorentz) so we expect (and find) 6 "first class" constraints. The remaining 8 are "second-class". Hence physical phase space has (local) dimension $24 - 2 \times 6 - 8 = 4$. **Agrees with linearized limit** (each mode adds 2 to dimension) and extends it to full theory: $\Rightarrow$ no Boulware-Deser ghost.

The CS-like formulation of NMG suggests that it might be a limit of some theory with two dreibein 1-forms, e^a and f^a; the h^a field could then be a limit of a second spin-connection.

Let $\left(e_I^a, \omega_I^a\right)$ be a pair of EC 1-form fields ($I = 1, 2$), with EC Lagrangian 3-forms

$$L_I = -\sigma_I M_I e_1^a \wedge R_I^a - \frac{1}{6}\alpha_I m^2 M_I \epsilon_{abc} e^a \wedge e^b \wedge e^c \,,$$

where $\sigma_I = (1, \sigma = \pm 1)$ and $\alpha_I = (\alpha_1, \alpha_2)$ are arbitrary constants. The masses $M_I = (M_1, M_2)$ are assumed to be positive.

➽ $L = L_1 + L_2$ describes two uncoupled EC models with **independent** EC gauge invariances. We need a coupling that breaks this to the 'diagonal' EC gauge group.

Zwei-Dreibein Gravity

Couple the two EC Lagrangians with

$$L_{12} = \tfrac{1}{2} m^2 M_{12} \epsilon_{abc} \left(\beta_1 e_1^a \wedge e_1^b \wedge e_2^c + \beta_2 e_1^a \wedge e_2^b \wedge e_2^c \right).$$

where $M_{12} = (\sigma M_1 M_2)/(\sigma M_1 + M_2)$, and $\beta_I = (\beta_1, \beta_2)$ are two more parameters.

We now have the ZDG Lagrangian 3-form

$$L_{ZDG}(\alpha_I, \beta_I, \eta; \sigma) = L_1 + L_2 + L_{12} \qquad (\eta = M_1/M_2).$$

It depends on five continuous parameters and a sign.

$ZDG \to NMG$

Take $\sigma = -1$, and $\beta_I = (0, 1)$ and

$$\alpha_1 = -\lambda_0\lambda + \tfrac{1}{\lambda}\,, \qquad \alpha_2 = 2\left(1 + \tfrac{1}{\lambda}\right)$$
$$M_1 = 2\left(1 + \tfrac{1}{\lambda}\right) M_P\,, \qquad M_2 = \tfrac{2}{\lambda} M_P\,.$$
$$e_2^a = e_1^a + \tfrac{\lambda}{m^2} f^a\,, \qquad \omega_2^a = \omega_1^a - \lambda h^a\,.$$

Substitute into L_{ZDG} and take $\lambda \to 0$ limit. Result is

$$L_{ZDG} \to 2M_P\, L_{NMG}$$

When account is taken of the 3D Newton constant, M_P is the 3D Planck mass for NMG.

Linearization of ZDG

Linearize ZDG about a vacuum dreibein $\bar{e}^a$ with cosmological constant Λ by setting

$$e_1^a = \bar{e}^a + h_1^a\,, \qquad e_2^a = \gamma\left(\bar{e}^a + h_1^a\right) \qquad \omega_I = \bar{\omega}(\bar{e}) + v_I^a$$

Cancellation of linear term fixes both Λ and parameter γ in terms of ZDG parameters (α_1, α_2). Diagonalizing $\mathcal{L}^{(2)}$, and defining $M_{\text{crit}} = \sigma M_1 + \gamma M_2$, we get

$$\mathcal{L}^{(2)} = M_{\text{crit}}\mathcal{L}_{EH}^{(2)} - \frac{\sigma\gamma M_1 M_2}{M_{\text{crit}}}\mathcal{L}_{FP}(\mathcal{M})\,,$$

where $\mathcal{M}^2 = m^2\,(\beta_1 + \gamma\beta_2)\,\frac{M_{\text{crit}}}{\sigma M_1 + M_2}$.

➻ Can now read off conditions for bulk unitarity.

Physical parameter space of ZDG

Assume AdS vacuum. For ZDG, boundary CFT central charge is $c = 12\pi\ell\, M_{\rm crit}$: boundary unitarity requires $M_{\rm crit} > 0$.

➽ Bulk unitarity is compatible with $M_{\rm crit} > 0$ only if $\sigma = 1$. Then all unitarity conditions are satisfied by

$$M_1 = M_2\,, \qquad \beta_1 = \beta_2 = 1\,, \qquad \alpha_1 = \alpha_2 = \frac{3}{2} + \zeta\,, \qquad (\zeta > 0)$$

and this gives a vacuum with $\gamma = 1$ and $\Lambda/m^2 = -\zeta$.

➽ The unitarity conditions are inequalities on ZDG parameters that have been **satisfied but not saturated**. No fine tuning is needed to achieve both boundary and bulk unitarity.

Summary and open questions

➽ NMG is generally covariant ghost-free non-linear extension of the 3D Fierz-Pauli theory for massive spin-2. It propagates a parity doublet of massive gravitons, has no other local degrees of freedom, and is unitary.

➽ But (i) unitarity requires fine tuning, and so is potentially unstable against quantum corections and (ii) BTZ black holes of NMG have negative mass. Equivalently, $c < 0$ for holographically dual CFT.

➽ ZDG solves both problems, but no ghost-free parity-violating extension of ZDG is known.

➽ Can adS/CFT be established for ZDG beyond semi-classical limit?