Constant Scalar Invariant Geometries Solving Massive Gravity Theories

Ph. SPINDEL
Université de Mons

work with K. Siampos,
Class. Quantum Grav. 30 145014 (2013), [arXiv:1302.6250 [hep-th]].

7th Aegean Summer School
Beyond Einstein's theory of Gravity

Short version (less than one minute!)

We solve massive gravity theory equations in the framework of
locally homogeneous spaces and of vanishing scalar invariant spaces.

These spaces are the building blocks of the constant scalar invariant spaces.

A variant of Petrov classification for 3D spaces will constitute a useful tool to identify new solutions

PLAN

1. MOTIVATIONS
2. MASSIVE GRAVITY THEORIES
   - Topological Massive Gravity
   - New Massive Gravity
   - Generalized Massive Gravity
3. SIMPLY TRANSITIVE HOMOGENEOUS SPACES
   - Invariant frame, connection, curvature and all that ...
4. SOLVING THE FIELD EQUATIONS
   - Strategy
   - Bianchi Classification : Unimodular Lie Algebras
   - Bianchi Classification : Non-unimodular Lie Algebras
5. EXPLICIT SOLUTIONS
   - Type *I*
   - Type *III* : Algebraic construction
   - Type *SII* : Analytic construction
6. COSET SPACETIMES
7. VANISHING SCALAR INVARIANT SPACES
   - GMG, Type *III* Solutions
8. DISCUSSION & CONCLUSIONS

MOTIVATIONS

Inspired by D. D. K. Chow, C. N. Pope and E. Sezgin, Class. Quant. Grav. **27** (2010) 105001 [arXiv:0906.3559 [hep-th]].

1. All the solutions that they found in the literature at this time were of Petrov types D or N, (squashed AdS3 or AdS pp-wave metrics),
2. All Petrov type D solutions of TMG actually are biaxially squashed AdS3 metrics,

we tried to obtain new exact solutions, vacuum candidates of 3d-massive gravity theories

1. Admitting some symmetries,
2. Free of scalar singularities,
3. Analyticaly calculable

Massive Gravity Theories

Einstein gravity :

$$S_{\mathrm{EH}} = \frac{1}{\kappa^2} \int \sqrt{|g|}(R - 2\Lambda)\, \mathrm{d}^3x \qquad , \qquad g := \det(g_{\mu\nu})$$

Topological Massive Gravity : *add a Chern-Simon term*

S. Deser, R. Jackiw and G. 't Hooft, Annals Phys. **152** (1984) 220, S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48** (1982) 975, Annals Phys. **140** (1982) 372 [Erratum-ibid. **185** (1988) 406] [Annals Phys. **185** (1988) 406] [Annals Phys. **281** (2000) 409].

$$S_{\mathrm{TMG}} := S_{\mathrm{EH}} + \frac{1}{\mu\,\kappa^2} S_{\mathrm{CS}} \qquad ,$$

$$S_{\mathrm{CS}} = \frac{1}{2} \int g\, \varepsilon^{\lambda\mu\nu}\, \Gamma_{\lambda\sigma}{}^{\rho} \left(\partial_\mu \Gamma_{\rho\nu}{}^{\sigma} + \frac{2}{3} \Gamma_{\mu\tau}{}^{\sigma} \Gamma_{\nu\rho}{}^{\tau} \right) \mathrm{d}^3x \qquad .$$

MASSIVE GRAVITY THEORIES

New Massive Gravity : *add a "R^2" term*

E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. **102** (2009) 201301 [arXiv:0901.1766 [hep-th]], Phys. Rev. D **79** (2009) 124042 [arXiv:0905.1259 [hep-th]].

$$S_{\mathrm{NMG}} := S_{\mathrm{EH}} - \frac{1}{\xi\,\kappa^2}\, S_{\mathrm{QC}} \qquad ,$$

$$S_{\mathrm{QC}} = \int \sqrt{|g|}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right)\,\mathrm{d}^3x \qquad .$$

Massive Gravity Theories

Generalized Massive Gravity: *add both terms*

$$S_{\text{GMG}} := S_{\text{EH}} + \frac{1}{\mu\,\kappa^2}\,S_{\text{CS}} - \frac{1}{\xi\,\kappa^2}\,S_{\text{QC}} \qquad ,$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \frac{1}{\mu}\,C_{\mu\nu} - \frac{1}{2\xi}\,K_{\mu\nu} = 0 \qquad .$$

$$C_{\mu\nu} := \sqrt{|g|}\,\varepsilon_{\mu\rho\sigma}\nabla^{\rho}\left(R_{\nu}{}^{\sigma} - \frac{1}{4}\delta_{\nu}{}^{\sigma}R\right) \qquad ,$$

$$K_{\mu\nu} = 2\nabla^2 R_{\mu\nu} - \frac{1}{2}\nabla_{\mu}\nabla_{\nu}R + \frac{9}{2}R\,R_{\mu\nu} - 8R_{\mu}{}^{\kappa}R_{\nu\kappa} + g_{\mu\nu}\left(3R_{\kappa\lambda}R^{\kappa\lambda} - \frac{1}{2}\nabla^2 R - \frac{13}{8}R^2\right) \qquad .$$

M_3 & AdS_3 background stability : $\quad \xi(\xi + 4\mu^2) \geqslant 2\Lambda\mu^2$

Y. Liu and Y. W. Sun, Phys. Rev. D **79** (2009) 126001 [arXiv:0904.0403 [hep-th]].

HOMOGENEOUS SPACES

Ph. S., "Gravity before Supergravity", in *Supersymmetry*, Proceedings of NATO Advanced Study Institute , **125 B** Bonn 1984 (Plenum, New York, 1984), pp 455-533.

Homogeneous space $\mathcal{H} : \exists\, G : \mathcal{H} \mapsto \mathcal{H}\ |\forall (x,\, y) \in \mathcal{H} \times \mathcal{H},\, \exists\, g \in G \wedge g[x] = y$

Things are simple when the group is simply-transitive : $\mathcal{H} \equiv G$.

Homogeneous Spaces

Homogeneous space $\mathcal{H} : \exists\, G : \mathcal{H} \mapsto \mathcal{H} \,|\forall(x,\, y) \in \mathcal{H} \times \mathcal{H},\, \exists\, g \in G \wedge g[x] = y$

Things are simple when the group is simply-transitive : $\mathcal{H} \equiv G$.
Suppose ξ_α to be the left generators

$[\xi_\alpha, \xi_\beta] = -C^{\gamma}_{\alpha\,\beta}\, \xi_\gamma \quad , \quad \sigma^{\alpha}[\xi_\beta] = \delta^{\alpha}_{\beta}$ *right invariant basis*

Left invariant basis

$\mathcal{L}_{\xi_\alpha} \theta^{\beta} = 0 \quad ,$

$\mathcal{L}_{\xi_\alpha} ds^2 = 0 \Leftrightarrow \quad , \quad \xi_\mu[g_{\alpha\,\beta}] = 0 \quad ,$ *constant metric components with respect to* $\{\theta^{\alpha}\}$

$\omega_{\alpha\,\beta\,\gamma} = \frac{1}{2}(C_{\beta\,\gamma\,\alpha} - C_{\alpha\,\gamma\,\beta} - C_{\alpha\,\beta\,\gamma}) \quad ,$ *constant components*

$R_{\alpha\,\beta} = \omega^{\gamma}{}_{\delta\,\gamma}\, \omega^{\delta}{}_{\alpha\,\beta} - \omega^{\gamma}{}_{\alpha\,\delta}\, \omega^{\delta}{}_{\beta\,\gamma} \quad ,$ *constant components*

$\nabla_\gamma R_{\alpha\,\beta} = -R_{\delta\,\beta}\, \omega^{\delta}{}_{\alpha\,\gamma} - R_{\alpha\,\delta}\, \omega^{\delta}{}_{\beta\,\gamma} \quad \ldots$

STRATEGY

$$C^{\alpha}_{\beta\,\gamma} = \varepsilon_{\beta\,\gamma\,\zeta} n^{\zeta\,\alpha} + k_\beta\,\delta^{\alpha}_{\gamma} - k_\gamma\,\delta^{\alpha}_{\beta} \quad , \quad k_\alpha = \frac{1}{2}\,C^{\beta}_{\alpha\,\beta} \quad , \quad k_\zeta\,n^{\zeta\,\alpha} = 0 \quad , \quad (Jacobi)$$

Basis are defined up to linear transformations !!!

Using a $GL(3,\mathbb{R})$ transformation : $\eta = diag.(-1,\ 1,\ 1)$, canonical Lorentz metric

Problem: Find all inequivalent tensor densities $n^{\zeta\,\alpha}$ and vectors k_α, up to $O(2,1)$ transformations

ALL FIELD EQUATIONS ARE ALGEBRAIC !!!

Bianchi Classification : Unimodular Lie Algebras : $k_\alpha = 0$

$$n_\alpha^{\beta} V_\beta = \lambda V_\alpha \qquad , \qquad (n_\alpha^{\beta}) \text{ is } \underline{\text{not}} \text{ a symmetric matrix}$$

3 e.v. :

$$(n_I^{\alpha\beta}) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{pmatrix} ,$$

I, *II*, VI_0, VII_0, *VIII*, *IX*

1 double null e.v. +1 e.v. :

$$(n_{II}^{\alpha\beta}) = \begin{pmatrix} \nu + a & \nu & 0 \\ \nu & \nu - a & 0 \\ 0 & 0 & b \end{pmatrix} \text{ with } \nu = \pm 1 ,$$

II($a = b = 0$); VI_0 $a \neq 0$ or $a = 0$ and $\nu\, b < 0$;

VII_0 ($a = 0$ and $\nu\, b > 0$), *VIII*

1 triple null e.v. :

$$(n_{III}^{\alpha\beta}) = \begin{pmatrix} a & 1 & 0 \\ 1 & -a & 1 \\ 0 & 1 & -a \end{pmatrix} ,$$

VI_0 if $a = 0$; *VIII* if $a \neq 0$

1 single e.v. :

$$(n_{IV}^{\alpha\beta}) = \begin{pmatrix} 0 & \nu & 0 \\ \nu & a & 0 \\ 0 & 0 & b \end{pmatrix} \text{ with } a^2 < 4\,\nu^2$$

VI_0 if $b = 0$ or *VIII* if $b \neq 0$

Bianchi Classification : Non-unimodular Lie Algebras : $k_\alpha \neq 0$

k_α timelike + 2 e.v. :

$$k_\alpha = (k, 0, 0)\ ,\ (n_T^{\alpha\,\beta}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{pmatrix}$$

V if $b = c = 0$, IV if $b = \pm c \neq 0$,

$III = VI_{-1}$, VI_h if $c^2 > b^2$, or VII_k if $c^2 < b^2$, $(h = k^2/(b^2 - c^2))$

k_α spacelike + 2 e.v. :

$$k_\alpha = (0, 0, k)\ ,\ (n_{SI}^{\alpha\,\beta}) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

V if $a = b = 0$, IV if $a \neq 0$ or $b \neq 0$,

VI_h if $a\,b < 0$, or VII_k if $a\,b > 0$, $(h = k^2/a\,b)$

Bianchi Classification : Non-unimodular Lie Algebras : $k_\alpha \neq 0$

k_α spacelike + 1 null double e.v. :

$$(n_{SII}^{\alpha\,\beta}) = \begin{pmatrix} \nu + a & \nu & 0 \\ \nu & \nu - a & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ with } \nu = \pm 1 \,,$$

IV if $a = 0$, VI_h $(h = -k^2/a^2)$

k_α spacelike , no other e. v. :

$$(n_{SIII}^{\alpha\,\beta}) = \begin{pmatrix} 0 & \nu & 0 \\ \nu & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ with } 4\,\nu^2 > a^2 \,,$$

VI_h $(h = -k^2/\nu^2)$

Bianchi Classification : Non-unimodular Lie Algebras : $k_\alpha \neq 0$

k_α lightlike :

$$k_\alpha = (1,1,0)\,,\ (n^{\alpha\,\beta}) = \begin{pmatrix} a & -a & b \\ -a & a & -b \\ b & -b & c \end{pmatrix},$$

V if $a = b = c = 0$, IV if $b^2 = a\,c$, VI_h if $b^2 > a\,c$,
VII_h if $b^2 < a\,c$ $(h = 1/(a\,c - b^2))$

Petrov Classification

D. D. K. Chow, C. N. Pope and E. Sezgin, Class. Quant. Grav. **27** (2010) 105001 [arXiv:0906.3559 [hep-th]].

Suppose $S_{\alpha\,\beta}$ to be a symmetric, traceless 3D Minkowskian tensor

$$S^{\alpha}_{\beta}\, V^{\beta} = \lambda\, V^{\alpha}$$

$$P[\lambda] := \det[\lambda\, \delta^{\alpha}_{\beta} - S^{\alpha}_{\beta}] =: \lambda^3 + p\,\lambda + q$$

$$\Delta := \frac{q^2}{4} + \frac{p^3}{27}$$

- $\Delta > 0$ $(\lambda_1 \in \mathbb{R}, \lambda_2 = \bar{\lambda}_3 \in \mathbb{C})$: Petrov type $I_{\mathbb{C}}$: $\eta - 3\, s \otimes s \pm (\ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2)$
- $\Delta < 0$ $(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R})$: Petrov type $I_{\mathbb{R}}$: $\eta - 3\, s \otimes s \pm (\ell_1 \otimes \ell_1 + \ell_2 \otimes \ell_2)$
- $\Delta = 0, p \neq 0$ $(\lambda_1 = \lambda_2 \neq \lambda_3)$: Petrov type II : $\eta - 3\, s \otimes s + \ell \otimes \ell$ or D_s : $\eta - 3\, s \otimes s$, D_t : $\eta + 3\, t \otimes t$
- $\Delta = 0, p = 0$ $(\lambda_1 = \lambda_2 = \lambda_3 = 0)$: Petrov type III : $\ell \otimes s + s \otimes \ell$ or N : $\ell \otimes \ell$ or O.

Types I, D_t and D_s have diagonal Jordan form.

Example of Solutions : Type I, GMG theory

Starting point :

$$(n_I^{\alpha\,\beta}) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{pmatrix}$$

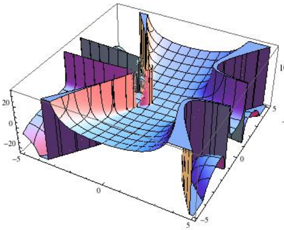
from the (algebraic) field equations …

$$\Lambda = -\frac{5\left(a^2+4\,a\,b+4\,c^2\right)^3}{8\left(a^2\,(3\,a^2-40\,a\,b-112\,b^2)-8\,(9\,a^2+20\,a\,b-16\,b^2)c^2-80\,c^4\right)}$$

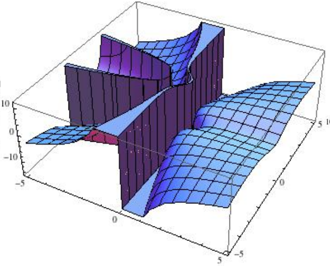
$$\mu = \frac{a^2\,(-3\,a^2+40\,a\,b+112\,b^2)+8\,(9\,a^2+20\,a\,b-16\,b^2)c^2+80\,c^4}{16\left(a^3+2\,a^2b-4\,a\,b^2-8\,bc^2\right)}$$

$$\xi = \frac{a^2\,(-3\,a^2+40\,a\,b+112\,b^2)+8\,(9\,a^2+20\,a\,b-16\,b^2)c^2+80c^4}{8\,(a^2+4\,a\,b+4\,c^2)}$$

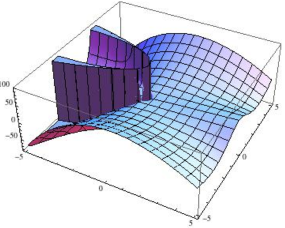
Λ, μ and ξ as functions of b/a and c/a



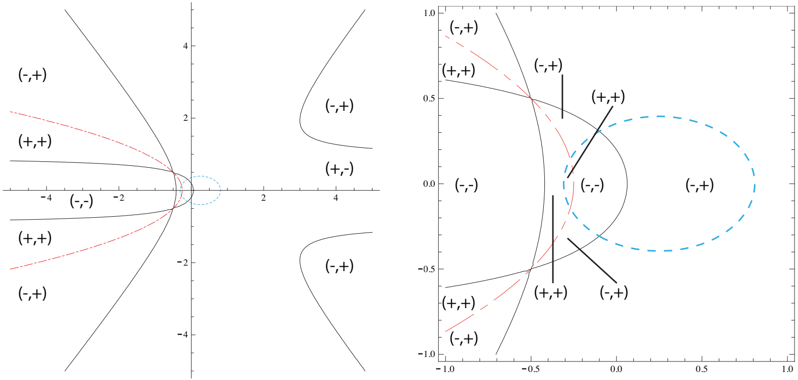
Λ



μ



ξ



$\Lambda = \infty$, $\xi = 0$, $\mu = 0$ on the solid curve; $\xi = \infty$, $\Lambda = 0$ on the dash-dotted curve; $\mu = \infty$ on the dotted curve. The pair of signs are those of Λ and μ; we always have $\Lambda\,\xi \geq 0$.

Starting point :

$$(n_{III}^{\alpha\,\beta}) = \begin{pmatrix} a & 1 & 0 \\ 1 & -a & 1 \\ 0 & 1 & -a \end{pmatrix}$$

$$\mu = -\frac{69}{80}\,a \qquad , \qquad \Lambda = -\frac{45}{184}\,a^2 \leqslant 0 \qquad , \qquad \xi = -\frac{23}{8}\,a^2 \leqslant 0 \qquad .$$

Canonical form of Bianchi *VIII* structure constants : $n_{BVIII} = diag.(-1, 1, 1)$

Problem : obtain Λ^p_α such that $(n_{BVIII})^{p\,q} = \det[\Lambda^p_\alpha]^{-1}\,\Lambda^p_\alpha\,\Lambda^q_\beta\,(n_{III})^{\alpha\,\beta}$

compute : $g_{p\,q} = \Lambda^{-1}{}^{\alpha}_{p}\Lambda^{-1}{}^{\beta}_{q}\,\eta_{\alpha\,\beta}$ and simplify using $Iso[1,2]$ transf.

$$ds^2 = a^{-2}(-(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2) + \gamma\,(\theta^1 + \theta^3)\,\theta^2 \qquad , \gamma \neq 0$$

$$\theta^1 = dt - \sinh x\,dy \quad , \quad \theta^2 = \cos t\,dx - \sin t\cosh x\,dy \quad , \quad \theta^3 = \sin t\,dx + \cos t\cosh x\,dy \quad .$$

Notice: this metric can't be obtained from a diagonal ansatz !

Example of Solutions : Type *SII*, GMG theory

Starting point :

$$k_\alpha = (0,0,k) \quad , \quad (n^{\alpha\,\beta}_{SII}) = \begin{pmatrix} \nu + a & \nu & 0 \\ \nu & \nu - a & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ with } \nu = \pm 1 \,,$$

Cartan equations :

$$[\xi_1,\ \xi_2] = 0\,,\ [\xi_1,\ \xi_3] = (k+\nu)\xi_1 + (\nu - a)\xi_2\,, [\xi_2,\ \xi_3] = -(a+\nu)\xi_1 + (k-\nu)\xi_2\,,$$

Isometry Lie Algebra generators :

$$\xi_1 = \partial_x\,,\ \xi_2 = \partial_y\,,\ \xi_3 = \partial_z + r(x,y)\partial_x + s(x,y)\partial_y \quad ,$$

$$r(x,y) = (k+\nu)\,x - (a+\nu)\,y \qquad \text{and} \qquad s(x,y) = (\nu - a)\,x + (k-\nu)\,y \qquad ,$$

Invariant basis ($\mathcal{L}_{\xi_k}\theta^p = 0$) :

$$\theta^1 = \mathrm{d}x - r(x,y)\,\mathrm{d}z \quad , \quad \theta^2 = \mathrm{d}y - s(x,y)\,\mathrm{d}z \quad , \quad \theta^3 = \mathrm{d}z \quad .$$

Metric :

$$\mathrm{d}s^2 = -\,(\mathrm{d}x - r(x,y)\,\mathrm{d}z)^2 + (\mathrm{d}y - s(x,y)\,\mathrm{d}z)^2 + \mathrm{d}z^2$$

Example of Solutions : Type *SII*, GMG theory

Metric :

$$\mathrm{d}s^2 = -\left(\mathrm{d}x - \left[x\,(k+\nu) - y\,(a+\nu)\right]\mathrm{d}z\right)^2 + \left(\mathrm{d}y - \left[x\,(\nu - a)\,x + y\,(k-\nu)\right]\mathrm{d}z\right)^2 + \mathrm{d}z^2$$

Right action of G (isometries) :

$$\begin{aligned} x \mapsto x \quad &, \quad y \mapsto y \quad , \quad z \mapsto z + \lambda_3 \\ x \mapsto x + p(z) \quad &, \quad y \mapsto y + q(z) \quad , \quad z \mapsto z \end{aligned}$$

$$p'(z) = (k+\nu)\,p(z) - (a+\nu)\,q(z) \quad , \quad q'(z) = (\nu - a)\,p(z) + (k-\nu)\,q(z) \quad ,$$

whose solution reads

$$\begin{aligned} p(z) &= \lambda_1\, e^{k\,z}\,(a\,\cosh(a\,z) + \nu\,\sinh(a\,z)) - \lambda_2\, e^{k\,z}\,(\nu + a)\,\sinh(a\,z) \\ &:= \lambda_1 p_1(z) + \lambda_2\, p_2(z) \quad , \\ q(z) &= \lambda_1\, e^{k\,z}\,(\nu - a)\,\sinh(a\,z) + \lambda_2\, e^{k\,z}\,(a\,\cosh(a\,z) - \nu\,\sinh(a\,z)) \\ &:= \lambda_1\, q_1(z) + \lambda_2\, q_2(z) \quad . \end{aligned}$$

EXAMPLE OF SOLUTIONS : TYPE *SII*, GMG THEORY

Fix a base point and use the group parameters $\lambda_{1,\,2}$ as coordinates (instead of x, y). After a (mainly linear) redefinition,

$$\lambda_{1,\,2} \mapsto \frac{\sigma\,\nu\,v + k^2\,(\nu \pm a)\,u}{2\,k^2\,a\,\sqrt{|a|}} \qquad , \qquad z \mapsto -\frac{1}{k}\ln\,\zeta \qquad , \qquad (\sigma := -\mathrm{sgn}(a)\,\nu)$$

pp-wave AdS metric or null warped AdS metric :

$$ds^2 = \frac{1}{k^2}\,\frac{\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\zeta^2}{\zeta^2} + \sigma\,\frac{\mathrm{d}u^2}{\zeta^{2(a/k+1)}} \qquad .$$

G. Moutsopoulos, *"Homogeneous anisotropic solutions of topologically massive gravity with cosmological constant and their homogeneous deformations,"* [arXiv:1211.2581 [gr-qc]].
G. Gibbons, C. Pope, and E. Sezgin, Class. Quant. Grav. **25** (2008) 205005, [arXiv:0807.2613 [hep-th]].

Coset spacetimes

Kantowsky–Sachs spacetimes are the exception in 3d of a simply-transitive group action.

R. Kantowski and R. K. Sachs, J. Math. Phys. **7** (1966) 443.

$$ds^2 = -dt^2 + R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

- It describes homogeneous spaces of the form $R \times S^2$.
- It is conformally flat, with traceless Ricci and $\widehat{K}_{\alpha\,\beta}$ tensors of Petrov type D_t.
- It satisfies the NMG and GMG with $\Lambda = \xi = \frac{1}{2R^2}$ and arbitrary μ.
- The flat and hyperbolic version, are within the simply-transitive groups.
- Lorentz group $SO(1,2)$ possesses two dimensional subgroups , whereas $SO(3)$ admits only $U(1)$ subgroups.
- The hyperbolic version is a special solution of Bianchi *III*, obtained by *S III* with $a = 0$ and $k = -\nu$.

Vanishing Scalar Invariant Solutions

A. Coley, S. Hervik and N. Pelavas, Class. Quant. Grav. **23** (2006) 3053 [gr-qc/0509113].

$$ds^2 = -2\,du\left[dv + \frac{1}{2}F(u,v,x)\,du + W(u,v,x)\,dx\right] + dx^2$$

$R_{\alpha\,\beta} = \phi\, l_\alpha l_\beta$ Petrov type *N* or $R_{\alpha\,\beta} = \psi\, l_{(\alpha} m_{\beta)}$ Petrov type *III*

Residual coordinates transformations :

$$u \mapsto \mathcal{U}[\tilde{u}] \quad , \quad v \mapsto \frac{\tilde{v}}{\dot{\mathcal{U}}[\tilde{u}]} + \mathcal{F}[\tilde{u},\,\tilde{x}] \quad , \quad x \mapsto \tilde{x} + \mathcal{G}[\tilde{u}]$$

$$\widetilde{F} = F\dot{\mathcal{U}}^2 + 2\left(\dot{\mathcal{F}}\dot{\mathcal{U}} + W\dot{\mathcal{G}}\dot{\mathcal{U}} - \tilde{v}\frac{\ddot{\mathcal{U}}}{\dot{\mathcal{U}}}\right) - \dot{\mathcal{G}}^2 \quad , \quad \widetilde{W} = W + \mathcal{F}' - \dot{\mathcal{G}} \quad ,$$

Type *III*

A : $F(u,v,x) = v f_1(u,x) + f_0(u,x) \quad , \quad W(u,v,x) = w_0(u,x) \quad .$

B : $F(u,v,x) = -\dfrac{v^2}{x^2} + v f_1(u,x) + f_0(u,x) \quad , \quad W(v,u,x) = -\dfrac{2v}{x} + w_0(u,x) \quad .$

GMG, TYPE *III*, VSI SOLUTIONS

D. D. K. Chow, C. N. Pope and E. Sezgin, Class. Quant. Grav. **27** (2010) 105002 [arXiv:0912.3438 [hep-th]].

The only nontrivial equations of motion are : $[u,x]$ & $[u,u]$; they are solved in three steps

By partial gauge fixing set :

$$w_0(x,y) = 0$$

From the $[u,x]$ eq. obtain :

$$f_1(u,x)$$

Plug it in the $[u,u]$ eq. and determine :

$$f_0(u,x)$$

and simplify them as much as possible, using the residual gauge freedom.

For example, in case A, the $[u,\ x]$ equation has for general solution:

$$f_1(u,x) = c_0(u) + c_1(u)\, e^{-\mu\, x} \implies f_0(u,x) = e^{-\mu\, x}$$

Then, the $[u,\ u]$ component has for general solution :

$$f_0(u,x) = q_0(u) + q_1(u)\, x + q_2(u)\, e^{-\mu\, x} + \frac{e^{-2\mu\, x}}{8\mu^2} \implies f_0(u,x) = q_0(u) + q_1(u)\, x + \frac{e^{-2\mu\, x}}{8\mu^2}$$

Notice the occurrence of arbitrary profile functions

Discussion & Conclusions

We have a complete list of C S I geometries, solution of MG theories

All the solutions obtained are not physically relevant
(compact spaces, with Lorentz metric, are causally pathological)

Some solutions require special values of the coupling constants

The writings of a given solution can be very different.
The Petrov type provides a useful invariant tool to discriminate
between inequivalent geometries.

It remains to analyse in detail the structure of the solutions obtained, with the hope to exhibit interesting properties: global structure, quotient, stability, asymptotic structure, holographic equivalent theories, ...