Embedding nonrelativistic physics inside a gravitational wave

Kevin MORAND

Laboratoire de Mathématiques et Physique Théorique (Tours)

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Eisenhart theorem

Eisenhart theorem (1929) :

Dynamical trajectories of nonrelativistic (NR) mechanics can always be lifted to geodesics of a *specific* relativistic spacetime (ST) with one dimension more.

Conversely, to any geodesic of this specific class of spacetimes corresponds a solution of a NR dynamical system.

Eisenhart theorem

More precisely, solutions of the NR system described by the Lagrangian :

$$\mathbf{L}(\mathbf{t},\mathbf{x},\dot{\mathbf{x}})=\frac{\mathbf{1}}{\mathbf{2}}\,\bar{\mathbf{g}}_{\mathbf{ij}}\,(\mathbf{t},\mathbf{x})\,\dot{\mathbf{x}}^{\mathbf{i}}\dot{\mathbf{x}}^{\mathbf{j}}+\bar{\mathbf{A}}_{\mathbf{i}}\,(\mathbf{t},\mathbf{x})\,\dot{\mathbf{x}}^{\mathbf{i}}-\bar{\mathbf{V}}\,(\mathbf{t},\mathbf{x})$$

are in one-to-one correspondence with geodesics of a relativistic ST endowed with the metric :

$$\mathbf{d}\bar{\mathbf{s}}^{\mathbf{2}}=\mathbf{2}\,\mathbf{dt}\left(\mathbf{du}+\bar{\mathbf{A}}_{\mathbf{i}}\,(\mathbf{t},\mathbf{x})\,\mathbf{dx}^{\mathbf{i}}-\bar{\mathbf{U}}\,(\mathbf{t},\mathbf{x})\,\mathbf{dt}\right)\,+\,\bar{\mathbf{g}}_{\mathbf{ij}}\,(\mathbf{t},\mathbf{x})\,\mathbf{dx}^{\mathbf{i}}\mathbf{dx}^{\mathbf{j}}$$

with $\bar{U}=\bar{V}-\frac{1}{2}\frac{M^2}{m^2}$.

Eisenhart spacetimes

Eisenhart metric :

$$\mathbf{d\bar{s}^2 = 2\,dt\left(du + \bar{A}_i\,(t,x)\,dx^i - \bar{U}\,(t,x)\,dt\right) + \bar{g}_{ij}\,(t,x)\,dx^i dx^j}$$

Remarks :

- The metric is *uniquely* determined by the form of the NR Lagrangian.
- The additional coordinate u is light-like and proportional to the NR action.
- Geometrically, this class corresponds to ST admitting a null ($\xi^2 = 0$) and parallel ($\nabla\xi = 0$) vector field.

Examples :

- Minkowski spacetime
- pp-waves : $d\bar{s}^2 = 2\,dt\left(du + \bar{A}_i\,(t,x)\,dx^i - \bar{U}\,(t,x)\,dt\right) + \delta_{ij}\,dx^i dx^j$

Lichnerowitz generalization of the Eisenhart lift (1955)

- The class of relativistic spacetimes allowing the Eisenhart lift has been enlarged by Lichnerowitz to the conformal class :

$$ds^2 = \Omega\left(t,x\right)\left[2\,dt\left(du + \bar{A}_i\left(t,x\right)dx^i - \bar{U}\left(t,x\right)dt\right) + \bar{g}_{ij}\left(t,x\right)dx^i dx^j\right].$$

- To a given NR Lagrangian corresponds now a whole class of ST, as the potential $\bar{U}$ now writes

$$\bar{U} = \bar{V} - \frac{1}{2}\frac{M^2}{m^2}\Omega$$

where the conformal factor $\Omega(t,x)$ is arbitrary.

Lichnerowitz spacetimes

Lichnerowitz metric :

$$ds^2 = \Omega\,(t,x)\left[2\,dt\left(du + \bar{A}_i\,(t,x)\,dx^i - \bar{U}\,(t,x)\,dt\right) + \bar{g}_{ij}\,(t,x)\,dx^i dx^j\right]$$

Examples :

- Anti de Sitter spacetime : $ds^2 = \frac{1}{z^2}[2\,dudt + dz^2 + d\vec{y}^2]$
- Siklos : $ds^2 = \frac{1}{z^2}\left[2\,dt\left(du - \bar{U}(t,z,\vec{y})dt\right) + dz^2 + d\vec{y}^2\right]$
    - Kaigorodov : $ds^2 = \frac{1}{z^2}\left[2\,dt\,(du \pm z^n dt) + dz^2 + d\vec{y}^2\right]$
    - Schrödinger : $ds^2 = \frac{1}{z^2}\left[2\,dt\left(du + z^{2(1-Z)}dt\right) + dz^2 + d\vec{y}^2\right]$

First results

Classical

- A Hamiltonian formulation of the Eisenhart-Lichnerowitz theorem shows that the class of spacetimes considered by Lichnerowitz is the most general allowing the Eisenhart lift.

First-quantized

- The curved Schrödinger equation is obtained via light-like dimensional reduction from the Klein-Gordon equation on Lichnerowitz background.

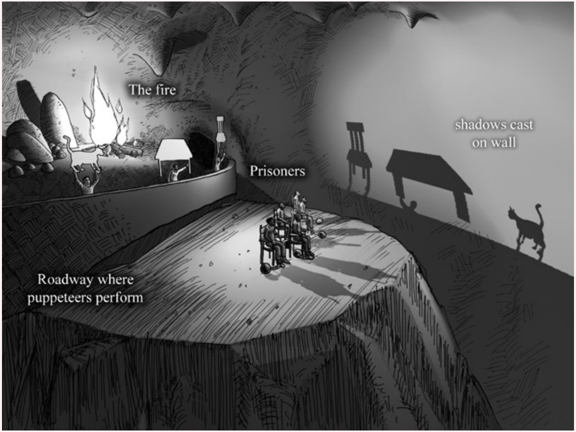
Geometrical interpretation

- Lichnerowitz spacetimes are geometrically characterized as conformally equivalent to Eisenhart spacetimes with preserved Killing vector.

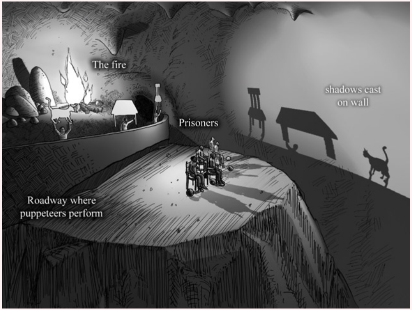
Plato allegory (Minguzzi,2006)

"To them, I said, the truth would be literally nothing but the shadows of the images. "

Plato, Book VII of The Republic

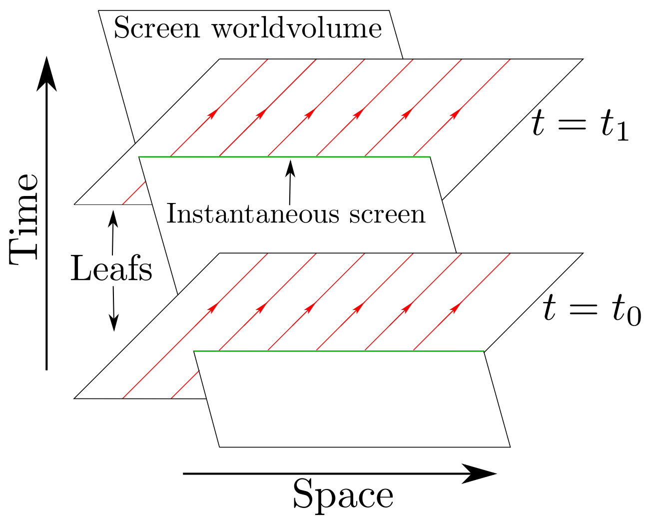


Plato allegory (Minguzzi,2006)

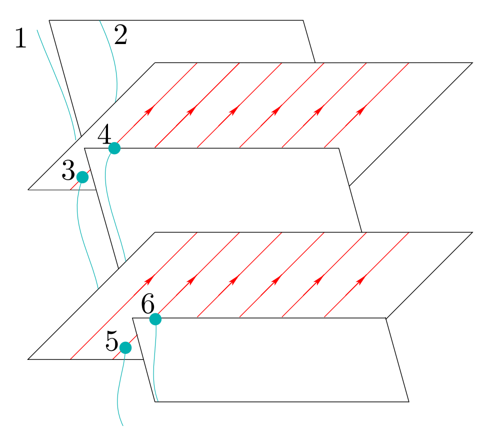


Allegory of the cave	Ambient approach
Cave	Ambient spacetime
Light-rays	Graviton worldlines
Wall	Screen
Prisoners	Physicists
Shadows	NR physics

Geometrical interpretation of the Eisenhart lift



Geometrical interpretation of the Eisenhart lift



1. Geodesic of ambient spacetime

2. Shadow of the ambient geodesic on the screen worldvolume

3. Emission of a graviton by the geodesic

4. Detection on the screen at $t = t_1$

5. Emission of a graviton by the geodesic

6. Detection on the screen at $t = t_0$

Ambient approach to gravity

Eisenhart lift can be extended to gravity in order to establish a correspondence between relativistic and nonrelativistic spacetimes :

- **1985 : Duval, Burdet, Künzle & Perrin**
  Ambient approach to gravity for Eisenhart spacetimes
- **1995 : Julia & Nicolai**
  Generalization of the ambient approach for gravity to Lichnerowitz spacetimes seen as possessing a null, hypersurface-orthogonal and Killing vector field

Results

Applications :

- Classification of Lichnerowitz spacetimes with constant scalar curvature invariants
- Generalization of results regarding geodesic completeness and causality :

Proposition

Lichnerowitz spacetimes with :

- *conformal factor* $\Omega(t,x)$,
- *minus the scalar potential* $-\bar{U}(x,t)$,
- *absolute value of the time derivative of the conformal factor* $|\partial_t \Omega(t,x)|$,
- *absolute value of the time derivative of the scalar potential* $|\partial_t \bar{U}(t,x)|$,

that grow at most quadratically at spatial infinity along finite times, are geodesically complete.

Thank you !