Asymptotically flat spacetimes in three-dimensional higher spin gravity.

Seventh Aegean Summer school: Beyond Einstein's theory of gravity

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Introduction

- Higher spin gravity has recently motivated a lot of activity.
- In 3 dimensions, the theory turns out to be simpler, since it can be described in terms of a Chern-Simons action.
- It is possible to have a consistent truncation to a finite number of higher spin gauge fields
- The theory in $d = 3$ can also be formulated in the case of vanishing cosmological constant Λ.
- The purpose of this talk is to show a set of asymptotic conditions for higher spin gravity in three dimensions in the case of $\Lambda = 0$.

Outline

- Higher spin theory with vanishing cosmological constant.
- Asymptotic symmetry Algebra.
- Vanishing cosmological constant limit of AdS_3.
- Summary.

Higher spin gravity in 3D with vanishing cosmological constant

- Higher spin gravity in three dimensions can be formulated in terms of a Chern-Simons action (Blencowe '89, Bergshoeff-Blencowe-Stelle '90), given by,

$$I[A] = \frac{k}{4\pi} \int \langle AdA + \frac{2}{3} A^3 \rangle \tag{1}$$

- In this case, the gauge field $A = A_\mu dx^\mu$ can be written as

$$A = \omega^a J_a + e^a P_a + W^{ab} J_{ab} + E^{ab} P_{ab} \ , \tag{2}$$

where the set $\{J_a, P_a, J_{ab}, P_{ab}\}$ spans the gauge group, and the generators P_{ab}, J_{ab} are symmetric and traceless.

- In the case of vanishing cosmological constant, the generators fulfill a generalization of the Poincaré algebra, which is obtained performing a Inönü-Wigner contraction of two copies of sl(3), and then taking the limit $l \to \infty$. The algebra is given by

$$\begin{aligned}
&[J_a, J_b] = \epsilon_{abc} J^c, \quad [P_a, J_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = 0, \\
&[J_a, J_{bc}] = \epsilon^m{}_{a(b} J_{c)m}, \quad [J_a, P_{bc}] = \epsilon^m{}_{a(b} P_{c)m}, \\
&[P_a, J_{bc}] = \epsilon^m{}_{a(b} P_{c)m}, \quad [P_a, P_{bc}] = 0, \\
&[J_{ab}, J_{cd}] = -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right) J^m, \\
&[J_{ab}, P_{cd}] = -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right) P^m, \\
&[P_{ab}, J_{cd}] = -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right) P^m, \quad [P_{ab}, P_{cd}] = 0,
\end{aligned} \tag{3}$$

where $k = \frac{1}{4G}$, and the nonvanishing components of the bracket $\langle \cdots \rangle$ are given by

$$\langle P_a J_b \rangle = \eta_{ab}, \quad \langle P_{ab} J_{cd} \rangle = \eta_{ac}\eta_{bd} + \eta_{ad}\eta_{cb} - \frac{2}{3}\eta_{ab}\eta_{cd} \,. \tag{4}$$

- Therefore, the action reduces to

$$I[e,\omega,E,W] = \frac{k}{2\pi}\int\Big[e^a\left(d\omega_a+\frac{1}{2}\epsilon_{abc}\omega^b\omega^c+2\epsilon_{abc}W^{bd}W^c{}_d\right) +2E^{ab}(dW_{ab}+2\epsilon_{cda}\omega^c W_b{}^d)\Big]\ , \quad (5)$$

- The field equations are given by

$$de^a+\epsilon^{abc}\omega_b e_c+4\epsilon^{abc}E_{bd}W^d{}_c = 0, \quad (6)$$

$$d\omega^a+\frac{1}{2}\epsilon^{abc}\omega_b\omega_c+2\epsilon^{abc}W_{bd}W^d{}_c = 0, \quad (7)$$

$$dE^{ab}+\epsilon^{cd(a|}\omega_c E_d{}^{|b)}+\epsilon^{cd(a|}e_c W_d{}^{|b)} = 0, \quad (8)$$

$$dW^{ab}+\epsilon^{cd(a|}\omega_c W_d{}^{|b)} = 0. \quad (9)$$

- The fields e^a, E^{ab}, and ω^a, W^{ab} are interpreted as a generalization of the dreibein and the spin connection, respectively.

Asymptotic symmetry Algebra.

- Asymptotically flat spacetimes in General Relativity with vanishing cosmological constant in three dimensions enjoy similar features as the ones by asymptotically AdS_3 geometries (Brown-Henneaux '86).
- Indeed, the asymptotic symmetry group is infinite dimensional (Ashtekar et.al '87), and its algebra, so-called BMS_3, also acquires a nontrivial central extension (Barnich-Compere '07, Barnich-Troessaert '10)

- It is then natural to look for asymptotic conditions in the case of higher spin gravity, such that they fulfill the following requirements:
  - (i) They reduce from the ones of pure gravity (Barnich-Compere '07, Barnich-Troessaert '10) when the higher spin fields are switched off, and
  - (ii) In presence of higher spin fields, they should correspond to the vanishing cosmological constant limit of the asymptotically AdS_3 conditions of (Henneaux-Rey '10 , Campoleoni-Fredenhagen-Pfenninger-Theisen et.al.'10 ,Gaberdiel-Hartman '11) .
- For simplicity the analysis is carried out in the case of spins $s = 2, 3$.

- It is possible to obtain from the field equations, that the connection satisfying (i) possesses the following asymptotic form

$$A = \left(\frac{1}{2}\mathcal{M}du - dr + \left(\mathcal{J} + \frac{u}{2}\partial_\phi\mathcal{M}\right)d\phi\right)P_0 + duP_1 + rd\phi P_2$$
$$+ \frac{1}{2}\mathcal{M}d\phi J_0 + d\phi J_1 + \left(\mathcal{W}du + \left(\mathcal{V} + u\partial_\phi\mathcal{W}\right)d\phi\right)P_{00} + \mathcal{W}d\phi J_{00}, \tag{10}$$

where r, ϕ correspond to the radial and angular coordinates, respectively, u is a null coordinate that plays the role of time, and $\mathcal{M}$, $\mathcal{J}$, $\mathcal{W}$ and $\mathcal{V}$ are arbitrary functions of ϕ.

- Now, we will calculate the asymptotic symmetries correspond to gauge transformations generated by a Lie-algebra-valued parameter
$$\lambda = \rho^a P_a + \eta^a J_a + \xi^{ab} P_{ab} + \Lambda^{ab} J_{ab} \ , \tag{11}$$
that preserves the form of the connection A, i.e.,
$$\delta A = d\lambda + [A, \lambda].$$

- One then finds that $\lambda = \lambda(\epsilon, y, w, v)$ depends on four independent functions of the angular coordinate, given by
$$\rho^1 = \epsilon + uy', \quad \eta^1 = y, \quad \xi^{11} = w + uv', \quad \Lambda^{11} = v, \tag{12}$$

- The arbitrary functions appearing in the asymptotic form transform according to the following rules

$$
\begin{aligned}
\delta\mathcal{M} &= y\mathcal{M}' + 2y'\mathcal{M} - 2y''' + 4\left(2v\mathcal{W}' + 3v'\mathcal{W}\right), \\
\delta\mathcal{J} &= y\mathcal{J}' + 2y'\mathcal{J} + \frac{1}{2}\epsilon\mathcal{M}' + \epsilon'\mathcal{M} - \epsilon''' \\
&\quad + 2\left(2w\mathcal{W}' + 3w'\mathcal{W} + 2v\mathcal{V}' + 3v'\mathcal{V}\right), \\
\delta\mathcal{W} &= y\mathcal{W}' + 3y'\mathcal{W} - \frac{1}{6}v\mathcal{M}''' - \frac{3}{4}v'\mathcal{M}'' - \frac{5}{4}v''\mathcal{M} - \frac{5}{6}v'''\mathcal{M} \\
&\quad + \frac{2}{3}(v\mathcal{M}\mathcal{M}' + v'\mathcal{M}^2) + \frac{1}{6}v^{(5)}, \\
\delta\mathcal{V} &= y\mathcal{V}' + 3y'\mathcal{V} + \epsilon\mathcal{W}' + 3\epsilon'\mathcal{W} - \frac{5}{3}v'''\mathcal{J} - \frac{5}{2}v''\mathcal{J}' - \frac{3}{2}v'\mathcal{J}'' \\
&\quad - \frac{1}{3}v\mathcal{J}''' + \frac{4}{3}v(\mathcal{J}\mathcal{M})' + \frac{8}{3}v'\mathcal{M}\mathcal{J} + \frac{2}{3}(w\mathcal{M}\mathcal{M}' + w'\mathcal{M}^2) \\
&\quad - \frac{1}{6}w\mathcal{M}''' - \frac{3}{4}w'\mathcal{M}'' - \frac{5}{4}w''\mathcal{M}' - \frac{5}{6}w'''\mathcal{M} + \frac{1}{6}w^{(5)},
\end{aligned}
\tag{13}
$$

which allows to find the form of the asymptotic symmetry algebra.

- The variation of the global charges that correspond to the asymptotic symmetries spanned by λ, in the canonical approach (Regge-Teitelboim '74), are given by

$$\delta Q[\lambda] = \frac{k}{2\pi} \int \langle \lambda \delta A_{\phi} \rangle d\phi. \tag{14}$$

- The charges can be integrated, and are given by

$$Q[\epsilon, y, w, v] = \frac{k}{4\pi} \int \left[\epsilon \mathcal{M} + 2y\mathcal{J} + 4(w\mathcal{W} + v\mathcal{V})\right] d\phi. \tag{15}$$

- Their algebra can then be obtained from the variation of the charges $\delta_{\lambda_2} Q[\lambda_1] = \{Q[\lambda_1], Q[\lambda_2]\}$, following the procedure explained in (Brown-Henneaux '86).
- In this case, the Poisson brackets read

$$\{Q(\epsilon_1, y_1), Q(\epsilon_2, y_2)\} = Q(\epsilon_{[1,2]}, y_{[1,2]}) + K[\epsilon_1, \epsilon_2, y_1, y_2], \qquad (16)$$

where the parameters $\epsilon_{[1,2]}$ y $y_{[1,2]}$ are given by

$$\epsilon_{[1,2]} = \epsilon_1 y_2' - \epsilon_2 y_1' - \epsilon_1' y_2 + \epsilon_2' y_1, \qquad (17)$$

$$y_{[1,2]} = y_1 y_2' - y_1' y_2, \qquad (18)$$

and the central charge K is

$$K[\epsilon_1, \epsilon_2, y_1, y_2] = \frac{k}{2\pi} \int [\epsilon_1' y_2'' - \epsilon_2' y_1''] \, d\phi. \qquad (19)$$

- Expanding in Fourier modes $P_n = Q(\epsilon = e^{in\phi})$ and $J_n = Q(y = e^{in\phi})$, the algebra acquires the form

$$i\{P_n, P_m\} = 0, \tag{20}$$
$$i\{J_n, J_m\} = (n-m)J_{n+m}, \tag{21}$$
$$i\{J_n, P_m\} = (n-m)P_{n+m} + kn^3\delta_{m+n}. \tag{22}$$

  As expected, the asymptotic symmetries associated to $\epsilon(\phi)$ and $y(\phi)$ span the BMS_3 algebra with the same central charge as in the case of General Relativity in three dimensions (Barnich-Compere '07).

- The brackets of the BMS_3 generators with the remaining charges associated to $w(\phi)$ and $v(\phi)$, are given by

$$i\{P_n, W_m\} = 0, \tag{23}$$
$$i\{J_n, W_m\} = (2n-m)W_{n+m}, \tag{24}$$
$$i\{P_n, V_m\} = (2n-m)W_{n+m}, \tag{25}$$
$$i\{J_n, V_m\} = (2n-m)V_{n+m}, \tag{26}$$

- And finally, the brackets between the higher spin generators are given by

$$i\{W_n, W_m\} = 0, \tag{27}$$

$$i\{W_n, V_m\} = \frac{1}{3}\left[\frac{8}{k}(n-m)\sum_{j=-\infty}^{\infty} P_j P_{n+m-j}\right. \tag{28}$$

$$\left. +(n-m)(2n^2+2m^2-mn)P_{m+n} + kn^5\delta_{m+n}\right], \tag{29}$$

$$i\{V_n, V_m\} = \frac{1}{3}\left[\frac{16}{k}(n-m)\sum_{j=-\infty}^{\infty} P_j J_{n+m-j}\right. \tag{30}$$

$$\left. +(n-m)(2n^2+2m^2-mn)J_{m+n}\right], \tag{31}$$

- In sum, the relations presented before provide the higher spin extension of the BMS_3 algebra.

Note: There is a paper which appeared simultaneously on arXiv and our results completely agree. (H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller, J. Rosseel, arXiv:1307.4768).

Vanishing cosmological constant limit of AdS_3 boundary conditions

- Now, we will show that our asymptotic conditions correspond to the vanishing cosmological limit of the asymptotically AdS_3 conditions.
- The asymptotic form of the gauge fields in the case of Higher spin gravity on AdS_3 is given by

$$A^{\pm} = b_{\pm}^{-1} a^{\pm} b_{\pm} + b_{\pm}^{-1} d b_{\pm}, \qquad (32)$$

where $b_{\pm} = e^{\pm \log(r/l) L_0}$, and

$$a^{\pm} = \pm(L_{\pm 1} - \Xi_{\pm} L_{\mp 1} - W_{\pm} W_{\mp 2}) dx^{\pm}.$$

here $\Xi_{\pm}$ and $W_{\pm}$ correspond to arbitrary functions of $x^{\pm} = \frac{t}{l} \pm \phi$.

- For our purposes, it is convenient to make a different gauge choice, so that the asymptotic form of the gauge fields read

$$A^{\pm} = {g_{\pm}}^{-1} a^{\pm} g_{\pm} + {g_{\pm}}^{-1} d g_{\pm},$$

with

$$g_{+} = b_{+} e^{-\log(\sqrt{2}\frac{r}{l})L_0} e^{\frac{r}{\sqrt{2}l}L_{-1}}, \tag{33}$$

$$g_{-} = b_{-} e^{-\log(\frac{1}{2\sqrt{2}}\frac{r}{l})L_0} e^{\frac{r}{\sqrt{2}l}L_{-1}} e^{\sqrt{2}\frac{l}{r}L_1}. \tag{34}$$

- Therefore, the components of the connections are given by

$$A^{\pm} = \frac{r}{l} dx^{\pm} L_0^{\pm} \pm \frac{1}{\sqrt{2}} \left[\frac{dr}{l} + \left(\frac{r^2}{2l^2} - 2\Xi_{\pm} \right) dx^{\pm} \right] L_{-1}^{\pm} \tag{35}$$

$$\pm \frac{dx^{\pm}}{\sqrt{2}} L_1^{\pm} \mp 2dx^{\pm} (W_{\pm} W_{-2}^{\pm}). \tag{36}$$

- It is also useful, before take the limit, change the basis according to

$$L^{\pm}_{-1} = -\sqrt{2}J^{\pm}_{0}, \quad L^{\pm}_{0} = J^{\pm}_{2}, \quad L^{\pm}_{1} = \sqrt{2}J^{\pm}_{1}, \tag{37}$$

$$W^{\pm}_{-2} = -2T^{\pm}_{00}, \quad W^{\pm}_{-1} = \sqrt{2}T^{\pm}_{02}, \quad W^{\pm}_{0} = -T^{\pm}_{22}, \tag{38}$$

$$W^{\pm}_{1} = -\sqrt{2}T^{\pm}_{12}, \quad W^{\pm}_{2} = -2T^{\pm}_{11}, \tag{39}$$

where the generators T_{ab} are traceless.

- Followed by

$$J^{\pm}_{a} = \frac{J_a \pm lP_a}{2} \qquad T^{\pm}_{ab} = \frac{J_{ab} \pm lP_{ab}}{2}, \tag{40}$$

- So that the full gauge field reads, for spin $s = 2$ and $s = 3$ case

$$A = A^{+} + A^{-} = \left(\frac{1}{2}\mathcal{M}d\phi + \frac{\mathcal{N}}{l^2}du - \frac{r^2}{2l^2}d\phi\right)J_0 + d\phi J_1 + \frac{r}{l^2}duJ_2$$
$$+ \left(-dr + \frac{1}{2}\mathcal{M}du + \mathcal{N}d\phi - \frac{r^2}{2l^2}du\right)P_0 + duP_1 + rd\phi P_2 \quad (41)$$
$$+ \left(\mathcal{W}d\phi + \frac{2}{l^2}\mathcal{Q}du\right)J_{00} + \left(\mathcal{W}du + 2\mathcal{Q}d\phi\right)P_{00}.$$

- Here, the arbitrary functions of u and ϕ have been conveniently redefined as

$$\mathcal{M} = 2(\Xi_{+} + \Xi_{-}), \quad \mathcal{N} = l(\Xi_{+} - \Xi_{-}), \quad (42)$$
$$\mathcal{W} = 2(W_{+} + W_{-}), \quad \mathcal{Q} = l(W_{+} - W_{-}), \quad (43)$$

- The chirality conditions read

$$\partial_u\mathcal{M} = \frac{2}{l^2}\partial_\phi\mathcal{N} \quad 2\partial_u\mathcal{N} = \partial_\phi\mathcal{M}, \quad (44)$$
$$\partial_u\mathcal{W} = \frac{2}{l^2}\partial_\phi\mathcal{Q} \quad 2\partial_u\mathcal{Q} = \partial_\phi\mathcal{W}. \quad (45)$$

- One of the main advantages of expressing the asymptotic form of the connection in our gauge choice is that the vanishing cosmological constant limit can be taken in a straightforward way.
- Then, in the limit $l \to \infty$, the chirality conditions implies that

$$\mathcal{M} = \mathcal{M}(\phi), \quad \mathcal{N} = \mathcal{J}(\phi) + \frac{u}{2}\partial_\phi \mathcal{M}, \tag{46}$$

$$\mathcal{W} = \mathcal{W}(\phi), \quad \mathcal{Q} = \frac{1}{2}(\mathcal{V}(\phi) + u\partial_\phi \mathcal{W}). \tag{47}$$

and we recover the connection

$$A = \left(\frac{1}{2}\mathcal{M}du - dr + \left(\mathcal{J} + \frac{u}{2}\partial_\phi \mathcal{M}\right) d\phi\right) P_0 + duP_1 + rd\phi P_2 + \frac{1}{2}\mathcal{M}d\phi J_0 + d\phi J_1 + (\mathcal{W}du + (\mathcal{V} + u\partial_\phi \mathcal{W})\, d\phi)\, P_{00} + \mathcal{W}d\phi J_{00}, \tag{48}$$

- In the case of $sl(3)$, it has been shown that the asymptotic symmetries are generated by two copies of the W_3 algebra (Henneaux-Rey '10 , Campoleoni-Fredenhagen-Pfenninger-Theisen '10, Gaberdiel-Hartman '11) , defined by

$$
\begin{aligned}
i\{\mathcal{L}_n^{\pm},\mathcal{L}_m^{\pm}\} &= (n-m)\mathcal{L}_{n+m}^{\pm} + \frac{kl}{2}n^3\delta_{m+n}, \\
i\{\mathcal{L}_n^{\pm},\mathcal{W}_m^{\pm}\} &= (2n-m)\mathcal{W}_{n+m}^{\pm}, \\
i\{\mathcal{W}_n^{\pm},\mathcal{W}_m^{\pm}\} &= \frac{1}{3}\Bigg[\frac{16}{kl}(n-m)\sum_{j=-\infty}^{\infty}\mathcal{L}_j^{\pm}\mathcal{L}_{n+m-j}^{\pm} \\
&\quad +(n-m)(2n^2+2m^2-mn)\mathcal{L}_{m+n}^{\pm} + \frac{kl}{2}n^5\delta_{m+n}\Bigg].
\end{aligned}
\tag{49}
$$

- It is also very simple to check that the asymptotic symmetries, described by two copies of the W_3 algebra, reduce to the spin-3 extension of BMS_3, by redefining the generators as

$$P_n = \frac{1}{l}(\mathcal{L}_n^+ + \mathcal{L}_{-n}^-), \tag{50}$$

$$J_n = \mathcal{L}_n^+ - \mathcal{L}_{-n}^-, \tag{51}$$

$$W_n = \frac{1}{l}(\mathcal{W}_n^+ + \mathcal{W}_{-n}^-), \tag{52}$$

$$V_n = \mathcal{W}_n^+ - \mathcal{W}_{-n}^-, \tag{53}$$

  and taking the limit $l \to \infty$.
- It is also simple to obtain, following the procedure described above, the higher spin extension of BMS_3 for the case $s \geq 2$, spanned by two copies of W_N, or $\mathrm{W}_\infty[\lambda]$, and redefining the generator in a suitable way.

Summary

- We described a higher spin theory with vanishing cosmological constant, using the Chern-Simons formalism
- We found the asymptotic symmetry algebra for the theory, which corresponds to a higher spin extension of the BMS_3 algebra with a nontrivial central extension.
- We described a procedure to perform a suitable flat limit ($\Lambda \to 0$), starting from the case of asymptotically AdS_3 theory.
- This procedure allows to extend the asymptotically flat conditions to the case of spins $s \geq 2$.