# Alternative Matching Conditions for Gravitating Defects

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$+ \ldots.$

- we study gravitating defects of any codimension $\delta$ in a $D$-dimensional spacetime
- we try a delta function approach instead of regularization schemes
- we study regular defects, i.e. the bulk metric on the brane is finite. Singular defects are different story since there is no notion of $h_{ij}$, $K_{ij}$

- Cod-1 : $ds^2 = dy^2 + g_{ij} d\chi^i d\chi^j$
  $\mathcal{G}_{ij} \sim g''_{ij} + ... = K'_{ij} + ... = \kappa_D^2 \mathcal{T}_{ij} \quad ('= \partial_y)$
  For regular bulk matter content $\mathcal{T}_{\mu\nu}$ (the matter content of the brane $T_{\mu\nu}$ is different thing), it is $g_{ij}(y=0) =$ finite.
  The inclusion $\mathcal{G}_{ij} = \kappa_D^2 \mathcal{T}_{ij} + \kappa_D^2 T_{ij} \delta(y)$ (even this is not true in our proposal) does not change the finiteness of $g_{ij}$.
  Electric potential of a charged plane is finite $\phi \sim y$.

- Cod-2 : $ds^2 = dr^2 + L^2 d\theta^2 + g_{ij} d\chi^i d\chi^j \qquad ('= \partial_r)$
  $\mathcal{G}_{ij} \sim K'_{ij} + \frac{L'}{L} K_{ij} - \frac{L''}{L} g_{ij} + ... = \frac{1}{L}(L K_{ij})' - \frac{L''}{L} g_{ij} + ... = \kappa_D^2 \mathcal{T}_{ij}$
  - $g_{ij}$ singular, e.g. $g_{ij} \sim \ln r \Rightarrow K_{ij} \sim \frac{1}{r} \Rightarrow L K_{ij} \sim 1$ (($\nexists$ induced metric $h_{ij}$)
    Electric potential of a charged rod is singular $\phi \sim \ln r$
  - $g_{ij}$ regular, e.g. the Vilenkin cosmic string

- Cod-3 : $ds^2 = dr^2 + L^2 d\theta^2 + M^2 d\varphi^2 + g_{ij} d\chi^i d\chi^j$
  $\mathcal{G}_{ij} \sim \frac{1}{L^2}(L^2 K_{ij})' - \frac{L''}{L^2} g_{ij} + \ldots = \kappa_D^2 \mathcal{T}_{ij} \quad ('= \partial_r)$
  - $g_{ij}$ singular, e.g. $g_{ij} \sim \frac{1}{r} \Rightarrow K_{ij} \sim \frac{1}{r^2} \Rightarrow L^2 K_{ij} \sim 1$ ($\nexists$ induced metric $h_{ij}$)
    Electric potential of a point charge is singular $\phi \sim \frac{1}{r}$,
    or Newtonian potential of a point mass
  - $g_{ij}$ regular

- In the braneworld context, only the regular solutions are interesting, since there is some finite metric in our 4-dimensional world

# Standard matching conditions

- Cod-1 : $ds^2 = dy^2 + g_{ij}(\chi, y) d\chi^i d\chi^j$

  $\mathcal{G}_{ij} + ... = K'_{ij} + ... = \kappa_D^2 \mathcal{T}_{ij} + \kappa_D^2 T_{ij} \delta(y) \quad (' = \partial_y)$

$K_{ij}$ discontinuous : $\mathrm{distr}\, \mathcal{G}_{ij} = -[(K_{ij} - K h_{ij})_+ + (K_{ij} - K h_{ij})_-]\delta(y)$

$(K_{ij} - K h_{ij})_+ + (K_{ij} - K h_{ij})_- + ... = -\kappa_D^2 T_{ij}$ Israel M.C.

$g_{ij}$ smooth $\Rightarrow K_{ij}^- = -K_{ij}^+ \Rightarrow \mathrm{distr}\, \mathcal{G}_{ij} = 0$

$\mathbb{Z}_2$ symmetric brane ($K_{ij}^- = K_{ij}^+$): $K_{ij} - K h_{ij} + ... = -\frac{\kappa_D^2}{2} T_{ij}$

- Cod-2 : $ds^2 = dr^2 + L(\chi, r)^2 d\theta^2 + g_{ij}(\chi, r) d\chi^i d\chi^j$ axial

  $\mathcal{G}_{ij} + ... = K'_{ij} + \frac{L'}{L} K_{ij} - \frac{L''}{L} g_{ij} + ... = \kappa_D^2 \mathcal{T}_{ij} + \kappa_D^2 T_{ij} \frac{\delta(r)}{2\pi\beta r} \quad ({}' = \partial_r)$

  $L(\chi, r) = \beta(\chi) r + \mathcal{O}(r^2) \Rightarrow L'(\chi, 0^+) = \beta \;;\; L'(\chi, 0) = 1$

  $L'$ discontinuous : distr $L'' = -(1-\beta)\delta(r)$

  $2\pi(\beta - 1) h_{ij} + ... = \kappa_D^2 T_{ij}$ Extended Israel M.C.

  $\Rightarrow$ *Inconsistent for arbitrary* $T_{ij}$

  • Idea [hep-th/0311074 Bostock, Gregory, Navarro, Santiago]:
  Add 6-dim Gauss-Bonnet

$$S = \frac{1}{2\kappa_6^2} \int_M d^6x \sqrt{|g|} \Big[ \mathcal{R} - 2\Lambda_6 + \alpha \big( \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\kappa\lambda}\mathcal{R}^{\mu\nu\kappa\lambda} \big) \Big]$$

$$+ \int_\Sigma d^4\chi \sqrt{|h|} \Big( \frac{r_c^2}{2\kappa_6^2} R - \lambda \Big) + \int_M d^6x\, \mathcal{L}_{mat} + \int_\Sigma d^4\chi\, L_{mat}$$

$$W^{ij} + \frac{1-\beta}{\eta-\beta}\left(1+\frac{r_c^2}{8\pi\alpha(1-\beta)}\right)G^{ij} + \frac{\kappa_6^2\lambda - 2\pi(1-\beta)}{8\pi\alpha(\eta-\beta)}h^{ij} - \frac{\kappa_6^2 T^{ij}}{8\pi\alpha(\eta-\beta)} = 0$$

$$W^{ij} = K^{\alpha ik}K_{\alpha k}{}^{j} - K_{\alpha k}{}^{k}K^{\alpha ij} - \frac{1}{2}(K_{\alpha k\ell}K^{\alpha k\ell} - K_{\alpha k}{}^{k}K^{\alpha}{}_{\ell}{}^{\ell})h^{ij}$$

- In [hep-th/0907.1640, Charmousis, G.K., Papazoglou]:

Consistency of M.C. with bulk equations checked for cosmology

# Equivalent Variational approach for standard M.C.

$$s_D = \int_M d^D x \sqrt{|g|}\, \mathcal{L}(g_{\mu\nu}) \quad , \; s_4 = \int_\Sigma d^4\chi \sqrt{|h|}\, L(h_{ij})$$

Vary with respect to $g_{\mu\nu}$:

$$\delta_g(s_D + s_4) = \int_M d^D x \sqrt{|g|}\, \mathrm{E}^{\mu\nu} \delta g_{\mu\nu} + \int_\Sigma d^4\chi \sqrt{|h|}\, \tau^{\mu\nu} \delta g_{\mu\nu}$$
$$= \int_M d^D x \sqrt{|g|}\, \mathrm{E}^{\mu\nu} \delta g_{\mu\nu} + \int_\Sigma d^4\chi \sqrt{|h|}\, \tau^{ij} \delta h_{ij}$$

$$\mathrm{E}^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} + \frac{\mathcal{L}}{2} g^{\mu\nu} \quad , \quad \tau^{\mu\nu} = \frac{\delta L}{\delta h_{\mu\nu}} + \frac{L}{2} h^{\mu\nu}$$

Because of the defect, $\exists\, \mathrm{distr}\mathrm{E}^{\mu\nu} = k^{\mu\nu}\delta^{(\delta)}$, or because the distributional terms appear in the parallel components $\exists\, \mathrm{distr}\mathrm{E}^{ij} = k^{ij}\delta^{(\delta)} \Rightarrow$

$$\delta_g(s_D + s_4) = \int_M d^D x \sqrt{|g|}\, \hat{\mathrm{E}}^{\mu\nu} \delta g_{\mu\nu} + \int_\Sigma d^4\chi \sqrt{|h|}\, (k^{\mu\nu} + \tau^{\mu\nu}) \delta g_{\mu\nu}$$
$$= \int_M d^D x \sqrt{|g|}\, \hat{\mathrm{E}}^{\mu\nu} \delta g_{\mu\nu} + \int_\Sigma d^4\chi \sqrt{|h|}\, (k^{ij} + \tau^{ij}) \delta h_{ij}$$

So, there are two independent variations $\delta g_{\mu\nu}|_{bulk}$, $\delta g_{\mu\nu}|_{brane}$ $(\delta h_{ij})$ which lead respectively to the bulk field equations and to the standard matching conditions :

$$\hat{\mathrm{E}}^{\mu\nu} = 0 \quad , \quad k^{ij} + \tau^{ij} = 0 \Leftrightarrow k^{\mu\nu} + \tau^{\mu\nu} = 0$$

$$k^{\mu\nu} = k^{ij} x^{\mu}_{,i} x^{\nu}_{,j} \; , \;\; \tau^{\mu\nu} = \tau^{ij} x^{\mu}_{,i} x^{\nu}_{,j}$$

# Proposal for alternative matching conditions

The eqm of the defect is decoupled from bulk metric variation.

• Bulk eqm : $\delta_g s_D = 0$ with $\delta g_{\mu\nu}$ arbitrary under some boundary variational condition $\delta g_{\mu\nu}|_{brane}$ (e.g. of Dirichlet type $\delta g_{\mu\nu}|_{brane} = 0$ or Neumann) $\Rightarrow \hat{\mathrm{E}}^{\mu\nu} = 0$

• Brane eqm : $\delta x^{\mu}$ arbitrary, but contribution from bulk terms is considered (back-reaction)

# Exact symbolic calculation for Alternative M.C.

- $s_4 = \int_\Sigma d^4\chi\sqrt{|h|}\,L(h_{ij})$ , $h_{ij} = g_{\mu\nu}\, x^\mu_{\ ,i}\, x^\nu_{\ ,j}$
  Embedding fields $x^\mu(\chi^i)$ , $\mu,\nu = 1,...,D$ , $i = 1,2,3,4$
  The variation $\delta x^\mu$ gives

$$\delta_x h_{ij} = g_{\mu\nu,\lambda} x^\mu_{\ ,i} x^\nu_{\ ,j} \delta x^\lambda + g_{\mu\nu} x^\mu_{\ ,i} \delta x^\nu_{\ ,j} + g_{\mu\nu} x^\nu_{\ ,j} \delta x^\mu_{\ ,i}$$
$$= x^\mu_{\ ,i} x^\nu_{\ ,j} (g_{\mu\nu,\lambda}\delta x^\lambda + g_{\mu\lambda}\delta x^\lambda_{\ ,\nu} + g_{\nu\lambda}\delta x^\lambda_{\ ,\mu})$$

- $\delta_x s_4 = \int_\Sigma d^4\chi\sqrt{|h|}\,\tau^{ij}\delta_x h_{ij}$ , $\tau^{ij} = \frac{\delta L}{\delta h_{ij}} + \frac{L}{2} h^{ij}$
- With $\delta x^\mu|_{\partial\Sigma} = 0$

$$\delta_x s_4 = -2\int_\Sigma d^4\chi\sqrt{|h|}\, g_{\mu\sigma}[(\tau^{ij} x^\mu_{\ ,i})_{|j} + \tau^{ij}\Gamma^\mu_{\ \nu\lambda} x^\nu_{\ ,i} x^\lambda_{\ ,j}]\delta x^\sigma$$
$$= -2\int_\Sigma d^4\chi\sqrt{|h|}\, g_{\mu\sigma}(\tau^{ij}_{\ \ |j}\, x^\mu_{\ ,i} - \tau^{ij} K^\alpha_{\ ij} n_\alpha^{\ \mu})\delta x^\sigma$$

- $\delta_x s_4 = 0 \Rightarrow \tau^{ij}{}_{|j}\, x^{\mu}{}_{,i} - \tau^{ij} K^{\alpha}{}_{ij} n_{\alpha}{}^{\mu} = 0$

  $\tau^{ij}{}_{|j} = 0 \quad , \qquad \tau^{ij} K^{\alpha}{}_{ij} = 0 \Leftrightarrow \tau^{ij}(x^{\mu}{}_{;ij} + \Gamma^{\mu}{}_{\nu\lambda} x^{\nu}{}_{,i} x^{\lambda}{}_{,j}) = 0$

## Examples

(1) Nambu-Goto:

$$L = 1 \;\Rightarrow \tau^{ij} = \tfrac{1}{2} h^{ij} \;\Rightarrow\; h^{ij} K^{\alpha}{}_{ij} = 0 \Leftrightarrow \Box_h x^{\mu} + \Gamma^{\mu}{}_{\nu\lambda} h^{\nu\lambda} = 0$$

(2) Regge-Teitelboim:

$$L = \tfrac{r_c^{D-4}}{2\kappa_D^2} R - \lambda + \tfrac{L_{mat}}{\sqrt{|h|}} \;\Rightarrow\; \tau^{ij} = -\tfrac{1}{2}\left(\tfrac{r_c^{D-4}}{\kappa_D^2} G^{ij} + \lambda h^{ij} - T^{ij}\right) \;\Rightarrow$$

$$T^{ij}{}_{|j} = 0 \;,\; \left(\tfrac{r_c^{D-4}}{\kappa_D^2} G^{ij} + \lambda h^{ij} - T^{ij}\right) K^{\alpha}{}_{ij} = 0$$

- Additionally $s_D = \int_M d^D x \sqrt{|g|}\, \mathcal{L}(g_{\mu\nu})$

$$\delta_x s_D = \int_M d^D x \sqrt{|g|}\, (\mathrm{distr}\,\mathrm{E}^{\mu\nu}) \delta_x g_{\mu\nu} = \int_M d^D x \sqrt{|g|}\, k^{ij} \delta^{(\delta)} \delta_x h_{ij}$$

$$= \int_\Sigma d^4 \chi \sqrt{|h|}\, k^{ij} \delta_x h_{ij}$$

$$\Rightarrow \delta_x (s_D + s_4)|_{brane} = \int_\Sigma d^4 \chi \sqrt{|h|}\, (k^{ij} + \tau^{ij}) \delta_x h_{ij}$$

$$\delta_x (s_D + s_4)|_{brane} = 0 \;\Rightarrow$$

$$(k^{ij} + \tau^{ij})_{|j} = 0\,,\; (k^{ij} + \tau^{ij}) K^\alpha{}_{ij} = 0 \Leftrightarrow (k^{ij} + \tau^{ij})(x^\mu{}_{;ij} + \Gamma^\mu{}_{\nu\lambda} x^\nu{}_{,i} x^\lambda{}_{,j}) = 0$$

"Gravitating Nambu-Goto matching conditions"

- Another way (effective) to get the same :
  Bulk coordinate change $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$
  Brane coordinates are changed $\delta x^\mu$ (only $\delta x^\mu|_{brane}$ matters)
  Due to distributional terms, $\delta_x s_D|_{brane} \neq 0$

  If the fields change under the *functional* variation :

$$\begin{aligned}\delta_x \phi^\mu_\nu &= \phi'^\mu_\nu(x^\rho) - \phi^\mu_\nu(x^\rho)\\ &= \phi^\lambda_\nu(x^\rho)\delta x^\mu_{,\lambda} - \phi^\mu_\lambda(x^\rho)\delta x^\lambda_{,\nu} - \phi^\mu_{\nu,\lambda}\delta x^\lambda = -\mathcal{L}_{\delta x}\phi^\mu_\nu\end{aligned}$$

- The embedding fields $x^\mu$ do not change functionally

$$\delta_x x^\mu = x'^\mu|_{x^\nu} - x^\mu = x^\mu - x^\mu = 0$$

- $$\begin{aligned}\delta_x g_{\mu\nu} &= g'_{\mu\nu}(x^\rho) - g_{\mu\nu}(x^\rho)\\ &= -(g_{\mu\nu,\lambda}\delta x^\lambda + g_{\mu\lambda}\delta x^\lambda{}_{,\nu} + g_{\nu\lambda}\delta x^\lambda{}_{,\mu}) = -\mathcal{L}_{\delta x} g_{\mu\nu}\end{aligned}$$

- $$\begin{aligned}\delta_x h_{ij} &= (\delta_x g_{\mu\nu}) x^\mu{}_{,i} x^\nu{}_{,j}\\ &= -x^\mu{}_{,i} x^\nu{}_{,j}(g_{\mu\nu,\lambda}\delta x^\lambda + g_{\mu\lambda}\delta x^\lambda{}_{,\nu} + g_{\nu\lambda}\delta x^\lambda{}_{,\mu})\end{aligned}$$

same as with $\delta x^\mu$ variation

- $\delta_x(s_D + s_4)|_{brane} = \int_\Sigma d^4\chi\sqrt{|h|}\,(k^{ij} + \tau^{ij})\delta_x h_{ij}$
  $\delta_x(s_D + s_4)|_{brane} = 0 \;\Rightarrow$

$$(k^{ij} + \tau^{ij})_{|j} = 0\,,\; (k^{ij} + \tau^{ij})K^{\alpha}{}_{ij} = 0 \Leftrightarrow (k^{ij} + \tau^{ij})(x^{\mu}{}_{;ij} + \Gamma^{\mu}{}_{\nu\lambda}x^{\nu}{}_{,i}x^{\lambda}{}_{,j}) = 0$$

# Problems/Possible problems with standard M.C.

- Generic cod-2 or 3 defects are not allowed in D=4 since Einstein gravity is insufficient and there are no higher Lovelock densities. Since 4 dimensions effectively describe spacetime at certain energy/length scales, the previous defects might be compatible.
- Even if the above is not necessary, this failure signals the failure of consistency for any defect with codimension $\delta > [\frac{D-1}{2}]$. I.e. the spirit of the proposal for the standard M.C. is to add higher and higher Lovelock densities to accommodate higher-codimension defects, but there are no Lovelock densities higher than $[\frac{D-1}{2}]$. So, for $D = 4$ the maximum consistent $\delta = 1$. For $D = 5, 6$ the maximum consistent $\delta = 2$, etc. What about generic cod-3 in 6-D EGB?

- The probe limit: Israel M.C. and generalizations do not obey the Nambu-Goto probe limit $h^{ij}K^{\alpha}{}_{ij} = 0 \Leftrightarrow \Box_h x^{\mu} + \Gamma^{\mu}{}_{\nu\lambda} h^{\nu\lambda} = 0$, which is the natural lowest-order dynamics of a test brane with tension.
  On the other hand there is the geodesic equation, not necessarily for point particles, which is a special case of NG equation, $K^{\alpha}{}_{ij} = 0 \Leftrightarrow x^{\mu}{}_{;ij} + \Gamma^{\mu}{}_{\nu\lambda} x^{\nu}{}_{,i} x^{\lambda}{}_{,j}$ [Of course for a point particle the geodesic and the NG equations coincide]
  The geodesic limit of the Israel M.C. cannot be acceptable since it is not preserved in other gravity theories/ or codimensions.
  In older works on the subject of the motion of probe particles, the geodesic motion arises under the assumption that the background metric remains fixed during passage to the limit of a small body. Ehlers and Geroch [gr-qc/0309074] have generalized for a body manifesting a (suitable small) gravitational field of its own, and derived that a material body in the limit of sufficiently small size and mass still moves along a geodesic.

At some point of the proof they say “We remark that since there was never used in the proof that the $G_{ab}$ in Eqn. (3) is the Einstein tensor of $g_{ab}$, thus, for example, the theorem above is also applicable to any metric theory of gravity.”

So the statement and the proof are kinematical, independent of the dynamics.

$\rightsquigarrow$ Alternative M.C. pass all the previous problems

- Where are the "garbage" terms?

For cod-1 it is

$$\delta_g \int_M d^5x\sqrt{|g|}\,(\mathcal{R}+...) - 2\delta_g \int_\Sigma d^4\chi\sqrt{|h|}\,(K+...)$$
$$= \Big[\int_M d^5x\sqrt{|g|}\,(-\mathcal{G}^{\mu\nu}+...)\delta g_{\mu\nu} + \int_M d^5x\sqrt{|g|}\,(...\delta g_{\mu\nu;\kappa\lambda}...)\Big]$$
$$+ \int_\Sigma d^4\chi\sqrt{|h|}\,(...\delta g_{\mu\nu;\kappa}...)$$
$$= \int_M d^5x\sqrt{|g|}\,(-\mathcal{G}^{\mu\nu}+...)\delta g_{\mu\nu} + \int_\Sigma d^4\chi\sqrt{|h|}\,(K^{\mu\nu}-Kh^{\mu\nu}+...)\delta g_{\mu\nu}$$

$$\delta_g\int_M d^5x\sqrt{|g|}\,(\mathcal{R}+...) = \int_M d^5x\sqrt{|g|}\,(-\mathcal{G}^{\mu\nu}+...)\delta g_{\mu\nu} + \int_M d^5x\sqrt{|g|}\,(...\delta g_{\mu\nu;\kappa\lambda}...)$$
$$= \Big[\int_M d^5x\sqrt{|g|}\,(-\hat{\mathcal{G}}^{\mu\nu}+...)\delta g_{\mu\nu} + \int_\Sigma d^4\chi\sqrt{|h|}\,(K^{\mu\nu}-Kh^{\mu\nu}+...)\delta g_{\mu\nu}\Big]$$
$$+ \int_M d^5x\sqrt{|g|}\,(...\delta g_{\mu\nu;\kappa\lambda}...)$$

For higher codimensions

$$\delta_g \int_M d^D x \sqrt{|g|}\,(\mathcal{R} + \ldots) + \delta_g \int_\Sigma d^4\chi \sqrt{|h|}\,(\not{K} + \ldots)$$
$$= \Big[ \int_M d^D x \sqrt{|g|}\,(-\mathcal{G}^{\mu\nu} + \ldots)\delta g_{\mu\nu} + \int_M d^D x \sqrt{|g|}\,(\ldots \delta g_{\mu\nu;\kappa\lambda} \ldots) \Big]$$
$$+ \int_\Sigma d^4\chi \sqrt{|h|}\,(W^{\mu\nu} + \ldots)\delta g_{\mu\nu}$$

$$\delta_g \int_M d^D x \sqrt{|g|}\,(\mathcal{R} + \ldots) = \int_M d^D x \sqrt{|g|}\,(-\mathcal{G}^{\mu\nu} + \ldots)\delta g_{\mu\nu} + \int_M d^D x \sqrt{|g|}\,(\ldots \delta g_{\mu\nu;\kappa\lambda} \ldots)$$
$$= \Big[ \int_M d^D x \sqrt{|g|}\,(-\hat{\mathcal{G}}^{\mu\nu} + \ldots)\delta g_{\mu\nu} + \int_\Sigma d^4\chi \sqrt{|h|}\,(W^{\mu\nu} + \ldots)\delta g_{\mu\nu} \Big]$$
$$+ \int_M d^D x \sqrt{|g|}\,(\ldots \delta g_{\mu\nu;\kappa\lambda} \ldots)$$

"Tube" around the defect

$$\delta_g \int_M d^D x \sqrt{|g|}\,(\mathcal{R} + \ldots) + \delta_g \int_\Sigma d^4\chi \sqrt{|h|}\,(\not{K} + \ldots)$$

$$= \Big[ \int_M d^D x \sqrt{|g|}\,(-\mathcal{G}^{\mu\nu} + \ldots)\delta g_{\mu\nu} + \int_{Tube} d^{D-1}\hat{x} \sqrt{|\hat{g}|}\,(\ldots\delta g_{\mu\nu;\kappa}\ldots)\Big]$$

$$+ \int_\Sigma d^4\chi \sqrt{|h|}\,(W^{\mu\nu} + \ldots)\delta g_{\mu\nu}$$

$\delta g_{\mu\nu}|_{bulk}$ arbitrary, $(\ldots\delta g_{\mu\nu;\kappa}\ldots)|_{Tube} = 0$, $\delta g_{\mu\nu}|_{brane}$ arbitrary.

$$\delta_g \int_M d^D x \sqrt{|g|}\,(\mathcal{R} + \ldots) + \delta_g \int_{Tube} d^{D-1}\hat{x} \sqrt{\hat{g}}\,(K + \ldots) + \delta_g \int_\Sigma d^4\chi \sqrt{|h|}\,(\not{K} + \ldots)$$

$$= \Big[ \int_M d^D x \sqrt{|g|}\,(-\mathcal{G}^{\mu\nu} + \ldots)\delta g_{\mu\nu} + \int_{Tube} d^{D-1}\hat{x} \sqrt{|\hat{g}|}\,(\ldots\delta g_{\mu\nu;\kappa}\ldots)\Big]$$

$$+ \int_{Tube} d^{D-1}\hat{x} \sqrt{|\hat{g}|}\,(\ldots\delta g_{\mu\nu;\kappa}\ldots) + \int_\Sigma d^4\chi \sqrt{|h|}\,(\ldots W^{\mu\nu} + \ldots)\delta g_{\mu\nu}$$

$$= \int_M d^D x \sqrt{|g|}\,(-\mathcal{G}^{\mu\nu} + \ldots)\delta g_{\mu\nu} + \int_{Tube} d^{D-1}\hat{x} \sqrt{|\hat{g}|}\,(K^{\mu\nu} - K h^{\mu\nu} + \ldots)\delta g_{\mu\nu}$$

$$+ \int_\Sigma d^4\chi \sqrt{|h|}\,(W^{\mu\nu} + \ldots)\delta g_{\mu\nu}$$

Using the method of distributional terms, the same:

$$\delta_g \int_M d^D x \sqrt{|g|}\,(\mathcal{R} + \ldots) = \int_M d^D x \sqrt{|g|}\,(-\mathcal{G}^{\mu\nu} + \ldots)\delta g_{\mu\nu} + \int_{Tube} d^{D-1}\hat{x}\sqrt{|\hat{g}|}\,(\ldots\delta g_{\mu\nu;\kappa})$$

$$= \Big[ \int_M d^D x \sqrt{|g|}\,(-\hat{\mathcal{G}}^{\mu\nu} + \ldots)\delta g_{\mu\nu} + \int_\Sigma d^4\chi \sqrt{|h|}\,(W^{\mu\nu} + \ldots)\delta g_{\mu\nu} \Big]$$

$$+ \int_{Tube} d^{D-1}\hat{x}\sqrt{|\hat{g}|}\,(\ldots\delta g_{\mu\nu;\kappa}\ldots)$$

$$\delta_g \int_M d^D x \sqrt{|g|}\,(\mathcal{R} + \ldots) + \int_{Tube} d^{D-1}\hat{x}\sqrt{\hat{g}}\,(K + \ldots)$$

$$= \Big[ \int_M d^D x \sqrt{|g|}\,(-\mathcal{G}^{\mu\nu} + \ldots)\delta g_{\mu\nu} + \int_M d^{D-1}\hat{x}\sqrt{|\hat{g}|}\,(\ldots\delta g_{\mu\nu;\kappa}\ldots) \Big]$$

$$+ \int_{Tube} d^{D-1}\hat{x}\sqrt{|\hat{g}|}\,(\ldots\delta g_{\mu\nu;\kappa}\ldots)$$

$$= \int_M d^D x \sqrt{|g|}\,(-\hat{\mathcal{G}}^{\mu\nu} + \ldots)\delta g_{\mu\nu} + \int_\Sigma d^4\chi \sqrt{|h|}\,(W^{\mu\nu} + \ldots)\delta g_{\mu\nu}$$

$$+ \int_{Tube} d^{D-1}\hat{x}\sqrt{|\hat{g}|}\,(K^{\mu\nu} - K h^{\mu\nu} + \ldots)\delta g_{\mu\nu}$$

Alternative M.C. : "Outside" problem (bulk equations) well-posed. "Interior" problem (M.C.) well-posed because $\delta x^\mu$ variation independent.

$$\delta_x \int_M d^D x \sqrt{|g|}\,(\mathcal{R} + ...)$$
$$= \int_M d^D x \sqrt{|g|}\,(-\mathcal{G}^{\mu\nu} + ...)\delta_x g_{\mu\nu} + \int_M d^D x \sqrt{|g|}\,(...\delta g_{\mu\nu;\kappa\lambda}...)$$
$$= \int_\Sigma d^4 \chi \sqrt{|h|}\,(W^{\mu\nu} + ...)\delta_x g_{\mu\nu} \;\; + \;\; 0 \;\; = \;\; \text{...Alternative M.C...}$$

# 6-dim Einstein-Gauss-Bonnet gravity

- $$S = \frac{1}{2\kappa_6^2}\int_M d^6x\sqrt{|g|}\Big[\mathcal{R} - 2\Lambda_6 + \alpha\big(\mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\kappa\lambda}\mathcal{R}^{\mu\nu\kappa\lambda}\big)\Big]$$
$$+\int_\Sigma d^4\chi\sqrt{|h|}\Big(\frac{r_c^2}{2\kappa_6^2}R - \lambda\Big) + \int_M d^6x\,\mathcal{L}_{mat} + \int_\Sigma d^4\chi\, L_{mat}$$
- $\delta g_{\mu\nu}|_{bulk}$ with $\delta g_{\mu\nu}|_{brane} = 0$:
$$\mathcal{G}_{\mu\nu} + 2\alpha\big(\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\kappa}\mathcal{R}_\nu{}^\kappa - 2\mathcal{R}_{\mu\kappa\nu\lambda}\mathcal{R}^{\kappa\lambda} + \mathcal{R}_{\mu\kappa\lambda\sigma}\mathcal{R}_\nu{}^{\kappa\lambda\sigma}\big)$$
$$-\frac{\alpha}{2}\big(\mathcal{R}^2 - 4\mathcal{R}_{\kappa\lambda}\mathcal{R}^{\kappa\lambda} + \mathcal{R}_{\kappa\lambda\rho\sigma}\mathcal{R}^{\kappa\lambda\rho\sigma}\big)g_{\mu\nu} = \kappa_6^2\mathcal{T}_{\mu\nu} - \Lambda_6 g_{\mu\nu}$$
- $\delta x^\mu$:
$$S_c = \int_\Sigma d^4\chi\,\sqrt{|h|}\,\Big[\lambda^{ij}(h_{ij} - g_{\mu\nu}x^\mu{}_{,i}x^\nu{}_{,j}) + \lambda^{\alpha i}n_{\alpha\mu}x^\mu{}_{,i}$$
$$+\lambda^{\alpha\beta}(g_{\mu\nu}n_\alpha{}^\mu n_\beta{}^\nu - \delta_{\alpha\beta})\Big]$$

►

$$
\begin{aligned}
\delta(S+S_c)\big|_{brane} =& \int_\Sigma d^4\chi\,\sqrt{|h|}\,\left(\lambda^{\alpha i}x^{\mu}_{\ ,i} + 2\lambda^{\alpha\beta}n_{\beta}^{\ \mu}\right)\delta n_{\alpha\mu} \\
&+ \int_\Sigma d^4\chi\,\sqrt{|h|}\left[\lambda^{ij} + \frac{1}{2}(T^{ij} - \lambda h^{ij}) - \frac{r_c^2}{2\kappa_6^2}G^{ij}\right]\delta h_{ij} \\
&- \int_\Sigma d^4\chi\sqrt{|h|}\,\left(\lambda^{ij}x^{\mu}_{\ ,i}x^{\nu}_{\ ,j} + \lambda^{\alpha\beta}n_{\alpha}^{\ \mu}n_{\beta}^{\ \nu}\right)\delta g_{\mu\nu} \\
&- \frac{1}{2\kappa_6^2}\int_M d^6x\sqrt{|g|}\,\left\{\mathcal{G}^{\mu\nu} + \alpha\mathcal{J}^{\mu\nu} - \kappa_6^2\mathcal{T}^{\mu\nu} + \Lambda_6 g^{\mu\nu}\right\}\delta g_{\mu\nu}\Big|_{brane} \\
&+ \frac{1}{\kappa_6^2}\int_M d^6x\sqrt{|g|}\,\Big\{g^{\mu[\kappa}g^{\lambda]\nu} + 2\alpha\Big(\mathcal{R}^{\mu\nu\kappa\lambda} + 2\mathcal{R}^{\nu[\kappa}g^{\lambda]\mu} \\
&\qquad - 2\mathcal{R}^{\mu[\kappa}g^{\lambda]\nu} + \mathcal{R}g^{\mu[\kappa}g^{\lambda]\nu}\Big)\Big\}(\delta g_{\nu\kappa})_{;\lambda\mu}\Big|_{brane}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}^{\mu\nu} =& 2\mathcal{R}\mathcal{R}^{\mu\nu} - 4\mathcal{R}^{\mu\kappa}\mathcal{R}^{\nu}_{\ \kappa} - 4\mathcal{R}^{\mu\kappa\nu\lambda}\mathcal{R}_{\kappa\lambda} + 2\mathcal{R}^{\mu\kappa\lambda\sigma}\mathcal{R}^{\nu}_{\ \kappa\lambda\sigma} \\
&- \frac{1}{2}\left(\mathcal{R}^2 - 4\mathcal{R}_{\kappa\lambda}\mathcal{R}^{\kappa\lambda} + \mathcal{R}_{\kappa\lambda\rho\sigma}\mathcal{R}^{\kappa\lambda\rho\sigma}\right)g^{\mu\nu}
\end{aligned}
$$

- Axially symmetric ansatz
  $ds_6^2 = dr^2 + L^2(\chi, r)d\theta^2 + g_{ij}(\chi, r)d\chi^i d\chi^j$
- 2 sources of discontinuity:
  – cone discontinuity: $L(\chi, r) = \beta(\chi)r + \mathcal{O}(r^2)$,
  $L'(\chi, 0^+) = \beta(\chi)$, $L'(\chi, 0) = 1$
  – extrinsic curvature discontinuity, $\mathcal{K}_{ij}(\chi, 0) = 0 \neq \mathcal{K}_{ij}(\chi, 0^+)$
  $\mathrm{distr}\, L''(\chi, r) = -(1 - \beta(\chi))\, \delta(r)$
  $\mathrm{distr}\, \mathcal{K}'_{ij}(\chi, r) = (1 - \eta) K_{ij}(\chi)\, \delta(r)$
- four kinds of matching conditions:
  (i) pure cone or topological matching conditions [Charmousis-Zegers 2005], there is a conical singularity, but the extrinsic curvature has no jump $\mathcal{K}_{ij}(\chi, 0) = \mathcal{K}_{ij}(\chi, 0^+)$
  (ii) pure extrinsic curvature discontinuity, there is no cone, so $\beta = 1$
  (iii) cone plus extrinsic curvature discontinuity [Bostock, Gregory, Navarro, Santiago 2004]
  (iv) "smooth" matching conditions, neither conical singularity nor extrinsic curvature discontinuity

If $\mathcal{K}_{ij}(\chi,0)=\eta K_{ij}(\chi)$ , $K_{ij}(\chi)\equiv\mathcal{K}_{ij}(\chi,0^+)$:
(i) $\beta\neq 1$, $\eta=1$, (ii) $\beta=1$, $\eta=0$, (iii) $\beta\neq 1$, $\eta=0$, (iv) $\beta=1$, $\eta=1$

- $\mathrm{distr}\,\mathcal{G}^{ij}=-\frac{1-\beta}{\beta}h^{ij}\frac{\delta(r)}{r}$, $\mathrm{distr}\,\mathcal{J}^{ij}=4\left(\frac{\eta-\beta}{\beta}W^{ij}+\frac{1-\beta}{\beta}G^{ij}\right)\frac{\delta(r)}{r}$
  $W^{ij}=K^{ik}K_k^j-KK^{ij}-\frac{1}{2}(K_{k\ell}K^{k\ell}-K^2)h^{ij}$
- 

$$\delta(S+S_c)\big|_{brane}=\int_\Sigma d^4\chi\,\sqrt{|h|}\,(\lambda^{\alpha i}x^\mu_{\ ,i}+2\lambda^{\alpha\beta}n_\beta^{\ \mu})\,\delta n_{\alpha\mu}$$
$$+\int_\Sigma d^4\chi\,\sqrt{|h|}\left[\lambda^{ij}+\frac{1}{2}T^{ij}+\frac{2\pi(1-\beta)-\kappa_6^2\lambda}{2\kappa_6^2}h^{ij}-\right.$$
$$\left.\frac{4\pi\alpha(1-\beta)}{\kappa_6^2}\left(1+\frac{r_c^2}{8\pi\alpha(1-\beta)}\right)G^{ij}-\frac{4\pi\alpha(\eta-\beta)}{\kappa_6^2}W^{ij}\right]\delta h_{ij}$$
$$-\int_\Sigma d^4\chi\,\sqrt{|h|}\,(\lambda^{\alpha\beta}n_\alpha^{\ \mu}n_\beta^{\ \nu}+\lambda^{ij}x^\mu_{\ ,i}x^\nu_{\ ,j})\,\delta g_{\mu\nu}$$

-

$$\lambda^{\alpha i} x^{\mu}_{,i} + 2\lambda^{\alpha\beta} n_{\beta}{}^{\mu} = 0$$

$$\lambda^{ij} = \frac{4\pi\alpha(\eta-\beta)}{\kappa_6^2} W^{ij} + \frac{4\pi\alpha(1-\beta)}{\kappa_6^2}\left(1+\frac{r_c^2}{8\pi\alpha(1-\beta)}\right) G^{ij} + \frac{\kappa_6^2\lambda-2\pi(1-\beta)}{2\kappa_6^2} h^{ij} - \frac{1}{2} T^{ij}$$

$$\int_{\Sigma} d^4\chi \sqrt{|h|}\,(\lambda^{\alpha\beta} n_{\alpha}{}^{\mu} n_{\beta}{}^{\nu} + \lambda^{ij} x^{\mu}_{,i} x^{\nu}_{,j})\,\delta g_{\mu\nu} = 0$$

- Matching Conditions

$$\left[W^{ij} + \frac{1-\beta}{\eta-\beta}\left(1+\frac{r_c^2}{8\pi\alpha(1-\beta)}\right) G^{ij} + \frac{\kappa_6^2\lambda-2\pi(1-\beta)}{8\pi\alpha(\eta-\beta)} h^{ij} - \frac{\kappa_6^2 T^{ij}}{8\pi\alpha(\eta-\beta)}\right] K_{ij} = 0$$

$$T^{ij}{}_{|j} = \frac{2\pi}{\kappa_6^2}\beta_{,j}(h^{ij} - 4\alpha G^{ij}) + \frac{8\pi\alpha}{\kappa_6^2}\left[(\eta-\beta)W^{ij}\right]_{|j}$$

- + Bulk equations on the brane .....

- $T^{ij}{}_{|j} = -\eta \frac{8\pi\alpha}{\kappa_6^2} \frac{\beta_{,j}}{\beta} \left( W^{ij} + G^{ij} - \frac{1}{4\alpha} h^{ij} \right)$
  for case (i) there is energy exchange between brane and bulk, different than that of standard treatment

# Cosmology from 6 dimensions

$$ds_6^2 = dr^2 + L^2(t,r)d\theta^2 - n^2(t,r)dt^2 + a^2(t,r)\gamma_{\hat{i}\hat{j}}(\chi^{\hat{\ell}})d\chi^{\hat{i}}d\chi^{\hat{j}}$$

- Complicated system of effective equations at the brane for $a, a', a'', n, n', n'', \beta, \rho$ $(p = w\rho)$ $('=\partial_r)$, but consistent
- Essential equations which govern cosmic evolution:

$$\dot{\rho} + 3nH(\rho + p) = -\eta\frac{24\pi\alpha}{\kappa_6^2}\frac{\dot{\beta}}{\beta}\mathcal{X}$$

$$(1+f)\mathcal{X} + 2\mathcal{Y} = 0$$

$$\frac{\dot{A}}{nA} + H(1-f) = \frac{\dot{\beta}}{n\beta}\frac{\mathcal{X}}{2A^2}$$

$$\mathcal{X} = X - A^2 + \frac{1}{12\alpha}\ ,\ \mathcal{Y} = Y - fA^2 + \frac{1}{12\alpha}\ ,\ A = \frac{a'}{a}$$

$$X = H^2 + \frac{k}{a^2}\ \ ,\ \ Y = \frac{\dot{X}}{2nH} + X\ ,\ f = 3\,\frac{p - \sigma_2 + \sigma_1(X+2Y)}{\rho + \sigma_2 - 3\sigma_1 X}$$

$$\sigma_1 = \frac{r_c^2}{\kappa_6^2} + \frac{8\pi\alpha(1-\eta)}{\kappa_6^2}\,,\ \sigma_2 = \lambda - \frac{2\pi(1-\eta)}{\kappa_6^2}\,,\ \sigma = \sigma_2 + \frac{\sigma_1}{4\alpha} = \lambda + \frac{r_c^2}{4\alpha\kappa_6^2}$$

- Although a system of 3 equations for 4 unknowns, it can be integrated to a system for $a, \rho$ (containing also $\beta$):

$$\frac{\dot{X}}{nH}\left(1+6\sigma_1\frac{X+\frac{1}{12\alpha}}{3\sigma_1 X-\rho-\sigma_2}+\frac{9\sigma_1}{\eta\frac{24\pi\alpha}{\kappa_6^2}+c\beta}\right)+2\left(X+\frac{1}{12\alpha}\right)\left(4+3\frac{\rho+p}{3\sigma_1 X-\rho-\sigma_2}\right.$$
$$\left.+\frac{9(\rho+p)+8(3\sigma_1 X-\rho-\sigma_2)}{\eta\frac{24\pi\alpha}{\kappa_6^2}+c\beta}\right)=0$$

$$\dot{\rho}+3nH(\rho+p)=\eta\frac{24\pi\alpha}{\kappa_6^2}\frac{\dot{\beta}}{\beta\left(\eta\frac{24\pi\alpha}{\kappa_6^2}+c\beta\right)}(3\sigma_1 X-\rho-\sigma_2)$$

# Cosmology with $\sigma_1 \neq 0$, $\beta(t)$ =constant

- General solution
  $\frac{1}{\tilde{c}}\,\rho^{\frac{8}{3(1+w)}}\,(\rho+\sigma)^{-3}\,\xi^3 - 3\xi = 2\gamma$ cubic for $\xi$

$\xi = \frac{\rho+\sigma}{3\sigma_1 X - \rho - \sigma_2}$ , $\gamma = \frac{3}{2} + \frac{9\sigma_1}{2}\left(\eta \frac{24\pi\alpha}{\kappa_6^2} + c\right)^{-1}$ , $\tilde{c}$ : integration const.

- Branch I: $\tilde{c}(\rho+\sigma) > 0$ and $\tilde{c}\,(\rho+\sigma)^3 \leq \gamma^2 \rho^{\frac{8}{3(1+w)}}$

$$H^2 + \frac{k}{a^2} = \frac{\rho}{3\sigma_1} + \frac{\sigma_2}{3\sigma_1} + \mathrm{sgn}(\gamma)\,\frac{c_*\rho^{\frac{4}{3(1+w)}}}{6\sigma_1\sqrt{\rho+\sigma}}\,\cosh^{-1}\!\left[\frac{1}{3}\mathrm{arccosh}\left(\frac{|\gamma| c_* \rho^{\frac{4}{3(1+w)}}}{\sqrt{(\rho+\sigma)^3}}\right)\right]$$

  For $w = 1/3$ and a range of the integration constants $c_*, \gamma$, it avoids a cosmological singularity (both in density and curvature) and undergoes accelerated expansion near the minimum scale factor.

- Branch II: $\tilde{c}(\rho+\sigma) > 0$ and $\tilde{c}\,(\rho+\sigma)^3 \geq \gamma^2 \rho^{\frac{8}{3(1+w)}}$

$$H^2 + \frac{k}{a^2} = \frac{\rho}{3\sigma_1} + \frac{\sigma_2}{3\sigma_1} + \frac{c_*\rho^{\frac{4}{3(1+w)}}}{6\sigma_1\sqrt{\rho+\sigma}}\,\cos^{-1}\!\left[\frac{1}{3}\arccos\left(\frac{\gamma c_* \rho^{\frac{4}{3(1+w)}}}{\sqrt{(\rho+\sigma)^3}}\right) + \frac{2\pi m}{3}\right],\ m = 0, 1, 2$$

$$H^2 + \frac{k}{a^2} \approx \frac{\rho}{3\sigma_1} + \frac{\sigma_2}{3\sigma_1} + \frac{c_*}{6\sigma_1\sqrt{\sigma}}\cos^{-1}\!\left[\frac{\pi(1+4m)}{6}\right]\rho^{\frac{4}{3}}\ ,\ \rho \to 0$$

- Branch II: $\tilde{c}(\rho+\sigma)<0$

$$H^2+\frac{k}{a^2}=\frac{\rho}{3\sigma_1}+\frac{\sigma_2}{3\sigma_1}-\frac{c_*\rho^{\frac{4}{3(1+w)}}}{6\sigma_1\sqrt{\rho+\sigma}}\ \sinh^{-1}\left[\frac{1}{3}\text{arcsinh}\left(\frac{\gamma c_*\rho^{\frac{4}{3(1+w)}}}{\sqrt{(\rho+\sigma)^3}}\right)\right]$$

$$H^2+\frac{k}{a^2}\approx\frac{\rho}{3\sigma_1}\left(1-\frac{3}{2\gamma}\right)+\frac{1}{3\sigma_1}\left(\sigma_2-\frac{3\sigma}{2\gamma}\right)-\frac{2\gamma c_*^2}{27\sigma_1\sigma^2}\rho^{\frac{8}{3}}\ ,\ \rho\to 0$$

# Cosmology with $\sigma_1 = 0$

$$\frac{dy}{d\Omega} + 3(1+w)y - 3(1+w)\lambda\left(\frac{24\pi\alpha}{\kappa_6^2\,\beta} + c\right)^{-1} = 0$$

$$y = \frac{\rho+\lambda}{\frac{24\pi\alpha}{\kappa_6^2\,\beta}+c} \quad , \quad \Omega = \ln\frac{a}{a_0}$$

$$\frac{d\tilde{X}}{d\Omega} + 2\frac{(1-3w)\rho+4\lambda}{\rho+\lambda}\tilde{X} + \left[(1+9w)\rho-8\lambda\right]\left(\frac{24\pi\alpha}{\kappa_6^2}+c\beta\right)^{-1} = 0\ ,\ \tilde{X} = X + \frac{1}{12\alpha}$$

For $\beta(t)$ =constant :

$$H^2 + \frac{k}{a^2} = \frac{\kappa_6^2\,\rho}{24\pi\alpha(1-\bar{c})} + \left(\frac{\kappa_6^2\,\lambda}{24\pi\alpha(1-\bar{c})} - \frac{1}{12\alpha}\right) + \frac{\tilde{c}}{(\rho+\lambda)^2}\,\rho^{\frac{8}{3(1+w)}}$$

# Cosmology from 5 dimensions

# Conclusions

- Possible theoretical problems of the standard approach urge us to consider alternative matching conditions
- Here, a sort of “gravitating” Nambu-Goto matching conditions are proposed by varying with respect to the embedding fields consistently
- The various problems seem to be resolved
- There are found interesting 6 and 5-dim cosmologies