Black holes in an expanding universe: The McVittie metric

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- Black holes in the presence of self-gravitating matter

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- Black holes in the presence of self-gravitating matter
- Two competing effects:
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  - Accretion through Einstein equations;
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  - Consistency and stability analyses;
  - Experimental tests:
    - Structure formation;
    - Observations of galaxy clusters.

Overview

The McVittie metric
- Properties
- Apparent horizons

Causal structure
- Asymptotic behavior
- Penrose diagrams
- Dependence with cosmological history
- Example: Black hole in ΛCDM

Conclusions

- Cosmological black holes: McVittie solution [McVittie, MNRAS **93**,325 (1933)]

$$ds^2 = -\frac{\left(1 - \frac{m}{2a(t)\hat{r}}\right)^2}{\left(1 + \frac{m}{2a(t)\hat{r}}\right)^2} dt^2 + a^2(t)\left(1 + \frac{m}{2a(t)\hat{r}}\right)^4 (d\hat{r}^2 + \hat{r}^2 d\Omega^2)$$

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- $m = 0$: FLRW metric

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- Unique solution that satisfies
  - Spherical symmetry
  - Perfect fluid
  - Shear-free
  - Asymptotically FLRW
  - Singularity at the center

- Past spacelike singularity at $a = \frac{m}{2\hat{r}}$
- Event horizons only defined if $H \equiv \frac{\dot{a}}{a}$ constant as $t \to \infty$

Metric features

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- Pressure is inhomogeneous

$$p(\hat{r}, t) = \frac{1}{8\pi} \left[H^2 \frac{-5m + 2a\hat{r}}{m - 2a\hat{r}} + \frac{2\ddot{a}}{a} \frac{m + 2a\hat{r}}{m - 2a\hat{r}} \right]$$

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- Areal radius

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- McVittie in new (canonical) coordinates

[Kaloper, Kleban, Martin, PRD **81**,104044 (2010)]

$$\mathrm{d}s^2 = -R^2\mathrm{d}t^2 + \left[\frac{\mathrm{d}r}{R} - Hr\mathrm{d}t\right]^2 + r^2\mathrm{d}\Omega^2$$

where $\left(R = \sqrt{1 - \frac{2m}{r}}\right)$

- Apparent horizons: zero expansion of null radial geodesics

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_{\pm} = R\left(rH \pm R\right) = 0$$

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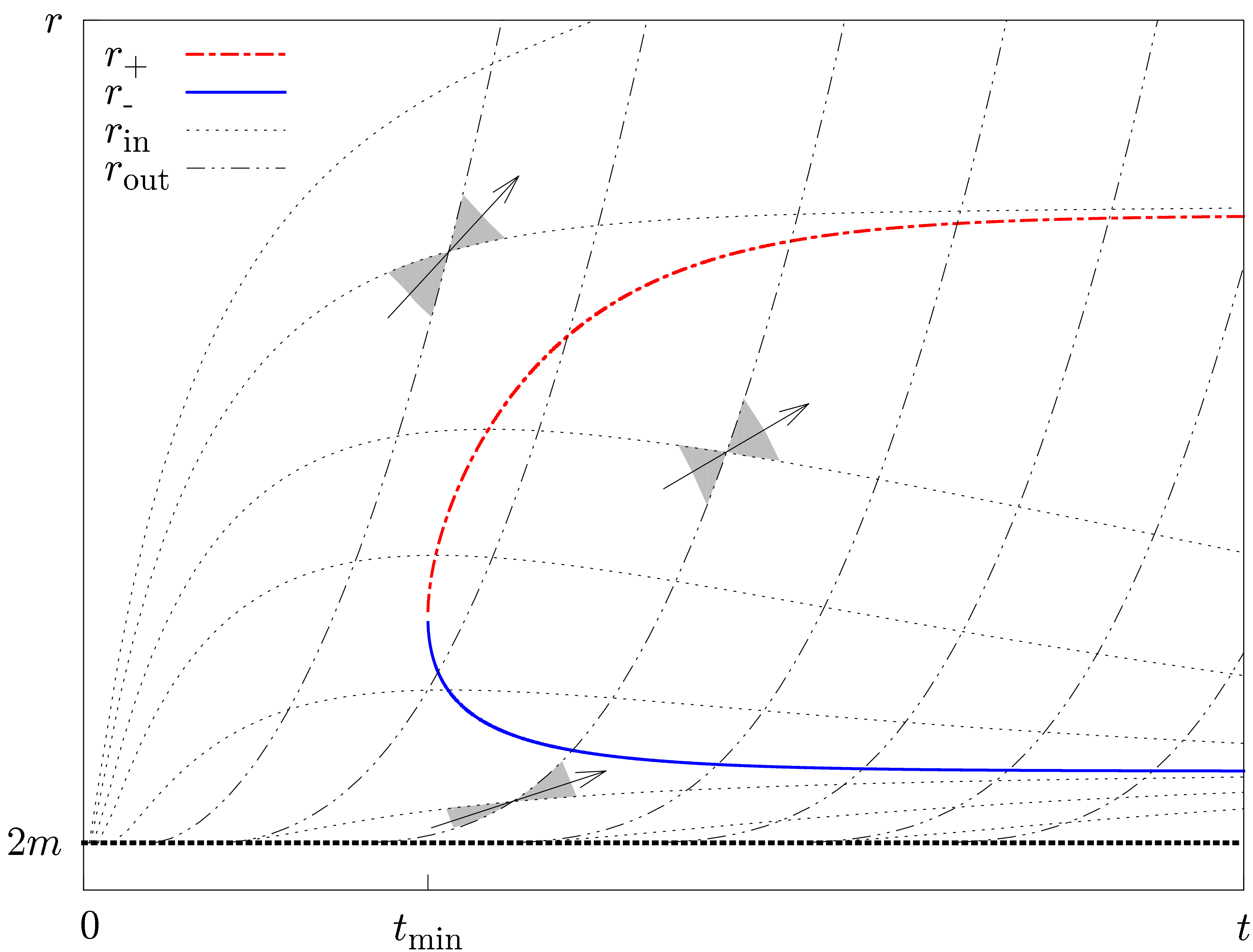
- Real positive solutions only exist if $\frac{1}{3\sqrt{3}m} > H > 0$
  - ☐ r_+ Outer (cosmological) horizon
  - ☐ r_- Inner horizon

- If $H(t) \to H_0$ for $t \to \infty$ apparent horizons become Schwarszchild–de Sitter event horizons ($r_- \to r_\infty$)

Light cones and apparent horizons



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Asymptotic behavior

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- Singular surface $r_* = 2m$ lies in the past of all events (McVittie big bang)

$$\frac{\mathrm{d}}{\mathrm{d}t}(r - r_*) = RrH + \mathcal{O}\left(R^2\right) > 0$$

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  - Black hole at Shwarzschild–de Sitter limit

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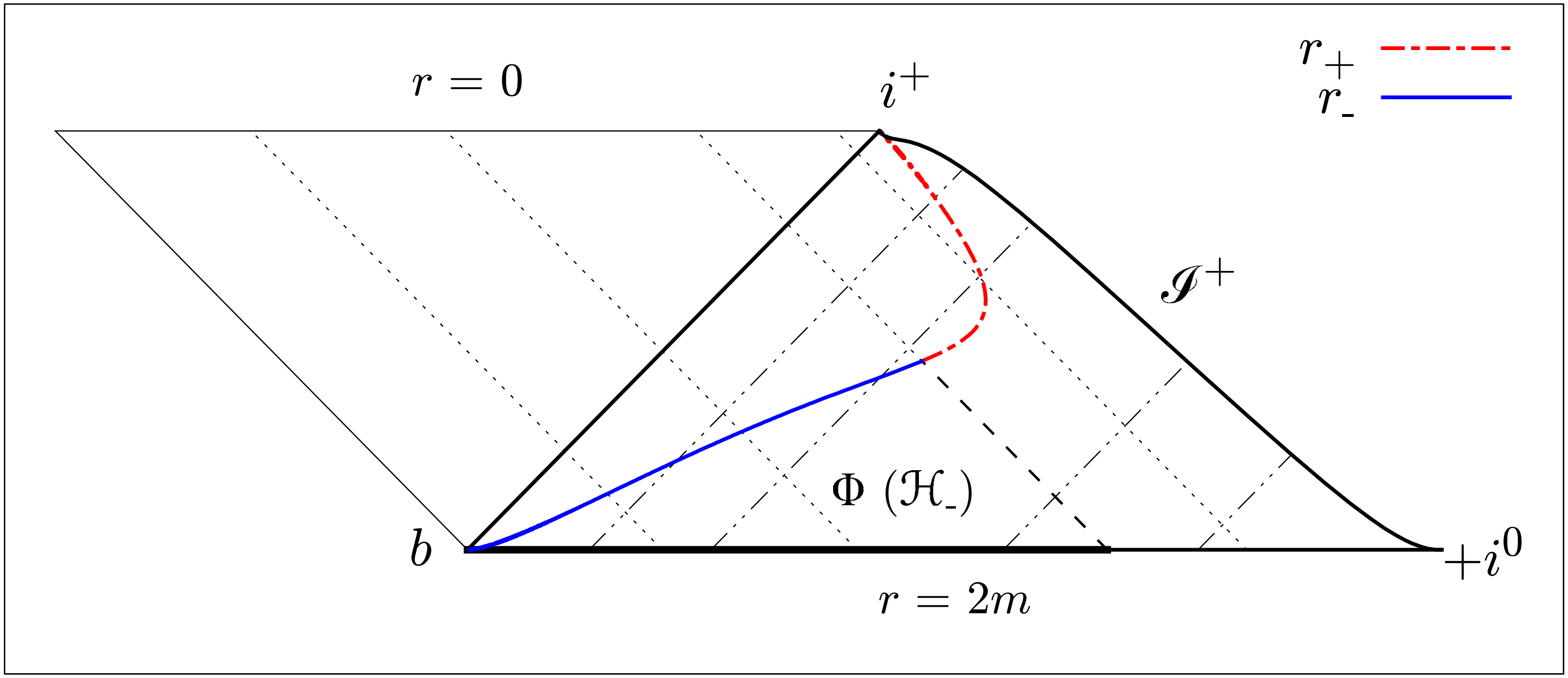
- If $H_0 = 0$, some curvature scalars become singular at r_∞
  - Possibly no BH interpretation at Schwarzschild limit

[Kaloper, Kleban, Martin, PRD **81**,104044 (2010)][Lake, Abdelqader, PRD **84**,044045 (2011)]

- Geodesic completion with Schwarzschild–de Sitter



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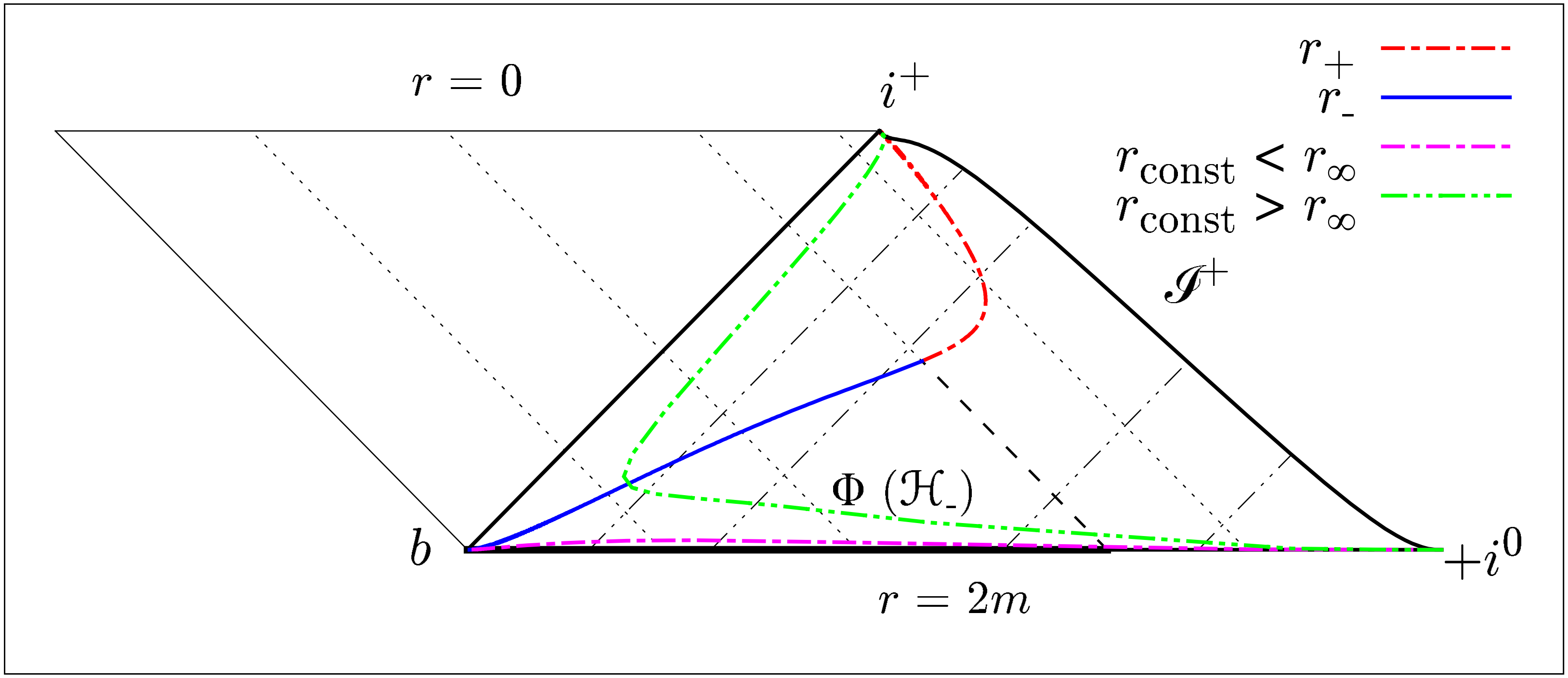

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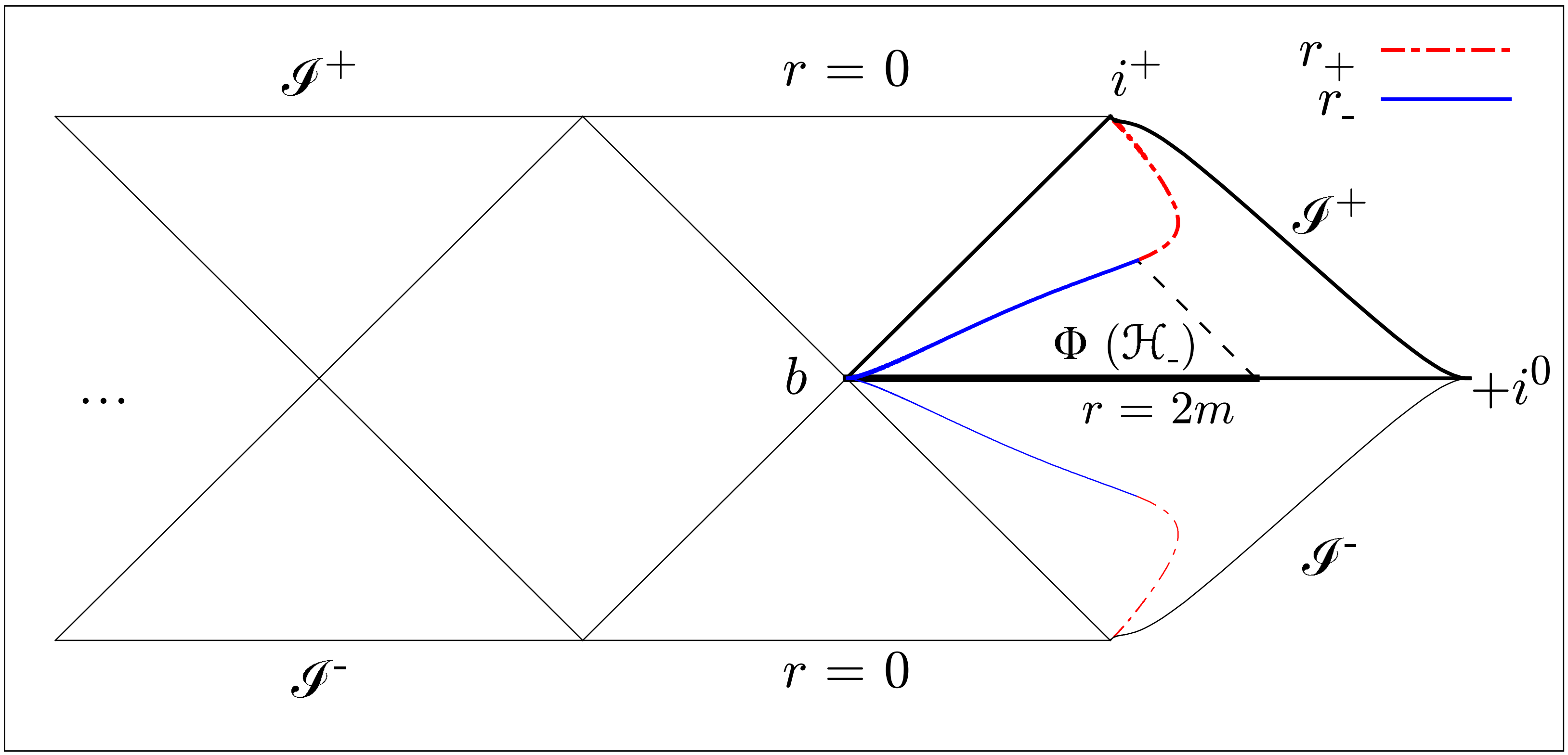

Convergence to H_0

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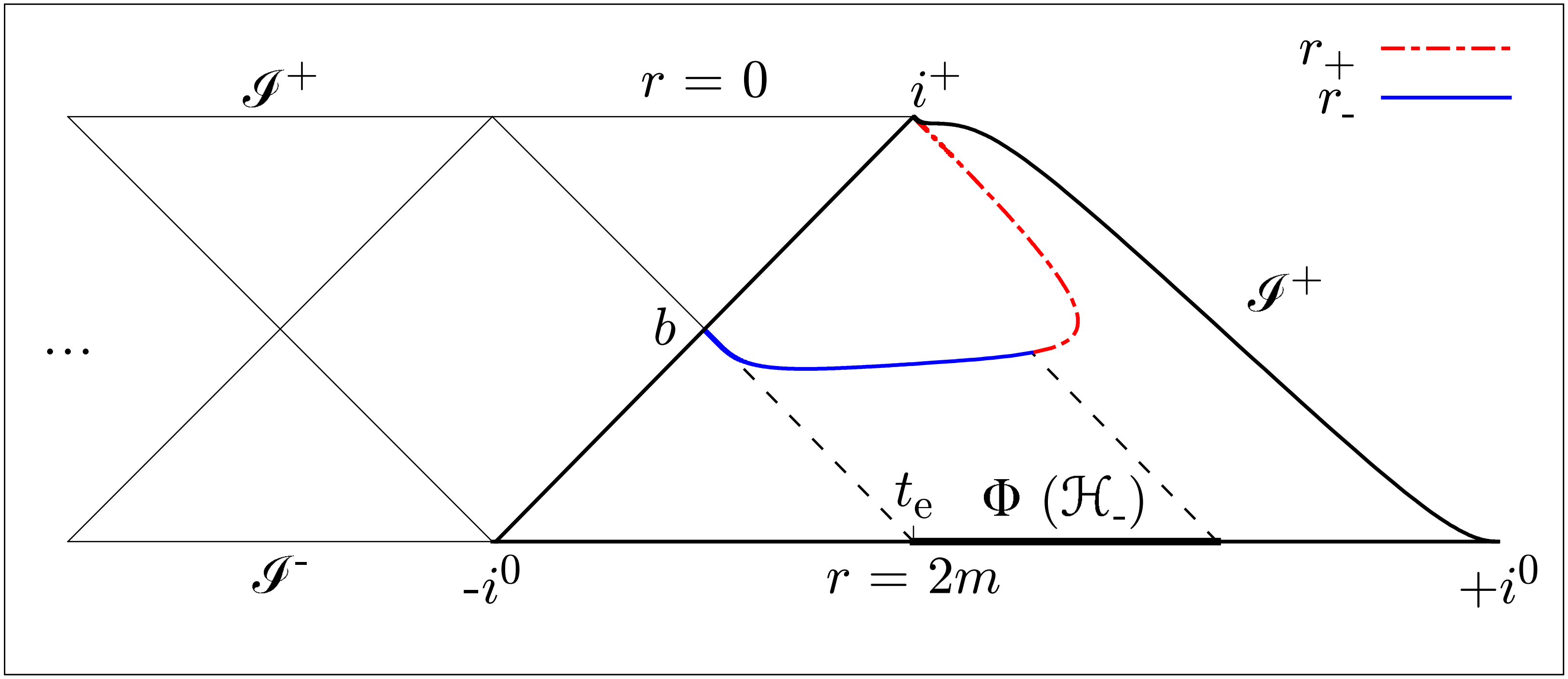
  - Φ compact
    - Causal curves may depart $r_\star$ and never reach r_-
    - Spacetime connects to both inner and middle regions of Schwarzschild–de Sitter


r_+
r_-
$\mathscr{I}^+$
$r = 0$
i^+
$\mathscr{I}^+$
b
...
t_e
$\Phi\,(\mathcal{H}_-)$
$\mathscr{I}^-$
$-i^0$
$r = 2m$
$+i^0$

Cosmological history

- We can determine the fate of null geodesics via the intermediate value theorem [da Silva, Fontanini, DCG, PRD **87**,064030 (2013)]
- Find known curves that bind the image of ingoing geodesics from above and below

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- $A = R(r_\infty) + r_\infty R'(r_\infty)$, $B = R(r_\infty)\left(R'(r_\infty) - H_0\right)$, $\Delta H = H - H_0$, $t_i > 0$

$$F_+(t_i, t) = \int_{t_i}^{t} e^{(B-\delta)u} e^{-A\int_{t_i}^{u} \Delta H(s)\mathrm{d}s} \Delta H(u)\mathrm{d}u$$

$$F_-(t_i, t) = \int_{t_i}^{t} e^{(B+\bar{\delta})u} e^{-A\int_{t_i}^{u} \Delta H(s)\mathrm{d}s} \Delta H(u)\mathrm{d}u$$

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Causal structure of McVittie depends on how fast $H \to H_0$.

Example: ΛCDM

- Source: dark matter and cosmological constant
- Einstein's equations result in the same H as in FLRW

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- Convergence depends on the sign of $B - 3H_0$

(a) $B - 3H_0 < 0 \implies F_-$ converges

(b) $B - 3H_0 > 0 \implies F_+$ diverges

(c) $B - 3H_0 = 0 \implies$ inconclusive

- Convergence depends on the sign of η

$$\eta \equiv \frac{B}{3H_0} - 1 = \frac{R(r_\infty)}{3}\left[\frac{R'(r_\infty)}{H_0} - 1\right] - 1$$

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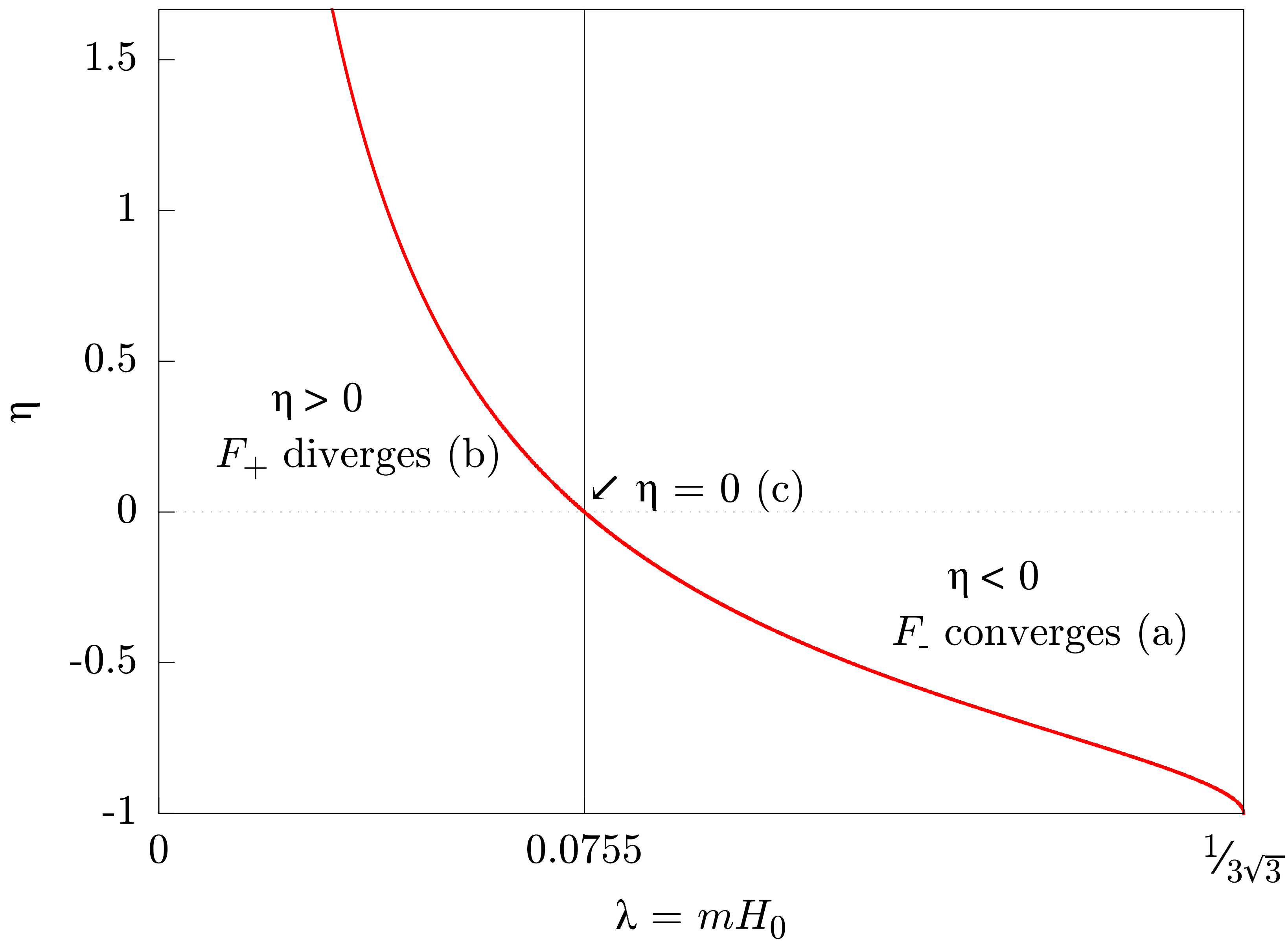
- Inner horizon on the limit $H \to H_0$

$$r_\infty = \frac{2}{H_0\sqrt{3}}\cos\left[\frac{\pi}{3} + \frac{1}{3}\arccos\left(3\sqrt{3}mH_0\right)\right]$$

- η depends only on the product $mH_0 \equiv \lambda$
- Non-extreme Schwarzschild–de Sitter at infinity

$$0 < \lambda < \frac{1}{3\sqrt{3}}$$


1.5
1
0.5
0
-0.5
-1
η
η > 0
F+ diverges (b)
↙ η = 0 (c)
η < 0
F- converges (a)
0
0.0755
1/3√3
λ = mH0

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  - Non empty space
  - Incompressible fluid
  - Shear-free motion

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  - Big Bang not on intuitive location
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  - Local physics dependent on global structure

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- Coupling to a scalar field opens possibilities for modified gravity
- A fluid with more degrees of freedom provides new mechanisms for interaction between the black hole and the environment (stay tuned for M. Fontanini's talk)